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SOME TOPOLOGICAL PROPERTIES OF BANACH SPACES AND RIEMANN INTEGRATION

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ABSTRACT. In this paper we establish some new characterizations of Schur property and H property of Banach spaces by using Riemann integration.

R. Gordon, in [**3**], showed that l_1 and the Tsirelson space have the property of Lebesgue, that is, every Riemann integrable mapping from the interval $[0,1]$ to the space is continuous almost everywhere, LP in short, but many "familiar" Banach spaces such as any infinite-dimensional, uniformly convex Banach spaces and almost all classical Banach spaces except l_1 do not have the property of Lebesgue. In [**5**], Wang Chonghu introduced the weak property of Lebesgue of a Banach space, that is, every Riemann integrable mapping from $[0,1]$ to the space is weakly continuous almost everywhere, WLP in short, and proved that any Banach space with separable dual has the weak property of Lebesgue, and so do many of classical Banach spaces and "familiar" Banach spaces. It is notable that the LP and the WLP of Banach spaces are topologically isomorphically invariant, and there are some relations between these two topological properties and some geometrical properties of Banach spaces.

In this paper we will describe some other topological properties, the H property and the Schur property, of Banach spaces using Riemann integration. We establish some new characterizations of the Schur property and the H property. These discussions are inspired by [**2**], which showed that a Banach space X is a Schur space if and only if for each weakly continuous mapping f from $[0,1]$ into X, $\|f(\cdot)\|$ is Riemann integrable.

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Throughout this paper let X be a real Banach space. We will use the same terminology and notation for Riemann integration as in [**3**]. Our definitions of the H property and the Schur property are a little more general, since we use a general linear space topology that is weaker than the norm topology in place of the weak topology on a Banach space.

Definition 1. Let τ be a locally convex linear space topology on X that is weaker than norm topology on X. X is said to have the H property with respect to τ if whenever $\{x_n\}$ is a sequence in X such that $x_n \to x \in X$ with respect to τ and $\|x_n\| \to \|x\|$, it follows that $\|x_n - x\| \to 0$.

Definition 2. Let τ be as above. X is said to have Schur property with respect to τ if whenever $\{x_n\}$ is a sequence in X such that $x_n \to x \in X$ with respect to τ, it follows that $\|x_n - x\| \to 0$.

Remark. If we put $\tau =$ weak topology on X, the above definitions are reduced to the usual ones of H property and Schur property. When X has Schur property or H property with respect to the weak topology on X, we call X a Schur space or say that X has the H property, respectively.

Now we start to describe the Schur property of Banach space.

Lemma 1. *Let (X,τ) be a real locally convex topological linear space and f be a vector valued mapping from $[0,1]$ to X with $f(t) = \sum_{n=1}^{\infty} f_n(t)x_n$ where $x_n \in X$, $f_n(t)$ is a continuous function from $[0,1]$ into (∞,∞), $n = 1,2,\dots,$ and the series $\sum_{n=1}^{\infty} f_n(t)x_n$ converges in (X,τ) for any $t \in [0,1]$. If $\sum_{n=1}^{\infty} f_n(t)x_n$ converges uniformly on $[0,1]$, that is, for any neighborhood U of 0 in (X,τ), there exists $N > 0$ such that whenever $m \geq N$, $\sum_{n=m}^{\infty} f_n(t)x_n \in U$ for all t in $[0,1]$, then $f(t)$ is continuous from $[0,1]$ into (X,τ).*

The proof is easy. We omit the details.

Theorem 2. *Let X be a Banach space and τ be a locally convex* *linear space topology on X which is weaker than the norm topology on X. Then the following statements are equivalent*:

(i) *X has Schur property with respect to τ.*

(ii) *If f is a mapping from $[0,1]$ to (X,τ) that is τ-continuous, it follows that f is Darboux integrable.*

(iii) *If f is a mapping from $[0,1]$ to (X,τ) that is τ-continuous, it follows that f is Riemann integrable.*

Proof. It is obvious that statement (i) implies statement (ii) and statement (ii) implies statement (iii).

We now prove statement (iii) implies (i). Assume that statement (i) is not true. We will prove that there exists a mapping f from $[0,1]$ to (X,τ) such that f is τ-continuous, but f is not Riemann integrable; this leads to a contradiction to statement (iii).

If X does not have the Schur property with respect to τ, there must be a sequence $\{x_n\}$ in X such that $x_n \xrightarrow{\tau} 0$, but $\|x_n\| \geq 1$.

Now we define a Cantor set H in $[0,1]$ having positive measure as follows.

Let $\gamma_1^{(1)}$ be the midpoint of the interval $[0,1]$. Take subinterval $A_1^{(1)}$ whose midpoint is $\gamma_1^{(1)}$ and whose length $d(A_1^{(1)}) = (1/3)$. Write $A_1^{(1)} = [a_1^{(1)}, \beta_1^{(1)}]$, $B_1^{(1)} = [0,1]$. Let $B_2^1 = [0, a_1^{(1)}]$, $B_2^{(2)} = [\beta_1^{(1)}, 1]$, the midpoint of $B_2^{(1)}$ be $\gamma_2^{(1)}$, the midpoint of $B_2^{(2)}$ be $\gamma_2^{(2)}$. Take subintervals $A_2^{(1)}$ and $A_2^{(2)}$ such that their midpoints are $\gamma_2^{(1)}$ and $\gamma_2^{(2)}$, and $d(A_2^{(1)}) = d(A_2^{(2)}) = (1/2)\cdot(1/3^2)$; clearly, $A_2^{(1)} \subset B_2^{(1)}$, $A_2^{(2)} \subset B_2^{(2)}, \ldots$, and so on.

For any $k = 1, 2, \ldots$, we have $A_k^{(1)}, A_k^{(2)}, \ldots, A_k^{(2^{k-1})}$, $B_k^{(1)}, B_k^{(2)}, \ldots$, $B_k^{(2^{k-1})}$ such that, for any $i = 1, 2, \ldots, 2^{k-1}$, $A_k^{(i)} \subset B_k^{(i)}$, the midpoints of $A_k^{(i)}$ and $B_k^{(i)}$ both are $\gamma_k^{(i)}$, $d(A_k^{(i)}) = (1/2^{k-1})\cdot(1/3^k)$, $d(B_k^{(i)}) = (1/2^{k-1})(1 - \sum_{n=1}^{k-1}(1/3^n))$.

Put $G = \cup_{k=1}^{\infty} \cup_{i=1}^{2^{k-1}} (a_k^{(i)}, \beta_k^{(i)})$, $H = [0,1]\backslash G$. Then it is easy to see that H is a perfect, nowhere dense subset of $[0,1]$ with the measure $\mu(H) = (1/2)$. Note that the open intervals $(a_k^{(i)}, \beta_k^{(i)})$, $k = 1, 2, \ldots$, $i = 1, 2, \ldots, 2^{k-1}$, are disjoint.

For any $k = 1, 2, \ldots$, $i = 1, 2, \ldots, 2^{k-1}$, put

$$\varphi_k^{(i)}(t) = \begin{cases} \dfrac{2}{\beta_k^{(i)} - a_k^{(i)}}(t - a_k^{(i)}) & \text{where } t \in \left[a_k^{(i)}, a_k^{(i)} + \dfrac{\beta_k^{(i)} - a_k^{(i)}}{2}\right] \\ \dfrac{2}{\beta_k^{(i)} - a_k^{(i)}}(\beta_k^{(i)} - t) & \text{where } t \in \left[a_k^{(i)} + \dfrac{\beta_k^{(i)} - a_k^{(i)}}{2}, \beta_k^{(i)}\right] \\ 0 & \text{elsewhere.} \end{cases}$$

Set $h_k(t) = \sum_{i=1}^{2^{k-1}} \varphi_k^{(i)}(t)$. Then

$$f(t) = \sum_{k=1}^{\infty} h_k(t)x_k : [0,1] \longrightarrow X$$

is desired.

First of all, we prove that $f : [0,1] \to (X, \tau)$ is τ-continuous by using Lemma 1.

Since for any k, $i = 1, 2, \ldots, 2^{k-1}$, $\varphi_k^{(i)}(t)$ is continuous, $h_k(t) = \sum_{i=1}^{2^{k-1}} \varphi_k^{(i)}(t)$ is continuous. By assumption, $x_k \xrightarrow{\tau} 0$, $k \to \infty$, then, for any balanced τ-neighborhood U of 0 in (X, τ), there exists K such that, if $k > K$, we have $x_k \in U$. Note that for any $t \in [0,1]$, if $t \in H \cup_{k=1}^{K} \cup_{i=1}^{2^{k-1}} (\alpha_k^{(i)}, \beta_k^{(i)})$, $\sum_{k=K+1}^{\infty} h_k(t)x_k = 0$; otherwise, there must be k_0, $k_0 > K$, such that $\sum_{k=K+1}^{\infty} h_k(t)x_k = h_{k_0}(t)x_{k_0}$. Since $0 \leq h_k(t) \leq 1$, $\sum_{k=K+1}^{\infty} h_k(t)x_k \in U$ for all $t \in [0,1]$, i.e., the series $\sum_{k=1}^{\infty} h_k(t)x_k$ converges uniformly on $[0,1]$. By Lemma 1, f is τ-continuous.

To prove that f is not Riemann integrable, from Theorem 5 in [**3**], it suffices to prove that for any $\delta > 0$, there exist two tagged partitions P_1 and P_2 with norms $|P_1| < \delta$, $|P_2| < \delta$ such that

$$\|f(P_1) - f(P_2)\| > \frac{1}{2}.$$

For any $\delta > 0$, take an integer m such that $(1/2^{m-1}) < \delta$. Let $B_m^{(i)} = [u_m^{(i)}, v_m^{(i)}]$. Since $d(B_m^{(i)}) = v_m^{(i)} - u_m^{(i)} < (1/2^{m-1})$, we can choose two tagged partitions

$$P_1 = \{(s_j, [t_{j-1}, t_j]) : \ 1 \leq j \leq n_m\}$$
$$P_2 = \{(s'_j, [t_{j-1}, t_j]) : \ 1 \leq j \leq n_m\}$$

such that, for any j, $1 \le j \le n_m$, $i = 1, 2, \dots, 2^{m-1}$

(1) $t_j - t_{j-1} < (1/2^{m-1})$

(2) $\{t_j : 0 \le j \le n_m\} \cap B_m^{(i)} = \{u_m^{(i)}, v_m^{(i)}\}$

(3) $s_j, s'_j \in [t_{j-1}, t_j]$ for $1 \le j \le n_m$

(4) whenever $[t_{j-1}, t_j] = B_m^{(i)}$, $s_j = \gamma_m^{(i)} = (u_m^{(i)} + v_m^{(i)})/2$, $s'_j = a_m^{(i)}$

(5) whenever $[t_{j-1}, t_j] \ne B_m^{(i)}$, $(s_j = 1, 2, \dots, 2^{m-1})$, $s_j = s'_j$.

So whenever $[t_{j-1}, t_j] = B_m^{(i)}$, we have $f(s_j) = h_m(s_j)x_m$, $f(s'_j) = 0$; whenever $[t_{j-1}, t_j] \ne B_m^{(i)}$, $i = 1, 2, \dots, 2^{m-1}$, $f(s_j) = f(s'_j)$.

Therefore,

$$\begin{aligned}\|f(P_1) - f(P_2)\| &= \left\| \sum_{j=1}^{n_m} (f(s_j) - f(s'_j))(t_j - t_{j-1}) \right\| \\ &= \left\| \sum_{i=1}^{2^{m-1}} x_m d(B_m^{(i)}) \right\| \\ &= \|x_m\| \cdot 2^{m-1} \frac{1 - \sum_{i=1}^{m-1} (1/3^i)}{2^{m-1}} > \frac{1}{2}.\end{aligned}$$

Note $|P_1| < \delta$, $|P_2| < \delta$. Hence, f is not Riemann integrable by Theorem 5 in [**3**].

The proof of Theorem 2 is completed.

Remark. From Theorem 2 we conclude that there exists a weakly continuous map $f : [0, 1] \to X$ that is not Riemann integrable if and only if X is not a Schur space. Previously, the existence of such maps was known only in special cases, see, e.g., [**1**] and [**6**].

Some quick conclusions follow.

Corollary 3 [**3**, Theorem 34]. *X is a Schur space and has* LP *if and only if every scalarly Riemann integrable function $f : [0, 1] \to X$ is Darboux integrable.*

Using Theorem 33 in [**3**] we have another necessary and sufficient condition for a Schur space.

Theorem 4. *X is a Schur space if and only if every mapping from $[0,1]$ to X which is scalarly Riemann integrable on $[0,1]$ is Riemann integrable on $[0,1]$.*

Next we describe the H property of Banach space by using Riemann integration.

Lemma 5. *Let X be a Banach space, $\{x_n\}_{n=1}^{\infty} \subset X$ such that $x_n \xrightarrow{w} x \in X$, $\|x_n\| = \|x\| = 1 (n = 1, 2, \dots)$. Then $\lim_{n\to\infty} d_n = 1$ where $d_n = \inf\{\|ax_n + (1-a)x\| : a \in [0,1]\}$.*

Proof. If $\lim_{n\to\infty} d_n = 1$ does not hold, without loss of generality we can assume that there exists $\delta > 0$ such that $d_n < 1 - \delta (n = 1, 2, \dots)$. Hence, we can choose a_n, $0 \le a_n \le 1$, $n = 1, 2, \dots$, such that

$$\|a_n x_n + (1-a_n)x\| < 1 - \delta.$$

Note that $0 \le a_n \le 1$, $a_n x_n + (1-a_n)x = a_n(x_n - x) + x$ and $x_n \xrightarrow{w} x$, it is easy to see that

$$a_n x_n + (1-a_n)x \xrightarrow{w} x.$$

Since, for each n, $\|a_n x_n + (1-a_n)x\| < 1 - \delta$, by Theorem 3.12 in [**4**] we have $\|x\| \le 1 - \delta < 1$, which contradicts $\|x\| = 1$.

Theorem 6. *Let X be a Banach space. Then the following statements are equivalent*:

(a) *X has* H *property.*

(b) *If $g : [0,1] \to X$ is weakly continuous on $[0,1]$, and $\|g(\cdot)\|$ is continuous on $[0,1]$, then g is Darboux integrable on $[0,1]$.*

(c) *If $g : [0,1] \to X$ is weakly continuous on $[0,1]$ and $\|g(\cdot)\|$ is continuous on $[0,1]$, then g is Riemann integrable on $[0,1]$.*

Proof. Clearly we only need to prove (c) implies (a).

If (a) does not hold, we will construct a mapping $g : [0,1] \to X$ which is weakly continuous on $[0,1]$, $\|g(\cdot)\|$ is continuous on $[0,1]$, but g is not

Riemann integrable on $[0,1]$. This leads to a contradiction to statement (c).

If X does not have the H property, there exists $\varepsilon_0 > 0$, $\{x_n\} \subset X$ such that $x_n \xrightarrow{w} x \in X$, $\|x_n\| = \|x\| = 1$, $\|x_n - x\| \geq \varepsilon_0$, $n = 1, 2, \dots$. Put $g(t) = x + \sum_{n=1}^{\infty} h_n(t)(x_n - x)$ where $h_n(t)$, $n = 1, 2, \dots$, are as in the proof of Theorem 2. It is easy to see that g is weakly continuous on $[0,1]$ and g is not Riemann integrable on $[0,1]$. So, to complete the proof, we only need to show that $\|g(\cdot)\|$ is continuous on $[0,1]$. We will use the same notation as in the proof of Theorem 2.

If $t_0 \in G$, then $t_0 \in (a_{k_0}^{(i_0)}, \beta_{k_0}^{(i_0)})$ for some k_0, i_0. Hence

$$\begin{aligned} g(t_0) &= h_{k_0}(t_0)(x_{k_0} - x) + x \\ g(t) &= h_{k_0}(t)(x_{k_0} - x) + x \quad \text{for } t \in (a_{k_0}^{(i_0)}, \beta_{k_0}^{(i_0)}) \end{aligned}$$

since $h_{k_0}(t)$ is continuous at t_0, $\|g(\cdot)\|$ is continuous at t_0. If $t_0 \in H$, $t_0 \neq a_k^{(i)}, \beta_k^{(i)}$, where $k = 1, 2, \dots, i = 1, 2, \dots, 2^{k-1}$. From Lemma 5, $d_n = \inf\{\|ax_n + (1-a)x\| : a \in [0,1]\} \to 1$, so, for each $\varepsilon > 0$, there exists a positive integer N such that $d_n > 1 - \varepsilon$ whenever $n > N$. Let $\delta = \min\{\rho(t_0, [a_n^{(i)}, \beta_n^{(i)}]) : 1 \leq n \leq N, 1 \leq i \leq 2^{n-1}\}$. Then $\delta > 0$. For any $\in [0,1]$, $|t - t_0| < \delta$, we have:

(i) if $t \in H$, then $g(t) = x$, $\|g(t)\| = 1$.

(ii) If $t \notin H$, then $t \in (a_n^{(i)}, \beta_n^{(i)})$ for some $n > N$, some i, $1 \leq i \leq 2^{n-1}$ and $\|g(t)\| = \|h_n(t)(x_n - x) + x\| > 1 - \varepsilon$.

So $\|g(\cdot)\|$ is continuous at t_0.

If $t_0 = a_{n_0}^{(i_0)}$, or $\beta_{n_0}^{(i_0)}$, for some n_0, i_0, $1 \leq i_0 \leq 2^{n_0-1}$, we can show by previous arguments that $\|g(\cdot)\|$ is left continuous and right continuous at t_0, i.e., $\|g(\cdot)\|$ is continuous at t_0.

Therefore, $\|g(\cdot)\|$ is continuous on $[0,1]$.

The proof of Theorem 6 is completed. $\square$

Finally one can ask if Theorem 6 is true for H property with respect to τ as in Definition 1?

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