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STABILITY THEOREM FOR THE FEYNMAN INTEGRAL VIA TIME CONTINUATION

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ABSTRACT. Lapidus proved a stability theorem for the Feynman integral as a bounded linear operator on $L_2(\mathbf{R}^d)$ with respect to potential functions. We establish a stability theorem for the Feynman integral with respect to measures whose positive and negative variations are in the generalized Kato class. This is a partial extension of Lapidus's result. In fact, we develop our stability theorem under a more general setting in the sense that potential functions in Lapidus's paper are involved in the Kato class and the measures in this paper are involved in the generalized Kato class which generalizes substantially the Kato class.

0. Introduction. The purpose of this paper is to study the stability of the analytic (in time) operator-valued Feynman integral with respect to certain functions determined by measures in the generalized Kato class. In 1984, Johnson proved the dominated convergence theorem for the Feynman integral as an operator from $L_2(\mathbf{R}^d)$ to $L_2(\mathbf{R}^d)$ [**10**]. As far as we know, this is the first stability theorem for the Feynman integral. Since then, many mathematicians have proved stability theorems for the Feynman integral as either $\mathcal{L}(L_p(\mathbf{R}^d), L_{p'}(\mathbf{R}^d))$ theory [**6, 12**] or $\mathcal{L}(L_1(\mathbf{R}), C_0(\mathbf{R}))$ theory [**5**], where p is a real number such that $1 < p \leq 2$ and d is a positive integer such that $d < 2p/(2-p)$ for $1 < p < 2$.

In [**15**], Lapidus proved a stability theorem for the Feynman integral as an $\mathcal{L}(L_2(\mathbf{R}^d))$ theory with respect to the potential functions V, V_m, $m = 1, 2, \dots$, satisfying the following conditions: V_m converges to V almost everywhere in $\mathbf{R}^d$ and there exist $U \in L^1_{\text{loc}}$ and $W \in (L^p_{\text{loc}})_{\tilde{u}}$ such that, for all $m \geq 1$, $V_m^+ \leq U$ almost everywhere and $V_m^- \leq W$

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almost everywhere where $p = 1$ if $d = 1$ and $p > (d/2)$ if $d \geq 2$. Let K_d denote the class of all Kato functions. (See [**11**] for precise definitions of K_d, L^1_{loc} and $(L^p_{\text{loc}})_{\tilde{u}}$). It is helpful to point out that K_d properly contains $(L^p_{\text{loc}})_{\tilde{u}}$ for the following discussion [**11**].

In this paper we introduce recent results of the existence theorem for the analytic (in time) operator-valued Feynman integral first, and then we prove a stability theorem for the Feynman integral with respect to signed measures μ, μ_n, $n = 1, 2, \dots$ satisfying the following conditions: For each Borel set E in $\mathbf{R}^d$, $\mu_n(E)$ converges to $\mu(E)$ and $\{\mu_n(E)\}_{n=1}^{\infty}$, $\{\mu_n^-(E)\}_{n=1}^{\infty}$ are nonincreasing sequences and there exist $\nu \in GK_d$ and $\eta \in GK_d$ such that $\mu_n^+ \leq \nu$ and $\mu_n^- \leq \eta$ for all $n \in \mathbf{N}$. Here GK_d stands for the generalized Kato class. GK_d is a substantial generalization of the Kato class K_d in the sense that if f is a Kato class function on $\mathbf{R}^d$, then $|f| \cdot m$ is in GK_d where m is the Lebesgue measure on $\mathbf{R}^d$. In fact, we develop our stability theorem under a more general setting in the sense that potential functions in Lapidus's paper are involved in K_d as we discussed in the above paragraph, and the measures in this paper which generalize the potential functions are in $GK_d - GK_d$. Theorem 3.6 and Theorem 3.3 play a key role to prove Theorem 4.2, which is our main theorem in this paper. The statement and the proof of Theorem 3.6 are quite close to those of Theorem 3.1 in [**15**]. Theorem 3.3 is borrowed from [**13**].

1. Preliminaries.

In this section we recall definitions and results related to Brownian motion, positive continuous additive functionals, measures in the generalized Kato class, closed forms and their associated operators.

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, X_t, P_x)$ be the canonical Brownian motion on $\mathbf{R}^d$ [**4**]. Let t be a nonnegative real number. For each ω in $\Omega = C([0, \infty), \mathbf{R}^d)$, the collection of all continuous functions from $[0, \infty)$ to $\mathbf{R}^d$, we define a function $\theta_t \omega : [0, \infty) \to \mathbf{R}^d$ by $(\theta_t \omega)(s) = \omega(t + s)$ for all s in $[0, \infty)$.

Definition 1.1. A function $A : [0, \infty) \times \Omega \to \mathbf{R}$ is called a positive continuous additive functional, abbreviated by PCAF, if $A(t, \cdot) = A_t$ is $\mathcal{F}_t$-measurable for each t and there exists $\Lambda \in \mathcal{F}$, called a defining set of A, satisfying the following properties:

(i) $P_x(\Lambda) = 1$ for all x in $\mathbf{R}^d$.

(ii) $\theta_t \omega \in \Omega$ for all ω in Λ.

(iii) For each ω in Λ, the function $A_{\cdot}(\omega) : [0, \infty) \to \mathbf{R}$ is continuous, increasing and vanishes at 0 and is additive in the sense that

$$A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t \omega)$$

for all $t, s \geq 0$.

For a nonnegative bounded Borel measurable function V on $\mathbf{R}^d$, we consider a function A^V defined on $[0, \infty) \times \Omega$ by

$$A^V(t, \omega) = A_t^V(\omega) = \int_0^t V(\omega(s))\, ds \tag{1.1}$$

for all (t, ω) in $[0, \infty) \times \Omega$. This is a typical example of a positive continuous additive functional.

Definition 1.2. A positive Borel measure μ on $\mathbf{R}^d$ is said to be in the generalized Kato class if

$$\lim_{\alpha \to 0^+} \sup_{x \in \mathbf{R}^d} \int_{|x-y| \leqslant \alpha} \frac{\mu(dy)}{|x-y|^{d-2}} = 0, \quad d \geqslant 3,$$

$$\lim_{\alpha \to 0^+} \sup_{x \in \mathbf{R}^d} \int_{|x-y| \leqslant \alpha} (\log |x-y|^{-1}) \mu(dy) = 0, \quad d = 2,$$

$$\sup_{x \in \mathbf{R}^d} \int_{|x-y| \leqslant 1} \mu(dy) < \infty, \quad d = 1.$$

We denote by GK_d the generalized Kato class.

Let $H^1(\mathbf{R}^d)$ be the standard Sobolev space, i.e.,

$$H^1(\mathbf{R}^d) \equiv \left\{ u \in L_2(\mathbf{R}^d, m) \middle| \, \frac{\partial u}{\partial x_i} \in L_2(\mathbf{R}^d, m), 1 \leqslant i \leqslant d \right\}, \tag{1.2}$$

where $L_2(\mathbf{R}^d, m)$ denotes the space of $\mathbf{R}$-valued functions on $\mathbf{R}^d$ which are square integrable with respect to the Lebesgue measure m and the derivatives are taken in the distributional sense. In this paper we adopt

$L_2(\mathbf{R}^d)$ instead of $L_2(\mathbf{R}^d, m)$. For a form q and an operator H, $D(q)$ and $D(H)$ stand for the domains of q and H, respectively. We let $\mathcal{E}$ denote the classical Dirichlet form, that is, the bilinear form acting on $D(\mathcal{E}) \equiv H^1(\mathbf{R}^d)$:

$$\mathcal{E}(u,v) = \frac{1}{2}\int_{\mathbf{R}^d} \nabla u \cdot \nabla v \, dm. \tag{1.3}$$

For a signed Borel measure $\mu = \mu^+ - \mu^-$ on $\mathbf{R}^d$ where μ^+ and μ^- are the usual positive and negative variations of μ, respectively, we say that μ is in $GK_d - GK_d$ if μ^+ and μ^- are both in GK_d. For μ in $GK_d - GK_d$, we define $\mathcal{Q}_\mu$ and $\mathcal{E}_\mu$ as follows:

$$\mathcal{Q}_\mu(u,v) \equiv \int_{\mathbf{R}^d} uv \, d\mu = \int_{\mathbf{R}^d} uv \, d\mu^+ - \int_{\mathbf{R}^d} uv \, d\mu^- \tag{1.4}$$

for all u, v in $D(\mathcal{Q}_\mu) \equiv L_2(\mathbf{R}^d, |\mu|) \cap L_2(\mathbf{R}^d)$ and

$$\mathcal{E}_\mu(u,v) \equiv \mathcal{E}(u,v) + \mathcal{Q}_\mu(u,v) \tag{1.5}$$

for all u, v in $D(\mathcal{E}_\mu) \equiv D(\mathcal{E}) \cap D(\mathcal{Q}_\mu)$.

For μ in $GK_d - GK_d$, let A^{μ^+} and A^{μ^-} be PCAFs corresponding to μ^+ and μ^-, respectively. (The existence of A^{μ^+} and A^{μ^-} is guaranteed by [**1**, Theorem 2.1.4].) We let $A_t^\mu = A_t^{\mu^+} - A_t^{\mu^-}$. Then $(A_t^\mu)_{t \geqslant 0}$ is a continuous additive functional which has finite variation on every bounded interval [**8**]. Let us introduce the notation

$$p_t^\mu f(x) = E_x[e^{-A_t^\mu} f(\omega(t))] \tag{1.6}$$

provided that the right-hand side in (1.6) makes sense for $f \in L_2(\mathbf{R}^d)$ where E_x stands for the expectation with respect to P_x and P_x is the probability measure associated with the Brownian paths in $\mathbf{R}^d$ which start at x at time 0.

Let $\mathcal{H}$ be a real or complex Hilbert space equipped with the inner product $\langle\, , \rangle$ and the norm $\|\cdot\|$. From [**13**], we have the following theorem.

Theorem 1.3. *Let q be a densely defined, symmetric closed form in $\mathcal{H}$ which is bounded below by γ. Then there exists a unique bounded* *below self-adjoint operator H satisfying that, for any $\xi \leqslant \gamma$, $D(q) = D((H-\xi)^{1/2})$ and $q(u,v) = \langle (H-\xi)^{1/2}u, (H-\xi)^{1/2}v\rangle + \xi\langle u,v\rangle$, for all u,v in $D(q)$. Furthermore, $q(u,v) = \langle Hu, v\rangle$ for all u in $D(H)$, v in $D(q)$.*

From [**1**, Proposition 3.4.3 and Proposition 3.4.4], we have the following proposition.

Proposition 1.4. *Let $\mu = \mu^+ - \mu^-$ be in $GK_d - GK_d$. Then*

(i) *$\mathcal{E}_\mu$ is a densely defined symmetric bilinear form with domain $D(\mathcal{E}_\mu) = D(\mathcal{E}) \cap D(Q_\mu)$.*

(ii) *$\mathcal{E}_\mu$ is closed and bounded below.*

(iii) *$(p_t^\mu)_{t\geqslant 0}$ is a strongly continuous symmetric semigroup on $L_2(\mathbf{R}^d)$.*

Moreover, let H^μ be the bounded below self-adjoint operator corresponding to $(\mathcal{E}_\mu, D(\mathcal{E}_\mu))$ whose existence is guaranteed by Theorem 1.3, and let $\tilde{H}^\mu$ be the infinitesimal generator of $(p_t^\mu)_{t\geqslant 0}$. Then

(iv) $H^\mu = \tilde{H}^\mu$,

and hence we have

$$p_t^\mu f(x) = e^{-tH^\mu} f(x) \tag{1.7}$$

for all f in $L_2(\mathbf{R}^d)$.

Remark. By (1.6) and (1.7), we obtain the Feynman-Kac formula

$$e^{-tH^\mu} f(x) = E_x[e^{-A_t^\mu(\omega)} f(\omega(t))] \tag{1.8}$$

for every f in $L_2(\mathbf{R}^d)$, m-almost everywhere x in $\mathbf{R}^d$ and for all $t \geq 0$.

Now we extend $\mathcal{E}_\mu$ to the subspace $D(\mathcal{E}_\mu^{\mathbf{C}}) \equiv D(\mathcal{E}_\mu) + iD(\mathcal{E}_\mu)$ of $L_2(\mathbf{R}^d, \mathbf{C}) \equiv L_2(\mathbf{R}^d) + iL_2(\mathbf{R}^d)$ where $i = \sqrt{-1}$.

Define $\mathcal{E}_\mu^{\mathbf{C}} : D(\mathcal{E}_\mu^{\mathbf{C}}) \longrightarrow \mathbf{C}$ by

$$\mathcal{E}_\mu^{\mathbf{C}}(u,v) \equiv \int_{\mathbf{R}^d} \nabla u \cdot \overline{\nabla v}\, dm + \int_{\mathbf{R}^d} u\bar{v}\, d\mu \tag{1.9}$$

for all u, v in $D(\mathcal{E}_\mu^{\mathbf{C}})$. From [**1**], we have the following propositions.

Proposition 1.5. *Let μ be in $GK_d - GK_d$. Then, for $u = u_1 + iu_2$, $v = v_1 + iv_2$ in $D(\mathcal{E}_\mu^{\mathbf{C}})$, $\mathcal{E}_\mu^{\mathbf{C}}$ is represented as follows:*

$$\mathcal{E}_\mu^{\mathbf{C}}(u,v) = \mathcal{E}_\mu(u_1,v_1) + \mathcal{E}_\mu(u_2,v_2) + i[\mathcal{E}_\mu(u_2,v_1) - \mathcal{E}_\mu(u_1,v_2)]. \tag{1.10}$$

Proposition 1.6. *Let $\mu = \mu^+ - \mu^-$ be in $GK_d - GK_d$. Then*

(i) *$\mathcal{E}_\mu^{\mathbf{C}}$ is a densely defined symmetric sesquilinear form.*

(ii) *$\mathcal{E}_\mu^{\mathbf{C}}$ is bounded below and closed.*

Moreover, let $H_{\mathbf{C}}^\mu$ be the bounded below self-adjoint operator corresponding to $(\mathcal{E}_\mu^{\mathbf{C}}, D(\mathcal{E}_\mu^{\mathbf{C}}))$ whose existence is guaranteed by Theorem 1.3. *Then we obtain*

$$(e^{-tH_{\mathbf{C}}^\mu}u)(x) = E_x[e^{-A_t^\mu(\omega)}u(\omega(t))] \tag{1.11}$$

for every u in $L_2(\mathbf{R}^d, \mathbf{C})$, m-almost everywhere x in $\mathbf{R}^d$ and for all $t \geqslant 0$.

2. The existence of the analytic (in time) operator-valued Feynman integral. Now we introduce the definition and the existence theorem of the analytic (in time) operator-valued Feynman integral of functions that we are especially interested in. Given ω in $\Omega = C([0,\infty), \mathbf{R}^d)$, let

$$F_t^\mu(\omega) = F^\mu(\omega) = e^{-A_t^\mu(\omega)} \tag{2.1}$$

where μ is in $GK_d - GK_d$ and A_t^μ is given in Section 1. Let $\mathcal{C}, \mathcal{C}_+$ and $\overline{\mathbf{C}}_+$ be the set of all complex numbers, all complex numbers with positive real part and all nonzero complex numbers with nonnegative real part, respectively.

Definition 2.1. Given $t > 0$, $u \in L_2(\mathbf{R}^d, \mathbf{C})$ and $x \in \mathbf{R}^d$, consider the expression

$$\begin{aligned}(J^t(F^\mu)u)(x) &= E_x\{e^{-A_t^\mu(\omega)}u(\omega(t))\}\\ &= \int_{\Omega_x} e^{-A_t^\mu(\omega)}u(\omega(t))\,dP_x(\omega),\end{aligned} \tag{2.2}$$

where Ω_x is the set of ω in $C([0,\infty),\mathbf{R}^d)$ such that $\omega(0)=x$ and P_x is the probability measure associated with the Brownian paths in $\mathbf{R}^d$ which start at x at time 0. We say that the operator-valued function space integral $J^t(F^\mu)$ exists for $t>0$ if (2.2) defines $J^t(F^\mu)$ as an element of $\mathcal{L}(L_2(\mathbf{R}^d,\mathbf{C}))$, the space of bounded linear operators on $L_2(\mathbf{R}^d,\mathbf{C})$. If $J^t(F^\mu)$ exists for every $t>0$ and, in addition, has an extension as a function of t to an analytic operator-valued function on $\mathbf{C}_+$, and a strongly continuous function on $\overline{\mathbf{C}}_+$, we say that $J^t(F^\mu)$ exists for all $t\in\overline{\mathbf{C}}_+$. When t is purely imaginary, $J^t(F^\mu)$ is called the analytic (in time) operator-valued Feynman integral of F^μ.

The following theorem comes from [**1**]. We state it and give a sketch of its proof for convenience.

Theorem 2.2. *Let $\mu=\mu^+-\mu^-$ be in GK_d-GK_d and let $\mathcal{E}_\mu^{\mathbf{C}}$ be given by (1.9) and $H_{\mathbf{C}}^\mu$ be the self-adjoint operator corresponding to $(\mathcal{E}_\mu^{\mathbf{C}},D(\mathcal{E}_\mu^{\mathbf{C}}))$. Then $J^t(F^\mu)$ exists for all $t\in\overline{\mathbf{C}}_+$ and has the representation*

$$J^t(F^\mu)=e^{-tH_{\mathbf{C}}^\mu} \tag{2.3}$$

for all $t\in\overline{\mathbf{C}}_+$, where $e^{-tH_{\mathbf{C}}^\mu}$ is given meaning via the spectral theorem applied to the self-adjoint operator $H_{\mathbf{C}}^\mu$. In particular, for $t\in\mathbf{R}$, the analytic (in time) operator-valued Feynman integral $J^{it}(F^\mu)$ exists and we have

$$J^{it}(F^\mu)=e^{-itH_{\mathbf{C}}^\mu} \tag{2.4}$$

where $\{e^{-itH_{\mathbf{C}}^\mu},\ t\in\mathbf{R}\}$ is the unitary group corresponding to the self-adjoint operator $H_{\mathbf{C}}^\mu$.

Proof. By Proposition 1.4, $\mathcal{E}_\mu$ given by (1.5) is a densely defined, symmetric closed bilinear form which is bounded below and the continuous additive functional A_t^μ is related to the operator H^μ by the Feynman-Kac formula (1.8). Hence in the light of Theorem 2.2.5 in [**1**], the proof is complete.

Remark 2.3. Let S denote the family of all smooth measures. It is well-known fact that GK_d is properly contained in S [**1, 3**]. Under certain conditions on $\mu = \mu^+ - \mu^-$ in $S - S$, the existence theorem of the analytic (in time) operator-valued Feynman integral of the function given by μ was proved in [**1**]. So we expect that the stability theorem which will be proved in Section 4 can be extended to a stability theorem for the Feynman integral with respect to certain functions determined by smooth measures.

3. Perturbation of forms. In order to prove Theorem 4.2, the main result of this paper, we use some known results in operator theory and a perturbation theorem which we prove in this section. In fact, the main theorem is closely related to perturbation theories for closed forms.

Unless otherwise specified, let $\mathcal{H}$ denote a complex Hilbert space with the inner product $\langle\, ,\, \rangle$ and the norm $\|\cdot\|$. Furthermore, for x_n, x in $\mathcal{H}$, let $x_n \to x$ denote that x_n is strongly convergent to x and $x_n \overset{w}{\to} x$ denote that x_n is weakly convergent to x. For operators A_n, A on $\mathcal{H}$, let $A_n \to A$ indicate that A_n converges to A in the strong operator topology.

Definition 3.1. Let A, A_m, $m = 1, 2, \dots$ be self-adjoint operators on $\mathcal{H}$. We say that $\{A_m\}_{m=1}^{\infty}$ converges to A in the strong resolvent sense if

$$[I + iA_m]^{-1} \longrightarrow [I + iA]^{-1},$$

where I denotes the identity operator and $i = \sqrt{-1}$.

From [**14**] and [**13**], we have the following two theorems, respectively.

Theorem 3.2 (Trotter, Kato, Rellich, Neveu). *Let H, H_m, $m = 1, 2, \dots$, be self-adjoint operators on $\mathcal{H}$. Then the following statements are equivalent*:

(a) $\{H_m\}_{m=1}^{\infty}$ *converges to H in the strong resolvent sense.*

(b) $e^{-itH_m} \to e^{-itH}$ *for all t in* $\mathbf{R}$.

(c) $[I + i\lambda H_m]^{-1} \to [I + i\lambda H]^{-1}$ *for all λ in $\mathbf{R}$, $\lambda \neq 0$.*

(d) $e^{-itH_m} \to e^{-itH}$, *uniformly in t on any compact subset of* $\mathbf{R}$.

If, in addition, the operators H_m and H are uniformly bounded below, *then* (a) *implies*:

(e) $e^{-tH_m} \to e^{-tH}$, *uniformly in* t *on any compact subset of* $[0, +\infty)$.

Theorem 3.3. *Let* $\{t_n\}$ *be a nonincreasing sequence of densely defined, closed symmetric forms in* $\mathcal{H}$ *which are uniformly bounded below by* γ. *If* H_n *is the self-adjoint operator associated with* t_n, *then* H_n *converges to a self-adjoint operator* $H \geq \gamma$ *strongly in the generalized sense. Furthermore,* $(H_n - \xi)^{1/2}u \xrightarrow{w} (H - \xi)^{1/2}u$ *for all* u *in* $\cup_n D(t_n)$ *and* $\xi < \gamma$. *If, in particular, the symmetric form* t *defined by* $t(u, u) = \lim_{n\to\infty} t_n(u, u)$ *with* $D(t) = \cup_n D(t_n)$ *is closable, then* H *is the self-adjoint operator associated with* $\tilde{t}$, *the closure of* t, *and* $(H_n - \xi)^{1/2}u \to (H - \xi)^{1/2}u$ *for all* u *in* $D(t)$ *and* $\xi < \gamma$.

Remark 3.4. Let q be a densely defined, symmetric closed form in $\mathcal{H}$ which is bounded below and let H be the self-adjoint operator corresponding to q. If x is in $D(q)$, z in $\mathcal{H}$ and $q(x, y) = \langle z, y\rangle$ for every y belonging to a core of q, then x is in $D(H)$ and $Hx = z$. (See [**13**, Theorem 2.1, p. 322].)

Remark 3.5. Let D' be a core of a closed form t and H be the bounded below (by α) self-adjoint operator corresponding to t. Suppose that $\langle u, y\rangle = \langle w, (H - \alpha)^{1/2}y\rangle$ for all y in D' where u, w are in $\mathcal{H}$. Then we can easily prove that $\langle u, y\rangle = \langle w, (H - \alpha)^{1/2}y\rangle$ for all y in $D(t)$.

Theorem 3.6. *Let* t, t_n, $n = 1, 2, \ldots$ *be densely defined, symmetric closed forms in* $\mathcal{H}$ *satisfying the following properties where* H *and* H_n *are the self-adjoint operators associated with* t *and* t_n, *respectively*:

(i) $D(t_n) \subset D(t)$, $n = 1, 2, \ldots$.

(ii) $t(u, u) \geq \gamma\langle u, u\rangle$, *for all* u *in* $D(t)$, $t_n(u, u) \geq \gamma\langle u, u\rangle$, *for all* u *in* $D(t_n)$, $n = 1, 2, \ldots$ *with* $\gamma < 0$.

(iii) *There is a core* D' *of* t *such that* $D' \subset \liminf D(t_n)$ *and, for some* $\alpha < \gamma$, $(H_n - \alpha)^{1/2}u \to (H - \alpha)^{1/2}u$ *for all* u *in* D'.

Then $\{H_n\}_{n=1}^{\infty}$ *converges to* H *in the strong resolvent sense.*

Proof. Fix $v \in \mathcal{H}$ and let $w_n = [I + iH_n]^{-1}(v)$. Then $w_n \in D(H_n)$

and

$$\|w_n\| \leq \|v\|. \tag{3.1}$$

Clearly

$$v = w_n + iH_n w_n.$$

By virtue of Theorem 1.3, for $\alpha < \gamma$, we have the following equalities:

$$\begin{gathered} D(t) = ((H-\alpha)^{1/2}), \\ t(u,v) = \langle (H-\alpha)^{1/2}u, (H-\alpha)^{1/2}v\rangle + \alpha\langle u,v\rangle, \forall\, u,v \in D(t), \\ D(t_n) = ((H_n-\alpha)^{1/2}) \quad \text{and} \\ t_n(u,v) = \langle (H_n-\alpha)^{1/2}u, (H_n-\alpha)^{1/2}v\rangle + \alpha\langle u,v\rangle, \forall\, u,v \in D(t_n). \end{gathered}$$

Consequently,

$$\langle v, w_n\rangle = \|w_n\|^2 + i\{\|(H_n-\alpha)^{1/2}w_n\|^2 + \alpha\|w_n\|^2\}. \tag{3.2}$$

By (3.1) and the Cauchy-Schwarz inequality,

$$|\mathrm{Im}\,(\langle v, w_n\rangle)| \leq |\langle v, w_n\rangle| \leq \|v\|^2. \tag{3.3}$$

Hence, by (3.2) and (3.3), we have

$$\|(H_n-\alpha)^{1/2}w_n\|^2 + \alpha\|w_n\|^2 \leq \|v\|^2.$$

Now we conclude that $\{(H_n-\alpha)^{1/2}w_n\}$ is a bounded sequence since $\alpha < 0$ and

$$\begin{aligned} \|(H_n-\alpha)^{1/2}w_n\|^2 &\leq \|v\|^2 - \alpha\|w_n\|^2 \\ &\leq (1-\alpha)\|v\|^2. \end{aligned} \tag{3.4}$$

Hence, by (3.1), (3.4) and the Banach-Alaoglu Theorem [**17**], there exist vectors w and u in $\mathcal{H}$ such that

$$w_n \xrightarrow{w} w \quad \text{and} \quad (H_n-\alpha)^{1/2}w_n \xrightarrow{w} u, \tag{3.5}$$

along some subsequence $\{n_j\} \to \infty$.

We claim that:

$$w \in D(H) \quad \text{and} \quad (H-\alpha)^{1/2}w = u. \tag{3.6}$$

To prove (3.6), let $y \in D'$. Then, by (3.5) and the hypothesis (iii), we have

$$\langle u, y\rangle = \lim_{n_j\to\infty} \langle w_{n_j}, (H_{n_j}-\alpha)^{1/2}y\rangle = \langle w, (H-\alpha)^{1/2}y\rangle.$$

Since D' is a core of t, it follows that by Remark 3.5

$$\langle u, y\rangle = \langle w, (H-\alpha)^{1/2}y\rangle$$
$$\text{for all } y \text{ in } D(t) = D((H-\alpha)^{1/2}).$$

Consequently,

$$w \in D(((H-\alpha)^{1/2})^*) = D((H-\alpha)^{1/2})$$
$$\text{and} \quad (H-\alpha)^{1/2}w = u$$

where $((H-\alpha)^{1/2})^*$ represents the adjoint operator of $(H-\alpha)^{1/2}$. To complete the proof of (3.6), it only remains to show that $w \in D(H)$. Now, let $y \in D'$. By (3.5), (3.6) and the hypothesis (iii), we have

$$\begin{aligned}\langle v, y\rangle &= \langle (I+iH_{n_j})w_{n_j}, y\rangle\\ &= \langle w, y\rangle + i\,[\lim_{n_j\to\infty} t_{n_j}(w_{n_j}, y)]\\ &= \langle w, y\rangle + i\,\{\langle (H-\alpha)^{1/2}w, (H-\alpha)^{1/2}y\rangle + \alpha\langle w, y\rangle\}\\ &= \langle w, y\rangle + i\,t(w, y).\end{aligned}$$

Note that we obtain the second equality in the above equation by Theorem 1.3 since $w_{nj} \in D(H_{nj})$. Hence $t(w,y) = \langle (v-w)/i, y\rangle$ for all $y \in D'$. Since D' is a core of t, we have $w \in D(H)$ and $Hw = (v-w)/i$ by Remark 3.4. This completes the proof of (3.6). Furthermore, we have

$$w = [I+iH]^{-1}v. \tag{3.7}$$

Now, it is sufficient to show that $w_n \to w$, that is,

$$[I+iH_n]^{-1}(v) \longrightarrow [I+iH\,]^{-1}(v) \tag{3.8}$$

to complete our proof.

In the light of (3.7), the limit w in (3.5) does not depend on the subsequence $\{w_{n_j}\}$. By a standard compactness argument, it follows that $w_n \stackrel{w}{\to} w$ as $n \to \infty$. Hence, by (3.2),

$$\lim_{n\to\infty} \|w_n\|^2 = \lim_{n\to\infty} \mathrm{Re}\,(\langle v, w_n\rangle) = \mathrm{Re}\,(\langle v, w\rangle) = \|w\|^2.$$

Consequently, we have $w_n \to w$ as desired. □

4. Stability theorem. A stability theorem for the analytic (in time) operator-valued Feynman integral, which is our main theorem in this paper, is proved in this section.

Proposition 4.1. *Let μ be a measure in $GK_d - GK_d$. Then $D(\mathcal{E}_\mu) \cap C_0(\mathbf{R}^d)$ is a core of $\mathcal{E}_\mu$ where $C_0(\mathbf{R}^d)$ denotes the family of all continuous functions on $\mathbf{R}^d$ with compact support and hence $H^1(\mathbf{R}^d) \cap C_0(\mathbf{R}^d)$ is a core of $\mathcal{E}_\mu$.*

Proof. Since μ is a measure in $GK_d - GK_d$, there exist $\lambda > 1$ and real constants c and β such that $||p_t^{\mu^+ - \lambda\mu^-} f||_2 \le ce^{\beta t}||f||_2$ for all $t \ge 0$ and $f \in L_2(\mathbf{R}^d)$. (See [**1**, Proposition 3.4.7 and Theorem 3.4.8]). Noting that $|\mu| = \mu^+ + \mu^-$ is a Radon measure and $\mathcal{E}_\mu$ is a closed form, we conclude that $D(\mathcal{E}_\mu) \cap C_0(\mathbf{R}^d) = H^1(\mathbf{R}^d) \cap C_0(\mathbf{R}^d)$ is a core of $\mathcal{E}_\mu$. (See [**2**, Theorem 5.8].) □

Theorem 4.2. *Let μ, μ_n, $n = 1, 2, \cdots$ be signed measures on $(\mathbf{R}^d, \mathcal{B}(\mathbf{R}^d))$ satisfying the following properties*:

(i) *For each Borel set E in δ, $\mu_n(E)$ converges to $\mu(E)$ as $n \to \infty$ and $\{\mu_n(E)\}_{n=1}^\infty$, $\{\mu_n^-(E)\}_{n=1}^\infty$ are nonincreasing sequences.*

(ii) *There exist $\nu \in GK_d$ and $\eta \in GK_d$ such that*

$$\mu_n^+ \le \nu, \quad \mu_n^- \le \eta$$

for all $n \in \mathbf{N}$.

For simplicity, let $t_n = \mathcal{E}_{\mu_n}^{\mathbf{C}}$ and $t = \mathcal{E}_\mu^{\mathbf{C}}$ where $\mathcal{E}_{\mu_n}^{\mathbf{C}}$ and $\mathcal{E}_\mu^{\mathbf{C}}$ are given in Section 1. Assume that t_n is uniformly bounded below by $\alpha < 0$. Then

$\{H_n\}_{n=1}^{\infty}$ converges to H in the strong resolvent sense where H_n and H are self-adjoint operators associated with t_n and t, respectively.

Remark 4.3. Using hypothesis (i) in the above theorem, we get $\{\mu_n^+(E)\}_{n=1}^{\infty}$ is a nonincreasing sequence for each Borel set E in $\mathbf{R}^d$. Then the definition of GK_d and hypothesis (ii) imply that, for all $n \in \mathbf{N}$, μ_n is in $GK_d - GK_d$.

Remark 4.4. A simple proof shows that the limiting measure μ is in $GK_d - GK_d$. To prove this, let $E \in \mathcal{B}(\mathbf{R}^d)$. Then, $\mu_n^+(E) \to \inf\{\mu_n^+(E)\}$ and $\mu_n^-(E) \to \inf\{\mu_n^-(E)\}$ as $n \to \infty$ by the monotone convergence theorem for sequences. Since $\mu_n(E) \to \mu(E) = \mu^+(E) - \mu^-(E)$, we get $\mu = \mu^+ - \mu^- = \inf \mu_n^+ - \inf \mu_n^-$ and this implies that $\inf \mu_n^+ \geq \mu^+$ and $\inf \mu_n^- \geq \mu^-$. Hence we conclude that $\mu^+ \in GK_d$ and $\mu^- \in GK_d$.

Proof of Theorem 4.2. For each $n \in \mathbf{N}$, t_n is a densely defined, closed symmetric form which is bounded below by Remark 4.3 and Proposition 1.6. Using (1.10) and hypotheses on measures μ_n, a direct calculation shows that t_n is a nonincreasing sequence of forms. Since t_n is uniformly bounded below by α, we can define

$$q(f, f) = \lim_{n\to\infty} t_n(f, f) \tag{4.1}$$

for all f in $D(q) = \cup_n D(t_n)$. Let $f = g + ih$ be in $\cup_n D(t_n)$. By (1.10) and (4.1), we have

$$\begin{aligned} q(f, f) &= \mathcal{E}(g, g) + \mathcal{E}(h, h) \\ &\quad + \lim_{n\to\infty} \left[\int_{\mathbf{R}^d} |g|^2 \, d\mu_n + \int_{\mathbf{R}^d} |h|^2 \, d\mu_n \right]. \end{aligned} \tag{4.2}$$

We claim that $q \subset t$. In fact, $D(q) \subset D(t)$. (See Remark 4.4.) And so, for the proof of $q \subset t$, it remains to show that, for all $f = g + ih$ in $D(q)$,

$$\lim_{n\to\infty} \int_{\mathbf{R}^d} |g|^2 \, d\mu_n = \int_{\mathbf{R}^d} |g|^2 \, d\mu \tag{4.3}$$

and

$$\lim_{n\to\infty} \int_{\mathbf{R}^d} |h|^2 \, d\mu_n = \int_{\mathbf{R}^d} |h|^2 \, d\mu. \tag{4.4}$$

If $g = \chi_E$, where χ_E denotes the characteristic function of a Borel set E, (4.3) is true by hypotheses on measures μ_n and μ. For a simple function g, (4.3) is easily proved by using the case of characteristic functions. Suppose that g is a nonnegative Borel measurable function. Then there exists a nonnegative and nondecreasing sequence $\{g_m\}$ of simple functions converging to g. By the monotone convergence theorem, we have

$$\lim_{m\to\infty} \int_{\mathbf{R}^d} |g_m|^2 \, d\mu_n = \int_{\mathbf{R}^d} |g|^2 \, d\mu_n \tag{4.5}$$

for all sufficiently large n. And so,

$$\lim_{n\to\infty} \left[\lim_{m\to\infty} \int_{\mathbf{R}^d} |g_m|^2 \, d\mu_n \right] = \lim_{n\to\infty} \int_{\mathbf{R}^d} |g|^2 \, d\mu_n. \tag{4.6}$$

Using the iterated limit theorem for a double sequence and the case of simple functions, we have

$$\begin{aligned} \lim_{n\to\infty} \left[\lim_{m\to\infty} \int_{\mathbf{R}^d} |g_m|^2 \, d\mu_n \right] &= \lim_{m\to\infty} \left[\lim_{n\to\infty} \int_{\mathbf{R}^d} |g_m|^2 \, d\mu_n \right] \\ &= \lim_{m\to\infty} \left[\int_{\mathbf{R}^d} |g_m|^2 \, d\mu \right] \\ &= \int_{\mathbf{R}^d} |g|^2 \, d\mu. \end{aligned} \tag{4.7}$$

By (4.6) and (4.7), we conclude that

$$\lim_{n\to\infty} \int_{\mathbf{R}^d} |g|^2 \, d\mu_n = \int_{\mathbf{R}^d} |g|^2 \, d\mu. \tag{4.8}$$

For a Borel measurable function $g = g^+ - g^-$, we easily get (4.3) by using the case of nonnegative Borel measurable functions. By essentially the same method as in the proof of (4.3), we can prove (4.4).

Now note that t is a closed form. Hence q is closable. By Proposition 4.1, $D = H^1(\mathbf{R}^d) \cap C_0(\mathbf{R}^d)$ is a core of $\mathcal{E}_\mu$ and hence $D' = D + iD$ is a core of $t = \mathcal{E}_\mu^{\mathbf{C}}$. Furthermore, $D' \subset D(q) = \cup_n D(t_n) \subset D(t)$. Consequently, it is easy to show that t is the closure of q and t is bounded below with lower bound α. Then in the light of Theorem 3.3, $(H_n - \xi)^{1/2}u \to (H - \xi)^{1/2}u$ for all u in $D(q)$ and $\xi < \alpha$. Hence we conclude that $\{H_n\}_{n=1}^\infty$ converges to H in the strong resolvent sense by Theorem 3.6.

Corollary 4.5. *Under the same conditions as in Theorem* 4.2*,*

$$J^{it}(F^{\mu_n}) \to J^{it}(F^\mu) \tag{4.9}$$

for all $t \in \mathbf{R}$.

Proof. By virtue of Theorem 2.2, we get

$$J^{it}(F^{\mu_n}) = e^{-itH_{\mathbf{C}}^{\mu_n}} \quad \text{and} \quad J^{it}(F^\mu) = e^{-itH_{\mathbf{C}}^{\mu}} \tag{4.10}$$

where $H_{\mathbf{C}}^{\mu_n}$ and $H_{\mathbf{C}}^{\mu}$ are self-adjoint operators associated with $\mathcal{E}_{\mu_n}^{\mathbf{C}}$ and $\mathcal{E}_{\mu}^{\mathbf{C}}$, respectively. By Theorem 4.2 and Theorem 3.2, we get (4.9).

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