ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 29, Number 4, Winter 1999

ON HEIGHT ORTHOGONALITY IN NORMED LINEAR SPACES

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ABSTRACT. We study geometric properties of an orthogonality relation defined in normed linear spaces and based on a classical property of right triangles.

In a real inner product space we can state the usual notion of orthogonality in many different ways that are also meaningful in real normed linear spaces. For example, the vectors x and y in a real inner product space are orthogonal if and only if $\|x+y\| = \|x-y\|$, where $\|\ \|$ denotes the norm induced by the inner product. That property led James [**11**] to say that in a real normed linear space two vectors x and y are *orthogonal in the isosceles sense* if the above identity holds. Similar properties gave rise to a great number of different concepts of generalized orthogonalities: x is *Birkhoff orthogonal* to y if $\|x+\lambda y\| \geq \|x\|$ for every $\lambda \in \mathbf{R}$ [**7**]; x is *Pythagorean orthogonal* to y if $\|x-y\|^2 = \|x\|^2 + \|y\|^2$ [**11**]; and so on, see [**1, 2**]. However, properties like symmetry, homogeneity, additivity, etc., of the orthogonality in inner product spaces do not always carry over to generalized orthogonalities. For example, isosceles and Pythagorean orthogonalities are homogeneous only in inner product spaces, whereas Birkhoff orthogonality is homogeneous in any normed linear space. In sum, it seems that orthogonality in normed linear spaces should provide a good framework for developing studies of the geometric structure of such spaces.

In this present paper we shall study an orthogonality introduced by Alsina, Guijarro and Tomas [**4**] which is based on a well-known property of right triangles: The height onto the hypotenuse in a right triangle divides it into two similar triangles.

Let E be a real normed linear space of dimension at least two. If $x, y \in E$, then x is *orthogonal to y in the height sense*, $x \perp^H y$, if either

Received by the authors on July 28, 1997, and in revised form on May 15, 1998.
Research partially supported by D.G.E.S. project PB96-1462.

$\|x\|\|y\| = 0$ or

$$\|x-y\| = \left\| \|y\|\frac{x}{\|x\|} + \|x\|\frac{y}{\|y\|} \right\|.$$

Known [**4**] properties of H-orthogonality are the following:

(a) *Symmetry*: $x \perp^H y$ if and only if $y \perp^H x$.

(b) *Simplification*: $x \perp^H y$ implies $\lambda x \perp^H \lambda y$ for every $\lambda \in \mathbf{R}$.

(c) *If $x, y \in E \setminus \{0\}$ and $x \perp^H y$, then x and y are linearly independent.*

(d) *Existence*: For every $x \in E$ and every $t > 0$, there exists $y \in E$ such that $\|y\| = t$ and $x \perp^H y$.

(e) *Existence of diagonals*: For every $x, y \in E \setminus \{0\}$, there exists $\delta > 0$ such that $x + \delta y \perp^H x - \delta y$.

It is easy to see that H-orthogonality also has the property of *non-degeneracy*, which means that $\lambda x \perp^H \mu x$ if and only if $\lambda\mu x = 0$.

Property (d) can be improved in the sense that we can find the vector y in each two-dimensional subspace that contains x. This follows from the proof given in [**4**]. Proposition 1 includes a new and intuitive proof of this fact, but the interest of this proposition is centered rather on the uniqueness property.

Lemma 1. *Let S be the unit sphere of a norm in $\mathbf{R}^2$, and let*

$$x : \theta \in [0, 2\pi] \longrightarrow x(\theta) \in S$$

be a parametrization of S, where $x(\theta)$ is the point of S that is at an angle θ to an arbitrary point $x(0)$. Then, for every $\lambda > 0$, the functions

$$\theta \in [0, \pi] \longrightarrow \|x(0) + \lambda x(\theta)\|, \qquad \theta \in [0, \pi] \longrightarrow \|x(0) - \lambda x(\theta)\|$$

are, respectively, decreasing and increasing.

Proof. Let $x = x(0)$, $u = x(\theta_1)$ and $v = x(\theta_2)$, with $0 \le \theta_1 < \theta_2 \le \pi$. We shall see that $\|x + \lambda v\| \le \|x + \lambda u\|$. From the convexity of S it follows that $u = \rho(\mu x + (1-\mu)v)$, with $\rho \ge 1$ and $0 < \mu \le 1$. On the one hand, from the identity

$$x + \lambda v = \left(\frac{1}{1+\mu\rho\lambda}\right)(x + \lambda u) + \left(\frac{1 + \mu\rho\lambda - \rho(1-\mu)}{1+\mu\rho\lambda}\right)\lambda v$$

it follows that

$$\|x+\lambda v\| \leq \frac{\|x+\lambda u\|}{1+\mu\rho\lambda} + \frac{\lambda|1+\mu\rho\lambda-\rho(1-\mu)|}{1+\mu\rho\lambda}.$$

On the other hand, from the identity

$$\rho\mu(x+\lambda u) = (1+\mu\rho\lambda)u - \rho(1-\mu)v,$$

we get

$$\rho\mu\|x+\lambda u\| \geq |1+\mu\rho\lambda-\rho(1-\mu)|.$$

Therefore, $\|x+\lambda v\| \leq \|x+\lambda u\|$. In a similar way it can be proved that $\|x-\lambda v\| \geq \|x-\lambda u\|$. $\square$

Proposition 1. (i) Existence. *Let $x \in E$ and $t > 0$. For every two-dimensional subspace $L \subset E$, with $x \in L$, there exists $y \in L$ such that $\|y\| = t$ and $x \perp^H y$.*

(ii) Uniqueness. *Let $x, y \in E \setminus \{0\}$ be such that $x \perp^H y$. If $x \perp^H \alpha x + \beta y$, with $\beta > 0$ and $\|\alpha x + \beta y\| = \|y\|$, then $\alpha = 0$ and $\beta = 1$.*

Proof. (i) We can assume without loss of generality that the space L is $\mathbf{R}^2$ endowed with a norm whose unit sphere is S. By property (b) we can suppose that $x \in S$. Let $x(\theta)$ be a parametrization of S as considered in Lemma 1, with $x = x(0)$. The function $F : [0, 2\pi] \to \mathbf{R}$, defined by

$$F(\theta) = \|x(0) - tx(\theta)\| - \|tx(0) + x(\theta)\|$$

is continuous and

$$F(0) = |1-t| - |1+t| = -F(\pi) = F(2\pi).$$

Then there exist $\theta_0 \in (0, \pi)$ and $\theta_1 \in (\pi, 2\pi)$ such that $F(\theta_0) = F(\theta_1) = 0$, and we obtain that the vectors $y_0 = tx(\theta_0)$, $y_1 = tx(\theta_1)$ satisfy $\|y_0\| = \|y_1\| = t$, $x \perp^H y_0$ and $x \perp^H y_1$.

(ii) We can assume that $\|x\| = 1$. We can also assume that $\alpha \geq 0$ because if $\alpha < 0$ we could consider that $x \perp^H \bar{y}$ and $x \perp^H (-\alpha/\beta)x + (1/\beta)\bar{y}$, where $\bar{y} = \alpha x + \beta y$. From $\alpha = 0$ it follows that $\beta = 1$.

So we shall suppose that $\alpha > 0$ and we shall get a contradiction. By property (c) we know that x and y are linearly independent so that we can assume without loss of generality that the subspace spanned by x and y is $\mathbf{R}^2$ endowed with a norm. Again, let $x(\theta)$ be a parametrization of its unit sphere S with $x = x(0)$, as described in Lemma 1. For the sake of brevity we shall write $t = \|y\| = \|\alpha x + \beta y\|$. We shall assume that the sense of θ is such that $\alpha x + \beta y = tx(\theta_1)$ and $y = tx(\theta_2)$, with $0 < \theta_1 < \theta_2 < \pi$. Then $x(0) \perp^H tx(\theta_1)$ and $x(0) \perp^H tx(\theta_2)$, which gives $F(\theta_1) = F(\theta_2) = 0$, with F as defined in (i). Therefore, from Lemma 1 it follows that

$$\begin{aligned} \|x(0) - tx(\theta_1)\| &= \|x(0) - tx(\theta_2)\| \\ &= \|tx(0) + x(\theta_2)\| \\ &= \|tx(0) + x(\theta_1)\|. \end{aligned} \tag{1}$$

Let us now consider the identity

$$tx(0) + x(\theta_1) = \gamma x(0) + \beta\big(tx(0) + x(\theta_2)\big), \tag{2}$$

where $\gamma = \alpha/t + t(1-\beta)$. If $\gamma = 0$, then it follows from (1) and (2) that $\beta = 1$ and then $\alpha = 0$, which contradicts the assumption. And, if $\alpha + \beta - 1 = 0$, then $tx(\theta_1) - x(0) = \beta(tx(\theta_2) - x(0))$, which also gives $\beta = 1$ and $\alpha = 0$. Therefore we can suppose that $\gamma \neq 0$ and $\alpha + \beta - 1 \neq 0$.

Now, the convex function

$$f(\lambda) = \big\|x(\theta_2) + \lambda\big(x(\theta_1) - x(\theta_2)\big)\big\|, \quad \lambda \in \mathbf{R},$$

satisfies $f(0) = f(1) = 1$, and from the identities

$$\begin{aligned} tx(0) + x(\theta_1) &= \frac{\gamma t}{\alpha}\left[x(\theta_2) + \left(\frac{t}{\gamma} + \frac{\alpha}{\gamma t}\right)\big(x(\theta_1) - x(\theta_2)\big)\right], \\ tx(0) + x(\theta_2) &= \frac{\gamma t}{\alpha}\left[x(\theta_2) + \frac{t}{\gamma}\big(x(\theta_1) - x(\theta_2)\big)\right], \\ tx(\theta_1) - x(0) &= \frac{t(\alpha+\beta-1)}{\alpha} \\ &\quad \cdot \left[x(\theta_2) + \left(\frac{\alpha-1}{\alpha+\beta-1}\right)\big(x(\theta_1) - x(\theta_2)\big)\right], \\ tx(\theta_2) - x(0) &= \frac{t(\alpha+\beta-1)}{\alpha} \\ &\quad \cdot \left[x(\theta_2) + \left(\frac{-1}{\alpha+\beta-1}\right)\big(x(\theta_1) - x(\theta_2)\big)\right], \end{aligned}$$

it follows that

$$\begin{aligned}
\|tx(0)+x(\theta_1)\| &= \frac{|\gamma|t}{\alpha} f\left(\frac{t}{\gamma}+\frac{\alpha}{\gamma t}\right),\\
\|tx(0)+x(\theta_2)\| &= \frac{|\gamma|t}{\alpha} f\left(\frac{t}{\gamma}\right),\\
\|tx(\theta_1)-x(0)\| &= \frac{t|\alpha+\beta-1|}{\alpha} f\left(\frac{\alpha-1}{\alpha+\beta-1}\right),\\
\|tx(\theta_2)-x(0)\| &= \frac{t|\alpha+\beta-1|}{\alpha} f\left(\frac{-1}{\alpha+\beta-1}\right).
\end{aligned}$$

Taking into account (1), we get

$$f\left(\frac{t}{\gamma}\right) = f\left(\frac{t}{\gamma}+\frac{\alpha}{\gamma t}\right), \qquad f\left(\frac{\alpha-1}{\alpha+\beta-1}\right) = f\left(\frac{-1}{\alpha+\beta-1}\right).$$

Remembering that $f(0)=f(1)=1$, we have from the convexity of f, independently of the signs of γ and $\alpha+\beta-1$, that $f(\lambda)=1$ for every λ in the smallest interval that contains the points

$$0, \quad 1, \quad \frac{t}{\gamma}, \quad \frac{t}{\gamma}+\frac{\alpha}{\gamma t}, \quad \frac{\alpha-1}{\alpha+\beta-1}, \quad \frac{-1}{\alpha+\beta-1}.$$

Therefore,

$$\|tx(0)+x(\theta_1)\| = \frac{|\gamma|t}{\alpha} = \frac{t|\alpha+\beta-1|}{\alpha},$$

so that $|\gamma| = |\alpha+\beta-1|$.

We end the proof by considering the separate cases. If $\gamma = 1-\alpha-\beta$, then, bearing in mind that $t = \|y\| = \|\alpha x+\beta y\|$, we get

$$0 \le t(1-\beta)+\alpha = \frac{1}{t}\big(t(1-\beta)-\alpha\big) \le 0,$$

so that $\alpha = 0$ and $\beta = 1$. On the other hand, if $\gamma = \alpha+\beta-1$, then

$$\gamma = \alpha + \frac{1}{t}\left(\frac{\alpha}{t}-\gamma\right),$$

from which it follows that $\gamma > 0$. From the identity

$$(t^2+\alpha+\beta)x(0) = t(tx(0)+x(\theta_1)) + \beta(x(0)-tx(\theta_2)),$$

we get

$$t^2+\alpha+\beta \le (t+\beta)\|tx(0)+x(\theta_1)\| = \frac{(t+\beta)t(\alpha+\beta-1)}{\alpha},$$

and then

$$\begin{aligned}\frac{\alpha^2}{t}+\frac{\alpha\beta}{t} &\le \beta(\alpha+\beta-1)+t(\beta-1)\\ &= \beta\gamma+t(\beta-1)=\beta\left(\frac{\alpha}{t}+t(1-\beta)\right)+t(\beta-1)\\ &= \frac{\alpha\beta}{t}-(\beta-1)^2t,\end{aligned}$$

which gives

$$\frac{\alpha^2}{t}+(\beta-1)^2t\le 0,$$

so that $\alpha=0$ and $\beta=1$, completing the proof. □

Another commonly studied kind of existence property, named α-*existence*, is the following: For every $x,y\in E$ is there some $\alpha\in\mathbf{R}$ such that $x\perp \alpha x+y$? If the orthogonality is *homogeneous*, i.e., $x\perp y$ implies $x\perp\lambda y$ for every $\lambda\in\mathbf{R}$, then α-existence and existence (as stated in Proposition 1) are equivalent properties. But, as we shall see later, H-orthogonality is not, in general, homogeneous.

Proposition 2. *For every $x,y\in E$ there exists $\alpha\in\mathbf{R}$ such that $x\perp^H \alpha x+y$.*

Proof. We can assume that x and y are linearly independent and that $\|x\|=1$. Let us consider the continuous function

$$G(\alpha)=\|\alpha x+y-x\|-\left\|\|\alpha x+y\|x+\frac{\alpha x+y}{\|\alpha x+y\|}\right\|,\quad \alpha\in\mathbf{R}.$$

Our aim is to see that there exist $\alpha_1,\alpha_2\in\mathbf{R}$ such that $\alpha_1\le\alpha_2$ and $G(\alpha_2)\le 0\le G(\alpha_1)$, which would imply that there exists $\alpha_0\in[\alpha_1,\alpha_2]$ such that $x\perp^H \alpha_0x+y$.

The function $\alpha \in \mathbf{R} \to f(\alpha) = \|\alpha x + y\|$ is positive, convex and such that $\lim_{\alpha\to\pm\infty} f(\alpha) = +\infty$. Hence, there exist $\beta_1, \beta_2 \in \mathbf{R}$, $\beta_1 \le \beta_2$, such that f is strictly decreasing in $(-\infty, \beta_1]$ and strictly increasing in $[\beta_2, +\infty)$. Taking into account that

$$\lim_{\alpha\to+\infty}\left\|x - \frac{\alpha x + y}{\|\alpha x + y\|}\right\| = 0 = \lim_{\alpha\to-\infty}\left\|x + \frac{\alpha x + y}{\|\alpha x + y\|}\right\|,$$

there exist $\gamma_1 \le \beta_1$ and $\gamma_2 \ge \beta_2$ such that

$$\left\|x + \frac{\alpha x + y}{\|\alpha x + y\|}\right\| \le 1 \quad \text{if } \alpha \le \gamma_1,$$

and

$$\left\|x - \frac{\alpha x + y}{\|\alpha x + y\|}\right\| \le 1 \quad \text{if } \alpha \ge \gamma_2.$$

Therefore, taking $\alpha_1 \le \gamma_1$ such that $f(\alpha_1) \ge 1$, it follows that

$$\begin{aligned}
\|\alpha_1 x + y - x\| &= f(\alpha_1 - 1) \ge f(\alpha_1)\\
&= \|(\|\alpha_1 x + y\| - 1)x\| + 1\\
&= \left\|\|\alpha_1 x + y\|x + \frac{\alpha_1 x + y}{\|\alpha_1 x + y\|} - \frac{\alpha_1 x + y}{\|\alpha_1 x + y\|} - x\right\| + 1\\
&\ge \left\|\|\alpha_1 x + y\|x + \frac{\alpha_1 x + y}{\|\alpha_1 x + y\|}\right\| - \left\|\frac{\alpha_1 x + y}{\|\alpha_1 x + y\|} + x\right\| + 1\\
&\ge \left\|\|\alpha_1 x + y\|x + \frac{\alpha_1 x + y}{\|\alpha_1 x + y\|}\right\|,
\end{aligned}$$

so that $G(\alpha_1) \ge 0$. On the other hand, taking $\alpha_2 \ge \gamma_2 + 1$, it follows that

$$\begin{aligned}
\|\alpha_2 x + y - x\| &= f(\alpha_2 - 1) \le f(\alpha_2) = \|(\|\alpha_2 x + y\| + 1)x\| - 1\\
&= \left\|\|\alpha_2 x + y\|x + \frac{\alpha_2 x + y}{\|\alpha_2 x + y\|} + x - \frac{\alpha_2 x + y}{\|\alpha_2 x + y\|}\right\| - 1\\
&\le \left\|\|\alpha_2 x + y\|x + \frac{\alpha_2 x + y}{\|\alpha_2 x + y\|}\right\| + \left\|x - \frac{\alpha_2 x + y}{\|\alpha_2 x + y\|}\right\| - 1\\
&\le \left\|\|\alpha_2 x + y\|x + \frac{\alpha_2 x + y}{\|\alpha_2 x + y\|}\right\|,
\end{aligned}$$

so that $G(\alpha_2) \le 0$, which completes the proof. $\square$

It is interesting to note that while H-orthogonality has the property of uniqueness in the sense of Proposition 1, as the next example will show, it does not have the property of α-*uniqueness*, i.e., the α in the α-existence property can be non-unique.

Example 1. Let us consider E to be the space $\mathbf{R}^2$ endowed with the norm

$$\|(z_1, z_2)\| = \begin{cases} \max\{|z_1|, |z_2|\} & \text{if } z_1(3z_1 + 4z_2) \ge 0, \\ |z_2| + \frac{1}{4}|z_1| & \text{otherwise,} \end{cases}$$

whose unit sphere is the hexagon of vertices $\pm(1,1)$, $\pm(0,1)$ and $\pm(-1,3/4)$. Taking $x = (1,1/2)$, $y = (-82/81, 164/81)$ and $\alpha = 574/1377$, we have $x \perp^H y$ and $x \perp^H \alpha x + y$.

Proposition 3. *Let $x, y \in E$ and $\alpha \in \mathbf{R}$ be such that $x \ne 0$ and $x \perp^H \alpha x + y$. Then,*

$$|\alpha| \le \frac{\|y\|}{\|x\|}\left(\frac{1+\sqrt{2}}{2}\right),$$

and the bound is sharp.

Proof. There is no loss of generality in assuming that $\alpha \ne 0$ and $\|x\| = 1$. Then, $x \perp^H \alpha x + y$ means that

$$\|(\alpha - 1)x + y\| = \left\| \frac{\alpha x + y}{\|\alpha x + y\|} + \|\alpha x + y\| x \right\|. \tag{3}$$

To simplify, we denote

$$u = (\alpha - 1)x + y, \qquad v = \frac{\alpha x + y}{\|\alpha x + y\|} + \|\alpha x + y\| x.$$

From the identities

$$\left(\frac{\|\alpha x + y\|^2 + \alpha}{\|\alpha x + y\|}\right)(\alpha x + y) = \alpha v + \|\alpha x + y\| y,$$

$$\alpha u = y + (\alpha - 1)(\alpha x + y),$$

$$v = \left(\frac{\alpha + \|\alpha x + y\|^2}{\|\alpha x + y\|}\right) x + \left(\frac{1}{\|\alpha x + y\|}\right) y,$$

we get the following inequalities

$$|\alpha|\|v\| - \|\alpha x + y\|\|y\| \le \big|\|\alpha x + y\|^2 + \alpha\big| \le |\alpha|\|v\| + \|\alpha x + y\|\|y\|, \tag{4}$$

$$\frac{|\alpha - 1|\|\alpha x + y\| - \|y\|}{|\alpha|} \le \|u\| \le \frac{|\alpha - 1|\|\alpha x + y\| + \|y\|}{|\alpha|}, \tag{5}$$

$$\frac{|\alpha + \|\alpha x + y\|^2| - \|y\|}{\|\alpha x + y\|} \le \|v\| \le \frac{|\alpha + \|\alpha x + y\|^2| + \|y\|}{\|\alpha x + y\|}. \tag{6}$$

We shall now consider four separate cases:

Case 1. Assume that $0 < \alpha \le 1$. Then, from (4) and the definition of u it follows that

$$\begin{aligned}\|\alpha x + y\|^2 + \alpha &\le \alpha\|u\| + \|\alpha x + y\|\|y\| \\ &\le \alpha(1 - \alpha + \|y\|) + \|\alpha x + y\|\|y\|,\end{aligned}$$

so that

$$\|\alpha x + y\|^2 - \|y\|\|\alpha x + y\| + \alpha(\alpha - \|y\|) \le 0. \tag{7}$$

This means that the polynomial $t^2 - \|y\|t + \alpha(\alpha - \|y\|)$ must have real roots and, therefore, a nonnegative discriminant. That is,

$$4\alpha^2 - 4\|y\|\alpha - \|y\|^2 \le 0,$$

from which it follows that

$$\|y\|\left(\frac{1 - \sqrt{2}}{2}\right) \le \alpha \le \|y\|\left(\frac{1 + \sqrt{2}}{2}\right).$$

Case 2. Assume that $1 < \alpha$. From (3), (5) and (6) we get

$$\frac{(\alpha + \|\alpha x + y\|^2) - \|y\|}{\|\alpha x + y\|} \le \frac{(\alpha - 1)\|\alpha x + y\| + \|y\|}{\alpha},$$

from which again (7) follows.

Case 3. Assume that $\alpha < 0$ and $\alpha + \|\alpha x + y\|^2 \geq 0$. In this case we have the inequality

$$\frac{(1-\alpha)\|\alpha x + y\| - \|y\|}{-\alpha} \leq \frac{(\alpha + \|\alpha x + y\|^2) + \|y\|}{\|\alpha x + y\|},$$

so that

$$\|\alpha x + y\|^2 - \|y\|\|\alpha x + y\| + \alpha(\alpha + \|y\|) \leq 0. \tag{8}$$

With the argument followed in Case 1 we now get

$$4\alpha^2 + 4\|y\|\alpha - \|y\|^2 \leq 0,$$

so that

$$-\|y\|\left(\frac{1+\sqrt{2}}{2}\right) \leq \alpha \leq \|y\|\left(\frac{-1+\sqrt{2}}{2}\right).$$

Case 4. Finally, assume that $\alpha < 0$ and $\alpha + \|\alpha x + y\|^2 \leq 0$. From (3) and (4) we have

$$\begin{aligned} -\|\alpha x + y\|^2 - \alpha &\geq -\alpha\|(\alpha - 1)x + y\| - \|\alpha x + y\|\|y\| \\ &\geq (-\alpha)(1 - \alpha - \|y\|) - \|\alpha x + y\|\|y\|, \end{aligned}$$

from which (8) follows.

To see that the bound is sharp we can consider the space $\mathbf{R}^2$ with the maximum norm. Taking $x = (1, 1-\sqrt{2})$, $y = (-1, 1)$ and $\alpha = (1+\sqrt{2})/2$, we have $x \perp^H \alpha x + y$ and $-x \perp^H (-\alpha)(-x) + y$. $\square$

It is well known that a normed linear space is an inner product space if and only if the *parallelogram equality*

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

holds for every $x, y \in E$. This basic characterization was strengthened by several authors: Day [**9**] showed that only unit vectors are needed and later Schoenberg [**12**] replaced equality by an inequality in either direction. Benítez and del Río [**6**] proved that E is an inner product space if and only if the *B-rectangle inequality* holds, i.e., if and only if

$$x, y \in E,\ x \perp^B y \Longrightarrow \|x+y\|^2 + \|x-y\|^2 \sim 2(\|x\|^2 + \|y\|^2),$$

where $\perp^B$ denotes Birkhoff orthogonality and $\sim$ denotes either $\le$ or $\ge$. Finally, Amir [**5**] observed that the proof given by Benítez and Del Río was valid for any orthogonality that has the existence of diagonals property. Hence, one has the following proposition.

Proposition 4. *A normed linear space E is an inner product space if and only if the H-rectangle inequality holds.*

Proposition 5. *If $x, y \in E \setminus \{0\}$ and $\delta \in \mathbf{R}$ are such that $x + \delta y \perp^H x - \delta y$, then*

$$\frac{\|x\|}{\|y\|\sqrt{2}} \le |\delta| \le \frac{\|x\|\sqrt{2}}{\|y\|},$$

and the bounds are sharp.

Proof. We can assume that $\|x\| = \|y\| = 1$. If $\|x+\delta y\|\|x-\delta y\| = 0$, then $|\delta| = 1$. Therefore, we can also assume that $\|x+\delta y\| \ge \|x-\delta y\| > 0$. Now, $x + \delta y \perp^H x - \delta y$ means that

$$2|\delta| = \left\| \|x-\delta y\| \left(\frac{x+\delta y}{\|x+\delta y\|} \right) + \|x+\delta y\| \left(\frac{x-\delta y}{\|x-\delta y\|} \right) \right\|.$$

Then

$$\begin{aligned} 2|\delta|\|x+\delta y\|\|x-\delta y\| &= \|\|x-\delta y\|^2(x+\delta y) + \|x+\delta y\|^2(x-\delta y)\| \\ &= \|(\|x-\delta y\|^2 + \|x+\delta y\|^2)x \\ &\qquad - (\|x+\delta y\|^2 - \|x-\delta y\|^2)\delta y\| \\ &\ge \|x-\delta y\|^2 + \|x+\delta y\|^2 \\ &\quad - (\|x+\delta y\|^2 - \|x-\delta y\|^2)|\delta|, \end{aligned}$$

from which it follows that

$$(1+|\delta|)\|x-\delta y\|^2 - 2|\delta|\|x+\delta y\|\|x-\delta y\| + (1-|\delta|)\|x+\delta y\|^2 \le 0.$$

Therefore, the polynomial

$$P(t) = (1+|\delta|)t^2 - 2|\delta|\|x+\delta y\|t + (1-|\delta|)\|x+\delta y\|^2$$

must have real roots, so that

$$|\delta|^2\|x+\delta y\|^2 - (1+|\delta|)(1-|\delta|)\|x+\delta y\|^2 \geq 0,$$

i.e., $|\delta| \geq 1/\sqrt{2}$.

To get the other bound, we have

$$\begin{aligned} 2|\delta| &= \left\| \|x-\delta y\| \left(\frac{x+\delta y}{\|x+\delta y\|} \right) + \|x+\delta y\| \left(\frac{x-\delta y}{\|x-\delta y\|} \right) \right\| \\ &= \left\| \left(\frac{\|x+\delta y\|^2 - \|x-\delta y\|^2}{\|x+\delta y\|} \right) \left(\frac{x-\delta y}{\|x-\delta y\|} \right) + \frac{2\|x-\delta y\|}{\|x+\delta y\|} x \right\| \\ &\leq \frac{\|x+\delta y\|^2 - \|x-\delta y\|^2 + 2\|x-\delta y\|}{\|x+\delta y\|}. \end{aligned}$$

Therefore,

$$\|x-\delta y\|^2 - 2\|x-\delta y\| + \|x+\delta y\|\big(2|\delta| - \|x+\delta y\|\big) \leq 0,$$

and the discriminant of the polynomial

$$Q(t) = t^2 - 2t + \|x+\delta y\|\big(2|\delta| - \|x+\delta y\|\big)$$

must be nonnegative, i.e.,

$$\|x+\delta y\|^2 - 2|\delta|\|x+\delta y\| + 1 \geq 0.$$

We want to see that $|\delta| \leq \sqrt{2}$, so that we can assume that $|\delta| > 1$. The roots of the polynomial $H(t) = t^2 - 2|\delta|t + 1$ are

$$t_+ = |\delta| + \sqrt{|\delta|^2 - 1}, \qquad t_- = |\delta| - \sqrt{|\delta|^2 - 1}.$$

Therefore, either $\|x+\delta y\| \leq t_-$ or $\|x+\delta y\| \geq t_+$. In the former case we have

$$|\delta| - 1 \leq \|x+\delta y\| \leq |\delta| - \sqrt{|\delta|^2 - 1},$$

so that $|\delta| \le \sqrt{2}$. In the latter case,

$$|\delta| + \sqrt{|\delta|^2 - 1} \le \|x + \delta y\| \le 1 + |\delta|,$$

and we also get $|\delta| \le \sqrt{2}$.

To see that the bounds are sharp, we can consider, as in Proposition 3, the space $\mathbf{R}^2$ with the maximum norm. Taking $x = (1,1)$, $y = (1,0)$ and $\delta = \sqrt{2}$, we have $x + \delta y \perp^H x - \delta y$ and $y + \delta^{-1}x \perp^H y - \delta^{-1}x$. $\square$

The next result shows that the lower bound in Proposition 5 is attained only in very special spaces.

Proposition 6. *Assume that there exist $x, y \in E \setminus \{0\}$ such that $x + \delta y \perp^H x - \delta y$, with $|\delta| = \|x\|/(\|y\|\sqrt{2})$. Then the subspace spanned by x and y is isometrically isomorphic to $(\mathbf{R}^2, \|\ \|_\infty)$.*

Proof. First, we can assume that $\|x\| = \|y\| = 1$ and $\delta = 1/\sqrt{2}$. Let $L = span(x,y)$. By means of an isometric isomorphism we can assume that $L = \mathbf{R}^2$, $x = (1,0)$ and $y = (1,1)$. Then we shall see that the unit sphere S of L is a square.

From the proof of Proposition 5 we know that $P(\|x - \delta y\|) \le 0$, where

$$P(t) = (1 + |\delta|)t^2 - 2|\delta|\|x + \delta y\|t + (1 - |\delta|)\|x + \delta y\|^2.$$

But now, the discriminant of this polynomial is zero. Therefore, $P(\|x - \delta y\|) = 0$, which yields

$$\|x + \delta y\| = (1 + \sqrt{2})\|x - \delta y\|. \tag{9}$$

Bearing in mind that $x + \delta y \perp^H x - \delta y$, we get $\|2x - y\| = 1$. Then, we have $\|(1,0)\| = \|(1,1)\| = \|(1,-1)\| = 1$, from which it follows that the line segment $[(1,-1), (1,1)]$ must be in S. Hence, $\|x + \delta y\| = \|(1 + \delta, \delta)\| = 1 + \delta$, and we get from (9) $\|x - \delta y\| = \delta$, which gives $\|(\sqrt{2} - 1, -1)\| = 1$. But $(\sqrt{2} - 1, -1)$ is a point in the line segment $[(-1,-1), (1,-1)]$. Therefore, S must be the square of vertices $\pm(1,1)$, $\pm(1,-1)$. $\square$

Remark 1. With regards to the upper bound in Proposition 5, we know that there are two-dimensional spaces, different from $(\mathbf{R}^2, \| \|_\infty)$, where it is attained. For example, consider in $\mathbf{R}^2$ a norm whose unit sphere is the hexagon of vertices $\pm(1,0)$, $\pm(0,1)$ and $\pm(1,\beta)$, where $\sqrt{2} \leq \beta \leq 2$. Then, taking $x = (1,0)$ and $y = (0,1)$, we have $x + \sqrt{2}y \perp^H x - \sqrt{2}y$.

Orthogonality in an inner product space is, obviously, homogeneous. However, as a general rule, we can say that all the generalized orthogonalities that have been defined are homogeneous either in every normed linear space (Birkhoff [**7**], Singer [**13**], Diminnie [**10**], area [**3**]) or only in inner product spaces (isosceles [**11**], Pythagorean [**11**], Carlsson [**8**]). But we shall see that this is not the case for H-orthogonality.

Definition 1. Let E be a real normed linear space of dimension at least two. We say that E has the $\pi/2$-property if for every two-dimensional subspace $L \subset E$, there exists an isometric isomorphism between L and $\mathbf{R}^2$ such that the unit sphere in $\mathbf{R}^2$ is invariant under rotations of angle $\pi/2$ radians.

Proposition 7. *Let L be a real normed linear space of dimension two.*

(i) *If there exist $x, y \in L \setminus \{0\}$ such that $x \perp^H \lambda y$ for every $\lambda \in \mathbf{R}$, then L has the $\pi/2$-property.*

(ii) *If L has the $\pi/2$-property, then the H-orthogonality is homogeneous.*

Proof. We can assume without loss of generality that L is $\mathbf{R}^2$ endowed with a norm whose unit sphere is parametrized by

$$x : \ \theta \in [0, 2\pi] \longrightarrow x(\theta) = |x(\theta)|(\cos\theta, \sin\theta) \in S,$$

where $| \ |$ denotes the usual modulus of a vector.

(i) We can assume that $\|x\| = \|y\| = 1$ and that the above parametrization satisfies $x = x(0) = (1,0)$ and $y = x(\pi/2) = (0,1)$. We shall see that $|x(\theta)| = |x(\theta + \pi/2)|$, for every θ. Now, $x(0) \perp^H \lambda x(\pi/2)$ is equivalent to the identity

$$\|x(0) - \lambda x(\pi/2)\| = \|x(\pi/2) + \lambda x(0)\|.$$

Hence, our hypothesis says that $\|(\lambda, 1)\| = \|(1, -\lambda)\|$ for every $\lambda \in \mathbf{R}$. But this yields

$$\begin{aligned}\|(\cos\theta, \sin\theta)\| &= \|(\sin\theta, -\cos\theta)\| \\ &= \left\|\left(\cos\left(\theta + \frac{\pi}{2}\right), \sin\left(\theta + \frac{\pi}{2}\right)\right)\right\|,\end{aligned}$$

for every θ.

(ii) In this case we can assume that $|x(\theta)| = |x(\theta + \pi/2)|$ for every θ. We shall see that $x(\theta) \perp^H \lambda x(\theta + \pi/2)$ for every $\lambda \in \mathbf{R}$, and then, bearing in mind Proposition 1, we shall have that the H-orthogonality is homogeneous.

Let $\lambda \in \mathbf{R}$. Then

$$\begin{aligned}x(\theta) - \lambda x\left(\theta + \frac{\pi}{2}\right) &= |x(\theta)|(\cos\theta + \lambda\sin\theta, \sin\theta - \lambda\cos\theta), \\ x\left(\theta + \frac{\pi}{2}\right) + \lambda x(\theta) &= |x(\theta)|(\lambda\cos\theta - \sin\theta, \lambda\sin\theta + \cos\theta).\end{aligned}$$

Also, let θ_1 and θ_2 be such that

$$\begin{aligned}x(\theta) - \lambda x\left(\theta + \frac{\pi}{2}\right) &= \left\|x(\theta) - \lambda x\left(\theta + \frac{\pi}{2}\right)\right\| |x(\theta_1)|(\cos\theta_1, \sin\theta_1), \\ x\left(\theta + \frac{\pi}{2}\right) + \lambda x(\theta) &= \left\|x\left(\theta + \frac{\pi}{2}\right) + \lambda x(\theta)\right\| |x(\theta_2)|(\cos\theta_2, \sin\theta_2).\end{aligned}$$

Then

$$\begin{aligned}\left\|x(\theta) - \lambda x\left(\theta + \frac{\pi}{2}\right)\right\| |x(\theta_1)| &= |x(\theta)|\sqrt{1 + \lambda^2} \\ &= \left\|x\left(\theta + \frac{\pi}{2}\right) + \lambda x(\theta)\right\| |x(\theta_2)|,\end{aligned}$$

and

$$\tan\left(\theta_1 + \frac{\pi}{2}\right) = -\frac{1}{\tan\theta_1} = \frac{\lambda\sin\theta + \cos\theta}{\lambda\cos\theta - \sin\theta} = \tan\theta_2,$$

from which it follows that $|x(\theta_2)| = |x(\theta_1 + \pi/2)|$. Hence $\|x(\theta) - \lambda x(\theta + \pi/2)\| = \|x(\theta + \pi/2) + \lambda x(\theta)\|$, which means that $x(\theta) \perp^H \lambda x(\theta + \pi/2)$. $\square$

It follows from Proposition 7 that in a two-dimensional space H-orthogonality is homogeneous if and only if there are two vectors x and y such that $x \perp^H \lambda y$ for every $\lambda \in \mathbf{R}$. The following result is then obvious.

Corollary 1. *A real normed linear space of dimension at least two has the $\pi/2$-property if and only if H-orthogonality is homogeneous.*

Conjecture 1. *A real normed linear space of dimension at least three is an inner product space if and only if H-orthogonality is homogeneous.*

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