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ISOMETRIC REFLECTIONS ON BANACH SPACES AFTER A PAPER OF A. SKORIK AND M. ZAIDENBERG

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ABSTRACT. Let E be a real Banach space. A norm-one element e in E is said to be an isometric reflection vector if there exist a maximal subspace M of E and a linear isometry $F : E \to E$ fixing the elements of M and satisfying $F(e) = -e$. We prove that each of the conditions (i) and (ii) below implies that E is a Hilbert space. (i) There exists a nonrare subset of the unit sphere of E consisting only of isometric reflection vectors, (ii) There is an isometric reflection vector in E, the norm of E is convex transitive, and the identity component of the group of all surjective linear isometries on E relative to the strong operator topology is not reduced to the identity operator on E.

1. Introduction. Throughout this paper E will be a real Banach space with dual denoted by E^*, $S = S(E)$ and $B = B(E)$ will be the unit sphere and the closed unit ball, respectively, of E, and $\mathcal{G} = \mathcal{G}(E)$ will stand for the group of all surjective linear isometries on E. A reflection on E is an operator on E of the form $s_{e,e^*} : x \to x - 2e^*(x)e$, where e and e^* are elements in E and E^*, respectively, satisfying $e^*(e) = 1$. For e in S there is at most one element e^* in E^* such that s_{e,e^*} is an isometric reflection, see [**23**, Remark 2.2.b] or the beginning of Section 2 of this paper, and, when there exists such an e^*, we have $\|e^*\| = 1$ and we say that e is a vector of isometric reflection and that e^* is the isometric reflection functional associated to e. The standard examples of isometric reflections are the so-called orthogonal reflections on a real Hilbert space H, namely the mappings of the form $x \to x - 2(x|e)e$ where e is any norm-one element in H.

In a recent paper [**23**], Skorik and Zaidenberg study in detail the behavior of the group $\mathcal{G}$ whenever E has some vector of isometric reflection. As a main result they prove that, if e is such a vector

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in E, if $\mathcal{G}_0$ denotes the identity component of $\mathcal{G}$ relative to the strong operator topology, if $\mathcal{G}_0(e) \neq \{e\}$, and if H denotes the closed linear hull of $\mathcal{G}_0(e)$, then H is a Hilbert space, $S(H)$ coincides with $\mathcal{G}_0(e)$ and E splits as a topological direct sum $E = H \oplus N$ in such a way that every surjective linear isometry on H can be extended to a surjective linear isometry on E whose restriction to N is the identity, and the elements of $\mathcal{G}$ leaving H invariant diagonalize relative to that splitting. Another result in the Skorik-Zaidenberg paper, relevant in its own right and also in order to motivate our approach, is that, if there exists a vector of isometric reflection in E, and if the norm of E is almost transitive, then E is a Hilbert space. (We recall that the norm of E is said to be almost transitive if, for every x in S, the orbit $\mathcal{G}(x) := \{F(x) : F \in \mathcal{G}\}$ is dense in S.) Natural questions about the limits of the above assertion are the following: does the result remain true whenever either the almost transitivity of the norm is relaxed to the convex transitivity or the almost transitivity of the norm and the existence of a vector of isometric reflection are replaced by the existence of a nonrare subset in S consisting of vectors of isometric reflection? (We recall that the norm of E is called convex transitive if, for every x in S, we have $\overline{\mathrm{co}}\,\mathcal{G}(x) = B$, where $\overline{\mathrm{co}}$ means closed convex hull, and that a subset R of a topological space T is said to be rare in T if the interior of the closure of R in T is empty.) In this paper we provide an affirmative answer to the second question, Theorem 2.2, and an "almost" affirmative answer to the former, Corollary 3.7, namely, if there exists an isometric reflection vector in E, if the norm of E is convex transitive, and if $\mathcal{G}_0$ is not reduced to the singleton consisting of the identity operator on E, then E is a Hilbert space. The proofs of our results involve a great part of the material, including the main result, in the paper of Skorik and Zaidenberg quoted above.

In fact the proof of our convex transitive extension of the Skorik-Zaidenberg theorem also involves an adaptation of the argument in Theorem 6.4 of the Kalton-Wood paper [**16**]. The Kalton-Wood result, as well as a related one due to Berkson [**3**], provides a characterization of complex Hilbert spaces among complex Banach spaces in terms of the abundance of Hermitian projections with one-dimensional range. We observe that, for a norm-one element e in a complex Banach space X, $\mathbf{C}e$ is the range of a Hermitian projection on X if and only if e is an isometric reflection vector in the real Banach space underlying X, Proposition 4.2. This observation allows us to rediscover and refine the results of Kalton-Wood and Berkson quoted above.

Finally, let us point out the following multiplicative characterization of real Hilbert spaces, that we obtain in Theorem 2.5 as an application of Theorem 2.2. Let us say that an element e in S acts as a unit on E if there exists a real Banach algebra containing E isometrically and having e as a unit. Then E is a Hilbert space whenever the set of all elements in S which act as a unit on E is nonrare in S. As a consequence, $\mathbf{R}$, $\mathbf{C}$ and $\mathbf{H}$ (the absolute-valued algebra of Hamilton quaternions) are the unique norm-unital real Banach algebras with almost transitive norm, compare [**14**].

2. Nonrarity of the set of vectors of isometric reflection. Let e be in S, and let e_1^*, e_2^* be in E^* such that s_{e,e_1^*} and s_{e,e_2^*} are isometric reflections on E. Then the operator T on E given by $T(x) = 2(e_2^* - e_1^*)(x)e$ satisfies $T^2 = 0$ and $\mathbf{1} + T = s_{e,e_1^*} \circ s_{e,e_2^*}$. Therefore, for all n in $\mathbf{N}$, $\mathbf{1} + nT = (\mathbf{1} + T)^n$ is an isometry, so $T = 0$ and so $e_1^* = e_2^*$. In this way we have shown that an isometric reflection vector in E has a unique associated isometric reflection functional.

Lemma 2.1. *The set R of all isometric reflection vectors in E is norm-closed in E and if, for e in R, e^* denotes the isometric reflection functional associated to e, then the mapping $e \to e^*$ from R to E^* is norm-to-weak* continuous.*

Proof. Let $\{e_n\}$ be a sequence in R norm-converging to some e in E. We show that e lies in R and $e^* = w^* - \lim\{e_n^*\}$. If x, y are in S and f, g are in $B(E^*)$, then we have

$$|f(x) - g(y)| \le \|x - y\| + |(f - g)(y)|,$$

hence, if $\mathbf{n}$ denotes the norm topology on E, then the mapping $(x, f) \to f(x)$ from $(S, \mathbf{n}) \times (B(E^*), w^*)$ to $\mathbf{R}$ is (jointly) continuous. By the w^*-compactness of $B(E^*)$, there exists a w^*-limit point f for the sequence $\{e_n^*\}$ in $B(E^*)$. It follows that $f(e)$ is a limit point of the sequence $\{e_n^*(e_n)\}$ in $\mathbf{R}$ and, since $e_n^*(e_n) = 1$ for all n in $\mathbf{N}$, we have $f(e) = 1$. On the other hand, for x in E, $2f(x)e - x$ is an $\mathbf{n}$-limit point of the sequence $\{2e_n^*(x)e_n - x\}$ in E, and therefore the reflection $s_{e,f}$

is isometric. Now e lies in R and the isometric reflection functional e^* associated to e coincides with f. Since f is an arbitrary w^*-limit point of $\{e_n^*\}$ in $B(E^*)$, again the w^*-compactness of $B(E^*)$ gives us that actually $e^* = f = w^* - \lim\{e_n^*\}$. □

Recall that a subspace M of E is said to be an L^2-summand of E if there is a linear projection π from E onto M satisfying

$$\|x\|^2 = \|\pi(x)\|^2 + \|x - \pi(x)\|^2$$

for all x in E. If M is an L^2-summand of E, then the projection π above is uniquely determined by M and is called the L^2-projection from E onto M. If e is in S and if $\mathbf{R}e$ is an L^2-summand of E, then e is an isometric reflection vector in E and the isometric reflection functional e^* associated to e is determined by the condition $\pi(x) = e^*(x)e$ for all x in E, where π is the L^2-projection from E onto $\mathbf{R}e$.

Theorem 2.2. *The following assertions are equivalent*:

(i) *The set of all elements e in S such that $\mathbf{R}e$ is an L^2-summand of E is not rare in S.*

(ii) *The set of all isometric reflection vectors in E is not rare in S.*

(iii) *E is a Hilbert space.*

Proof. The implications (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (i) are clear. Assume that (ii) holds, and let R denote the set of all isometric reflection vectors in E. By Lemma 2.1 the nonrarity of R in S provides us with an element e in S and a positive number $\varepsilon \leq (1/2)$ such that x lies in R whenever x is in S and $\|e - x\| < \varepsilon$. Let Π be an arbitrary two-dimensional subspace of E, $\Pi^\wedge$ the linear hull of $\Pi \cup \{e\}$, A the set

$$\{s_{x,x^*} : x \in S \cap \Pi^\wedge, \|e - x\| < \varepsilon\}$$

regarded as a set of isometric reflections on $\Pi^\wedge$ and W the group of isometries on $\Pi^\wedge$ generated by A. We recall that the Coxeter graph of W relative to A, $\Gamma_{W,A}$, has the set A as the set of vertices, and two vertices are connected by an edge whenever the corresponding reflections do not commute. But, for x, y in $S \cap \Pi^\wedge$ with $\|e - x\| < \varepsilon$ and

$\|e-y\| < \varepsilon$, we have $\|x-y\| < 1$ so that if $x \neq y$, then clearly s_{x,x^*} and s_{y,y^*} do not commute, and therefore $\Gamma_{W,A}$ is connected. Also we claim that, if z is in $\Pi^\wedge$ and remains fixed under W, then $z = 0$. Indeed, this is clear if the system $\{z, e\}$ is linearly dependent and, otherwise, the set

$$\{x \in S \cap \mathrm{Lin}\,\{z, e\}, \|e - x\| < \varepsilon\}$$

must contain two linearly independent vectors u and v so that, with the aid of an "invariant" inner product $\langle\cdot|\cdot\rangle$ on $\mathrm{Lin}\,\{z, e\}$, i.e., $\langle\cdot|\cdot\rangle$ makes isometric all the linear isometries on $(\mathrm{Lin}\,\{z, e\}, \|\cdot\|)$, [**22**, Theorem 9.5.1], we easily see that the conditions $s_{u,u^*}(z) = z$ and $s_{v,v^*}(z) = z$ imply $\langle u|z\rangle = \langle v|z\rangle = 0$, and hence $z = 0$. The claim just proved, together with the connectedness of $\Gamma_{W,A}$ and the existence of invariant inner products on $\Pi^\wedge$, allows us to apply [**6**, Chapter 5], see also [**23**, Lemma 1.1] to obtain that W acts irreducibly on $\Pi^\wedge$. Since clearly W is infinite, it follows from [**23**, Corollary 1.4] that $\Pi^\wedge$, and hence Π is Euclidean. Now every two-dimensional subspace of E is Euclidean, and therefore (iii) is true. $\square$

If the norm of E is almost transitive, and if there exists an isometric reflection vector e in E, then $\mathcal{G}(e)$ is a dense, hence nonrare, set in S consisting of isometric reflection vectors, and hence, by the theorem, E is a Hilbert space [**23**, Theorem 2a]. By the way, the eventual assumption on E that, for some e in S, $\mathcal{G}(e)$ is not rare in S is nothing but the almost transitivity of the norm [**2**]. Note also that Theorem 2.2 contains the result of Carlson and Hicks [**7**] that E is a Hilbert space whenever every one-dimensional subspace of E is an L^2-summand of E.

In what follows we will apply Theorem 2.2 to obtain a new multiplicative characterization of real Hilbert spaces, see [**21**] for related results. The geometry of norm-unital Banach algebras at their units is very peculiar, and every real Hilbert space enjoys this peculiarity (at any norm-one element): every real Hilbert space H can be converted into a (nonassociative) norm-unital Banach algebra with arbitrarily prefixed unit in the unit sphere of H, [**21**, Observation 1.3]. Moreover, by putting together [**21**, Claim 5.8], [**10**, Proposition 4.1] and [**19**, Theorem 27], see also [**20**, Section 2], one can see that the above property characterizes real Hilbert spaces among real Banach spaces. Although not all, see Remark 2.7 below, most properties of the geometry of a norm-unital Banach algebra at its unit are inherited by subspaces containing the unit. Then one can suspect that, if for every e in S, E can be regarded as a subspace of a norm-unital Banach algebra with unit e, then E is a Hilbert space. Actually the answer to this conjecture is affirmative as a consequence of Theorem 2.5 below. Let us say that an element e in E acts as a unit on E if there exist a norm-unital real Banach algebra $\mathcal{A}$ and a (possibly nonsurjective) linear isometry ϕ from E to $\mathcal{A}$ satisfying $\phi(e) = \mathbf{1}$, where $\mathbf{1}$ denotes he unit of $\mathcal{A}$. We do not assume the algebra $\mathcal{A}$ above to be associative but, actually, it can be chosen with that additional property (replace $\mathcal{A}$ by the norm-unital associative real Banach algebra $L(\mathcal{A})$ of all bounded linear operators on $\mathcal{A}$, and ϕ by the linear isometry $x \to L_{\phi(x)}$ from E to $L(\mathcal{A})$ where, for a in $\mathcal{A}$, L_a denotes the operator of left multiplication by a on $\mathcal{A}$).

Lemma 2.3. *The set U of all elements in E which act as units on E is norm-closed in E.*

Proof. Let $\{e_n\}$ be a sequence in U norm-converging to some e in E. For n in $\mathbf{N}$, there exist a norm-unital Banach real algebra $\mathcal{A}_n$ and a linear isometry ϕ_n from E to $\mathcal{A}_n$ satisfying $\phi_n(e_n) = \mathbf{1}_n$, where $\mathbf{1}_n$ denotes the unit of $\mathcal{A}_n$. Take an ultrafilter $\mathcal{U}$ on $\mathbf{N}$ refining the Fréchet ultrafilter, and consider the unital Banach algebra $(\mathcal{A}_n)_{\mathcal{U}}$, i.e., the Banach ultraproduct of the family of Banach spaces $\{\mathcal{A}_n : n \in \mathbf{N}\}$ relative to $\mathcal{U}$, cf. [**12**], regarded as a unital Banach algebra under the (well-defined) product $(a_n)(b_n) := (a_n b_n)$. Then the mapping $\phi : x \to (\phi_n(x))$ from E to $(\mathcal{A}_n)_{\mathcal{U}}$ is a linear isometry from E to $(\mathcal{A}_n)_{\mathcal{U}}$ satisfying $\phi(e) = \mathbf{1}$, where $\mathbf{1}$ denotes the unit of $(\mathcal{A}_n)_{\mathcal{U}}$. □

Lemma 2.4. *Let e act as a unit on E, and assume that E is smooth at e. Then $\mathbf{R}e$ is an L^2-summand of E.*

Proof. Taking a real Banach algebra $\mathcal{A}$ and a linear isometry ϕ from E to $\mathcal{A}$ satisfying $\phi(e) = \mathbf{1}$ and denoting by f the unique element in E^* satisfying $\|f\| = f(e) = 1$, for every g in $\mathcal{A}^*$ with $\|g\| = g(\mathbf{1}) = 1$ we have

$$\|g \circ \phi\| = g \circ \phi(e) = 1$$

and hence $g \circ \phi = f$. It follows that, for every x in $\operatorname{Ker}(f)$, the numerical range of $\phi(x)$ relative to $\mathcal{A}$, see [**5**, p. 42] for definition, is reduced to zero. But, for λ in $\mathbf{R}$ and a in $\mathcal{A}$ with zero numerical range, the equality

$$\|\lambda \mathbf{1} + a\|^2 = \lambda^2 + \|a\|^2$$

holds. (This is a well-known consequence of Sinclair's theorem [**5**, Corollary 26.6], cf. [**20**, Lemma 2.2] and [**19**, Lemma 3(b)] for details.) Finally, for λ in $\mathbf{R}$ and x in $\operatorname{Ker}(f)$ we have

$$\|\lambda e+x\|^2 = \|\phi(\lambda e+x)\|^2 = \|\lambda \mathbf{1}+\phi(x)\|^2 = \lambda^2+\|\phi(x)\|^2 = \lambda^2+\|x\|^2. \qquad \square$$

For e in S and x in E, the function $\alpha \to \|e + \alpha x\|$ from $\mathbf{R}$ to $\mathbf{R}$ is convex, hence the limit

$$\tau(e,x) := \lim_{\alpha \to 0^+} \frac{\|e + \alpha x\| - 1}{\alpha}$$

makes sense, and we have

$$\tau(e,x) := \inf\left\{ \frac{\|e + \alpha x\| - 1}{\alpha} : \alpha \in \mathbf{R}^+ \right\}.$$

Theorem 2.5. *Assume that the set U of all elements in E which act as units on E is not rare in S. Then E is a Hilbert space.*

Proof. For e in U, take a real Banach algebra $\mathcal{A}$ and a linear isometry ϕ from E to $\mathcal{A}$ satisfying $\phi(e) = \mathbf{1}$. As noticed in the proof of [**21**, Claim 5.8], the proofs of [**17**, Proposition 4.5] and [**1**, Theorem 5.1] give us that, for every positive number ε, we have

$$\frac{\|\mathbf{1} + \lambda a\| - 1}{\lambda} - \tau(\mathbf{1}, a) < \varepsilon$$

whenever a is in $B(A)$ and $0 < \lambda < \operatorname{Min}\{(\varepsilon/4), (1/2)\}$, hence also

$$\frac{\|e + \lambda x\| - 1}{\lambda} - \tau(e, x) < \varepsilon$$

whenever x is in B and $0 < \lambda < \operatorname{Min}\{(\varepsilon/4), (1/2)\}$. Now we will apply the lines of the argument in the proof of [**10**, Proposition 4.1] to show that E is smooth at every interior point of U. By the observation at the beginning of the proof, the convergence

$$\frac{\|e+\lambda x\|-1}{\lambda} \overset{\lambda\to 0^+}{\longrightarrow} \tau(e,x)$$

is uniform when the couple (e,x) runs in $U \times B$, hence the function $(e,x) \to \tau(e,x)$ from $U \times B$ to $\mathbf{R}$ is continuous, and consequently, for arbitrarily fixed x in E, the function $e \to \tau(e,x)$ from U to $\mathbf{R}$ is continuous. On the other hand, to show that E is smooth at a point $v \in \overset{\circ}{U}$, the interior of U relative to S, it is enough to prove that every two-dimensional subspace of E containing v is smooth at v. Let v be in $\overset{\circ}{U}$, and Y be a two-dimensional subspace of E containing v. Then $Y \cap \overset{\circ}{U}$ is an open subset of $S(Y)$ containing v, hence, by Mazur's theorem, see for example [**13**, p. 171], there exists a sequence $\{v_n\}$ in $Y \cap \overset{\circ}{U}$ converging to v and such that Y is smooth at v_n for all n in $\mathbf{N}$. Then, for every y in Y, the sequence $\{\tau(v_n,y)\}$ converges to $\tau(v,y)$. Since the smoothness of Y at v_n reads as $\tau(v_n,-y) = -\tau(v_n,y)$ for all y in Y, it follows that $\tau(v,-y) = -\tau(v,y)$ for all y in Y, i.e., Y is smooth at v. Now that we know that E is smooth at every point of $\overset{\circ}{U}$, we note that $\overset{\circ}{U}$ is not rare in S because U is closed, by Lemma 2.3, and nonrare in S. It follows from Lemma 2.4 and Theorem 2.2 that E is a Hilbert space. $\square$

Corollary 2.6. *Assume that E is a subspace of a norm-unital real Banach algebra $\mathcal{A}$ containing the unit of $\mathcal{A}$, and that the norm of E is almost transitive. Then E is a Hilbert space.*

In the above corollary the almost transitivity of the norm cannot be relaxed to the convex transitivity of the norm. Indeed, if E is the norm-unital (associative and commutative) Banach algebra of all real valued continuous functions on the Cantor set, then the norm of E is convex transitive [**18**]. Now let $\mathcal{A}$ be a norm-unital, possibly nonassociative, real Banach algebra with almost transitive norm. It follows from Corollary 2.6 that the norm of $\mathcal{A}$ derives from an inner product $(\cdot|\cdot)$, so that the structure of $\mathcal{A}$ is given by [**19**, Theorem 27]. As a consequence, for x, y in $\mathcal{A}$, we have

$$(xy+yx)/2 = (x \mid \mathbf{1})y + (y \mid \mathbf{1})x - (x \mid y)\mathbf{1},$$

where $\mathbf{1}$ denotes the unit of $\mathcal{A}$, hence $\mathcal{A}$ is a quadratic algebra and every nonzero element x of $\mathcal{A}$ has an inverse in the (associative) subalgebra of $\mathcal{A}$ generated by x. Therefore, by applying the Frobenius-Zorn theorem [**9**, pp. 229, 262], we obtain that, if $\mathcal{A}$ is alternative, then $\mathcal{A}$ is isometrically isomorphic to $\mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$ or $\mathbf{O}$ (the absolute-valued algebra of real octonions). This extends a result of Ingelstam in [**15**].

Remark 2.7. For e in S, consider the following conditions:

(a) There exists a norm-one continuous bilinear mapping $f : E\times E \to E$ satisfying $f(e,x) = f(x,e) = x$ for all x in E.

(b) $m(E,e) = 1$, where $m(E,e)$ means the infimum of the set of numbers of the form $\|f\|$ when f runs over the set of all continuous bilinear mappings $f : E \times E \to E$ satisfying $f(e,x) = f(x,e) = x$ for all x in E.

(c) $\operatorname{sm}(E,e) = 1$, where $\operatorname{sm}(E,e)$ stands for the infimum of the set of numbers of the form $\operatorname{Max}\{\|f\|, 1 + \|L_e^f - I_E\|, 1 + \|R_e^f - I_E\|\}$ when f runs over the set of all continuous bilinear mappings $f : E \times E \to E$. (Here I_E means the identity operator on E and, for f as above, L_e^f and R_e^f denote the operators on E given by $x \to f(e,x)$ and $x \to f(x,e)$, respectively.)

(d) e acts as a unit on E.

Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d); for the last implication see the proof of [**21**, Proposition 2.4]. It follows from Theorem 2.5 that if, for e in S, $\mathcal{P}_e$ denotes any one of the conditions (a), (b) or (c), and if the set $\{e \in S : e \text{ satisfies } \mathcal{P}_e\}$ is not rare in S, then E is a Hilbert space. However, the information contained in Theorem 2.5 is strictly stronger than the above consequence because, in general, condition (d) does not imply (c). This is shown in the following example. For every nonzero real Hilbert space H, put $Y := \{T \in L(H) : T^* = -T\}$, and let E be the closed subspace of $L(H)$ given by $E := \mathbf{R}e \oplus Y$, where e denotes the identity operator on H. Then, for λ in $\mathbf{R}$ and T in Y, we have

$$\begin{aligned}\|\lambda e + T\|^2 &= \|(\lambda e + T)^*(\lambda e + T)\| = \|(\lambda e - T)(\lambda e + T)\|\\ &= \|\lambda^2 + T^2\| = \lambda^2 + \|T\|^2,\end{aligned}$$

hence E contains $\mathbf{R}e$ as an L^2-summand and therefore E is smooth at e. Clearly e acts as a unit on E. However, for dimension of $H \geq 4$, we can choose pairwise orthogonal nonzero elements x, y, z, t in H so that the linear hull of $\{x \otimes y - y \otimes x, z \otimes t - t \otimes z\}$ in $L(H)$ is a copy of l_∞^2 contained in E, and therefore we cannot have $\mathrm{sm}\,(E, e) = 1$ since, otherwise, by the smoothness of E at e and [**21**, Theorem 2.5], E would be a Hilbert space.

3. Convex transitivity of the norm. Recall that $\mathcal{G}_0 = \mathcal{G}_0(E)$ denotes the connected component of $\mathcal{G}$ containing the identity operator on E when $\mathcal{G}$ is endowed with the strong operator topology.

Lemma 3.1. *Let e be in S such that the closed linear hull of $\mathcal{G}(e)$ is E. Then $\mathcal{G}_0$ is trivial, i.e., it reduces to the singleton consisting of the identity operator on E, if and only if $\mathcal{G}_0(e) = \{e\}$.*

Proof. Assume $F(e) = e$ for every F in $\mathcal{G}_0$. Since $\mathcal{G}_0$ is a normal subgroup of $\mathcal{G}$, we actually have $FG(e) = Ge$ for all F in $\mathcal{G}_0$ and G in $\mathcal{G}$. Since the closed linear hull of $\mathcal{G}(e)$ is E, it follows that $\mathcal{G}_0$ is trivial. $\square$

Corollary 3.2. *Assume that $\mathcal{G}_0$ is not trivial, and let e be a vector of isometric reflection in E such that the closed linear hull of $\mathcal{G}(e)$ is E. Then H, defined as the closed linear hull of $\mathcal{G}_0(e)$, is a Hilbert space, the unit sphere of H coincides with $\mathcal{G}_0(e)$, and there exists a contractive projection* (*uniquely determined under suitable conditions*) *from E onto H.*

Proof. Put together Lemma 3.1 and [**23**, Theorem 1]. $\square$

Recall that a family $\mathcal{F}$ of linear functionals on E is said to be total if for every x in $E\backslash\{0\}$ there exists f in $\mathcal{F}$ satisfying $f(x) \neq 0$.

Lemma 3.3. *Assume that the group $\mathcal{G}$ admits only the trivial $\mathcal{G}$-invariant closed subspaces and that there exists a vector of isometric reflection e in E. Then there exists a total family of isometric reflection functionals on E. More precisely, if e^* denotes the isometric reflection functional associated to e, then the family $\{e^* \circ F : F \in \mathcal{G}\}$ is a total* *family of isometric reflection functionals on* E.

Proof. Since e is a vector of isometric reflection and e^* is its associated isometric reflection functional, for every F in $\mathcal{G}$, $F^{-1}(e)$ is a vector of isometric reflection and $e^* \circ F$ is its associated isometric reflection functional. Moreover, the family $\{e^* \circ F : F \in \mathcal{G}\}$ is total because, for every x in $E \backslash \{0\}$, the closed linear hull of $\mathcal{G}(x)$ is E. □

Theorem 3.4. *Assume that* $\mathcal{G}_0$ *is nontrivial, that there exists a vector of isometric reflection in* E, *and that there is a* $\delta > 0$ *such that* $\overline{\text{co}}\, \mathcal{G}(x) \supseteq \delta B$ *for all* x *in* S. *Then* E *is isomorphic to a Hilbert space. More precisely, there exist a natural number* $n \leq \delta^{-2}$ *and pairwise isomorphic Hilbert subspaces* $H_1, \ldots, H_n$ *of* E *satisfying*:

(i) $E = \oplus_{i=1}^n H_i$.

(ii) *For* $i = 1, \ldots, n$ *and* F *in* $\mathcal{G}$, *there is* $j = 1, \ldots, n$ *with* $F(H_i) = H_j$.

(iii) *For* $i = 1, \ldots, n$, H_i *is invariant under* $\mathcal{G}_0$.

Proof. Let e be the vector of isometric reflection whose existence has been assumed. For x, y in $\mathcal{G}(e)$, define $x \simeq y$ if $\mathcal{G}_0(x) = \mathcal{G}_0(y)$. Then $\simeq$ is an equivalence relation on $\mathcal{G}(e)$, so that we may consider the quotient set $\mathcal{B} := \mathcal{G}(e)/\simeq \; (= \{\mathcal{G}_0(x) : x \in \mathcal{G}(e)\})$. Since, for x in $\mathcal{G}(e)$, we have $\overline{\text{co}}\, \mathcal{G}(x) \supseteq \delta B$, it follows from Corollary 3.2 that, for β in $\mathcal{B}$, H_β (defined as the closed linear hull of β) is a Hilbert space, the unit sphere of H_β coincides with β, and there exists a contractive projection p_β (uniquely determined under suitable conditions) from E onto H_β. Moreover, since $\overline{\text{co}}\, \mathcal{G}(e) \supseteq \delta B$, we have that $E = \text{clos}\,(\sum_{\beta \in \mathcal{B}} H_\beta)$ (where clos denotes closure). Now Lemma 3.3 allows us to apply [**23**, 5.3 and Lemma 5.4.c] so that we have $p_\alpha p_\beta = 0$ for all α, β in $\mathcal{B}$ with $\alpha \neq \beta$ and therefore $E = \text{clos}\,(\oplus_{\beta \in \mathcal{B}} H_\beta)$. From now on, we follow an argument in the proof of [**16**, Theorem 6.4]. For every F in $\mathcal{G}$, we have $F(e) \in \gamma := \mathcal{G}_0(F(e)) \in \mathcal{B}$, so that $F(e) \in H_\gamma$ for some γ in β. Hence,

$$\sum_{\beta \in \mathcal{B}} \| p_\beta (F(e)) \| = 1.$$

Now for every finite subset Γ of $\mathcal{B}$, the set $\{x \in E : \sum_{\beta \in \Gamma} \| p_\beta(x) \| \leq 1\}$ is closed and convex in E and contains $\mathcal{G}(e)$. Since $\overline{\text{co}}\, \mathcal{G}(e) \supseteq \delta B$, it follows that, for every x in E, the family $\{\|p_\beta(x)\|\}_{\beta\in\mathcal{B}}$ is summable in $\mathbf{R}$ and

$$\delta \sum_{\beta\in\mathcal{B}} \|p_\beta(x))\| \leq \|x\|. \tag{3.1}$$

We note that each F in $\mathcal{G}$ induces a bijection $F_* : \mathcal{B} \to \mathcal{B}$ given by $F_*(\beta) = F(\beta)$ for all β in $\mathcal{B}$, and, even more, the mapping $F \to F_*$ is an action of $\mathcal{G}$ on B, see [**23**, 5.3]. Moreover, with our definition of $\mathcal{B}$, $\mathcal{G}$ acts transitively on $\mathcal{B}$. As a consequence, the Hilbert spaces H_β, $\beta \in \mathcal{B}$, are pairwise isomorphic. Let $\beta_1, \dots, \beta_n$ be pairwise distinct elements in $\mathcal{B}$. Then, choosing $x_1 \in \beta_1, \dots, x_n \in \beta_n$, for every F in $\mathcal{G}$ we have

$$\max\{\|p_\beta(F(x_1 + \cdots + x_n))\| : \beta \in \mathcal{B}\} = 1.$$

Since $\delta\|x_1 + \cdots + x_n\|e \in \delta\|x_1 + \cdots + x_n\|B \subseteq \overline{\mathrm{co}}\,\mathcal{G}(x_1 + \cdots + x_n)$, we have

$$\delta\|x_1 + \cdots + x_n\| \max\{\|p_\beta(e))\| : \beta \in \mathcal{B}\} \leq 1.$$

But, clearly, $\max\{\|p_\beta(e))\| : \beta \in \mathcal{B}\} = 1$, and from (3.1) we obtain

$$n\delta = \delta \sum_{i=1}^{n} \|p_{\beta_i}(x_i)\| = \delta \sum_{\beta\in\mathcal{B}} \|p_\beta(x_1 + \cdots + x_n)\| \leq \|x_1 + \cdots + x_n\|.$$

It follows that $n\delta^2 \leq 1$. □

The following corollary is immediate.

Corollary 3.5. *Assume that $\mathcal{G}_0$ is nontrivial, that there exists a vector of isometric reflection in E, and that there is $\delta > 2^{-1/2}$ such that $\overline{\mathrm{co}}\,\mathcal{G}(x) \supseteq \delta B$ for all x in S. Then E is a Hilbert space.*

The assumption that $\mathcal{G}_0$ is not trivial cannot be removed in the above corollary. Indeed, if n is in $\mathbf{N}$, and if E denotes $\mathbf{R}^2$ under the norm whose closed unit ball is a regular polygon with $2(n+1)$ sides centered at zero, then, for n large enough, all assumptions in the corollary, except the nontriviality of $\mathcal{G}_0$, are satisfied by E.

We recall that the norm of E is called maximal if $\mathcal{G}$ cannot be strictly enlarged to the group of all surjective linear isometries on any equivalent renorming of E. It is well known that convex transitivity of the norm implies maximality of the norm, see, for instance, [**22**, Theorem 9.7.1]. Actually the norm of E is convex transitive if and only if the unique equivalent renormings of E which enlarge the group of surjective linear isometries are those obtained by multiplying the norm of E by a positive number [**8**].

Corollary 3.6. *Assume that $\mathcal{G}_0$ is nontrivial, that there exists a vector of isometric reflection in E, that the norm of E is maximal and that there is a $\delta > 0$ such that $\overline{\mathrm{co}}\,\mathcal{G}(x) \supseteq \delta B$ for all x in S. Then E is a Hilbert space.*

Proof. Let n and $H_1, \dots, H_n$ be the natural number and the Hilbert subspaces of E, respectively, given by Theorem 3.4. For $j = 1, \dots, n$, let π_j denote the projection from E onto H_j corresponding to the decomposition $E = \oplus_{i=1}^n H_i$ and consider the equivalent Hilbertian norm $|||\cdot|||$ on E given by $|||x||| := (\sum_{i=1}^n \|\pi_i(x)\|^2)^{1/2}$. By Property (ii) in the theorem, $\mathcal{G}(E, |||\cdot|||)$ enlarges $\mathcal{G}$. Assume $n > 1$. Then we can take nonzero elements x_1 and x_2 in H_1 and H_2, respectively, with $|||x_1 + x_2||| = 1$ and an element F in $\mathcal{G}(E, |||\cdot|||)$ satisfying $F(x_1) = |||x_1|||(x_1 + x_2)$. Again, by Property (ii) in the theorem, such an F cannot belong to $\mathcal{G}$. Therefore, $\mathcal{G}(E, |||\cdot|||)$ strictly enlarges $\mathcal{G}$, contradicting the fact that the norm of E is maximal. $\square$

Either from Corollary 3.5 or Corollary 3.6 we obtain the announced "almost" improvement of [**23**, Theorem 2a]:

Corollary 3.7. *Assume that $\mathcal{G}_0$ is nontrivial, that there exists a vector of isometric reflection in E and that the norm of E is convex transitive. Then E is a Hilbert space.*

We do not know if the assumption in Corollaries 3.6 and 3.7 that $\mathcal{G}_0$ is not trivial can be removed. In any case the convex transitivity of the norm of E does not imply that $\mathcal{G}_0$ is nontrivial. To see this, let E denote the Banach algebra of all real valued continuous functions on the Cantor set. Then we know that the norm of E is convex transitive. However, if the $\mathbf{1}$ denotes the unit of E, and if F is in $\mathcal{G}$ with $F(\mathbf{1}) \neq \mathbf{1}$, then the well-known description of surjective isometries on E shows that $\|\mathbf{1}-F(\mathbf{1})\|=2$. Therefore $\mathcal{G}(\mathbf{1})$ is discrete, hence totally disconnected. Since the mapping $G \to G(\mathbf{1})$ from $\mathcal{G}$ to $\mathcal{G}(\mathbf{1})$ is continuous when $\mathcal{G}$ is endowed with the strong operator topology, it follows that $\mathcal{G}_0(\mathbf{1})=\{\mathbf{1}\}$, and $\mathcal{G}_0$ becomes trivial in view of Lemma 3.1.

The rest of this section is devoted to prove that the number $2^{-1/2}$ in Corollary 3.5 is sharp. As a consequence, the assumption in Corollary 3.6 that the norm of E is maximal cannot be removed. The following lemma is a straightforward consequence of the Hahn-Banach separation theorem.

Lemma 3.8. *Let x be in S and ρ be a positive number. Then $\overline{\mathrm{co}}\,\mathcal{G}(x) \supseteq \delta B$ if and only if $\inf\{\sup\{f(F(x)) : F \in \mathcal{G}\} : f \in S(E^*)\} \geq \rho$.*

The next lemma is surely well known.

Lemma 3.9. *Let Ω and K be metrizable topological spaces with Ω locally compact and K compact, and let h be a continuous mapping from $\Omega \times K$ into $\mathbf{R}$. Then the mapping $s : t \to \inf\{h(t,k) : k \in K\}$ from Ω to $\mathbf{R}$ is continuous.*

Proof. To prove the continuity of s at some point t_0 of Ω is enough to show the continuity of the restriction of s to some compact neighborhood of t_0 in Ω, hence we may assume that Ω is compact. Then, by Heine's theorem, h is uniformly continuous on $\Omega \times K$ and therefore the family of functions $\{h(\cdot,k) : k \in K\}$ is equicontinuous on Ω. For $\varepsilon > 0$, let $\delta > 0$ be such that $|h(t_1,k)-h(t_2,k)| \leq \varepsilon$ whenever k is in K and t_1 and t_2 are in Ω with $d(t_1,t_2) \leq \delta$. Then, for k in K and t_1,t_2 in Ω with $d(t_1,t_2) \leq \delta$, we have $s(t_1) \leq h(t_1,k) \leq h(t_2,k)+\varepsilon$, hence $s(t_1) \leq s(t_2)+\varepsilon$, and by symmetry $|s(t_1)-s(t_2)| \leq \varepsilon$. □

Proposition 3.10. *For every positive number $\delta < 2^{-1/2}$ there exists a non-Hilbert real Banach space E with the following properties*:

(i) *$\mathcal{G}_0(E)$ is nontrivial.*

(ii) *There is a vector of isometric reflection in E.*

(iii) $\overline{\mathrm{co}}\,\mathcal{G}(E)(x) \supseteq \delta B(E)$ *for all* x *in* $S(E)$.

Proof. The Banach space E will be of the form $H \times H$, for an arbitrary real Hilbert space H of dimension ≥ 2, with norm $\|\cdot\|$ given by

$$\|(\eta, \xi)\| := (\|\eta\|^p + \|\xi\|^p)^{1/p}$$

for a suitable real number p with $1 < p \neq 2$. Clearly such a Banach space E is non-Hilbert and satisfies properties (i) and (ii) above. We will see in what follows that, for $\delta < 2^{-1/2}$, we may choose $1 < p = p(\delta) \neq 2$ in such a way that property (iii) also holds for E. Consider the continuous function

$$h : (p, \lambda, \mu) \longrightarrow \max\{\lambda(1-\mu^q)^{1/q} + (1-\lambda^p)^{1/p}\mu, \\ \lambda\mu + (1-\lambda^p)^{1/p}(1-\mu^q)^{1/q}\}$$

from $]1, \infty[\times [0,1] \times [0,1]$ into $\mathbf{R}$ where, for p in $]1, \infty[$, q denotes the real number determined by $(1/p)+(1/q) = 1$. According to Lemma 3.9, the function

$$s : p \longrightarrow \inf\{h(p, \lambda, \mu) : (\lambda, \mu) \in [0,1] \times [0,1]\}$$

from $]1, \infty[$ to $\mathbf{R}$ is continuous, and a not difficult computation shows that $s(2) = 2^{-1/2}$. Now for $\delta < 2^{-1/2}$, the existence of $1 < p = p(\delta) \neq 2$ with $s(p) \geq \delta$ is not in any doubt. We fix such a p and consider the corresponding Banach space E at the beginning of the proof. Let x and f be in $S(E)$ and $S(E^*)$, respectively. Then there exist η, ξ, χ, ζ in H satisfying $x = (\eta, \xi)$, hence $\|\eta\|^p + \|\xi\|^p = 1$, $\|\chi\|^q + \|\zeta\|^q = 1$, and $f((y,z)) = (\chi \mid y) + (\zeta \mid z)$ for all (y,z) in $H \times H = E$. Also we can find elements M, P, Q, T in $\mathcal{G}(H)$ with $\|\chi\|M(\eta) = \|\eta\|\chi$, $\|\zeta\|P(\xi) = \|\xi\|\zeta$, $\|\zeta\|Q(\eta) = \|\eta\|\zeta$ and $\|\chi\|T(\xi) = \|\xi\|\chi$. Therefore, since the mappings $(y,z) \to (M(y), P(z))$ and $(y,z) \to (T(z), Q(y))$ are surjective linear isometries on E, we have

$$\begin{aligned}
&\sup\{f(F(x)) : F \in \mathcal{G}\} \\
&\qquad \geq \max\{(\chi \mid M(\eta)) + (\zeta \mid P(\xi)), (\chi \mid T(\xi)) + (\zeta \mid Q(\eta))\} \\
&\qquad \geq \max\{\|\eta\|\|\chi\| + \|\xi\|\|\zeta\|, \|\xi\|\|\chi\| + \|\eta\|\|\zeta\|\} \\
&\qquad \geq \inf\{\max\{ac+bd, bc+ad\} : \\
&\qquad\qquad a, b, c, d \in \mathbf{R}_0^+, a^p + b^p = 1, c^q + d^q = 1\} \\
&\qquad = s(p) \geq \delta.
\end{aligned}$$

Since x and f are arbitrary elements in $S(E)$ and $S(E^*)$, respectively, Lemma 3.8 shows that property (iii) holds for E. $\square$

4. Applications to complex spaces. Throughout this section X will denote a complex Banach space, and we will discuss the results previously obtained for a real Banach space E when applied to $E = X_{\mathbf{R}}$, the real Banach space underlying X. The key tool will be the characterization of isometric reflection vectors in $X_{\mathbf{R}}$ provided by Proposition 4.2 below. The notation and definitions introduced in the previous sections for real Banach spaces will now be taken verbatim for complex Banach spaces. Sometimes some obvious formal changes are needed. For example, we will say that an element e in X acts as a unit on X if there exist a norm-unital *complex* Banach algebra $\mathcal{A}$ and a (complex-) linear isometry ϕ from X to $\mathcal{A}$ satisfying $\phi(e) = \mathbf{1}$.

If an element e in X acts as a unit on X, then clearly e acts as a unit on $X_{\mathbf{R}}$. Therefore, we may apply Corollary 2.5 with $E = X_{\mathbf{R}}$ and the Bohnenblust-Karlin theorem [**4**], asserting that the unit of a norm-unital complex Banach algebra $\mathcal{A}$ is a vertex of $B(\mathcal{A})$ to obtain that, if the set of elements in X which act as a unit in X is not rare in $S(X)$, then X is one-dimensional. As a consequence, if X is a subspace of a norm-unital complex Banach algebra $\mathcal{A}$ containing the unit of $\mathcal{A}$, and if the norm of X is almost transitive, then X is one-dimensional. Actually the last assertion can be refined as follows:

Proposition 4.1. *There exists a universal constant $C > 1$ such that, for every complex Banach space Y with almost transitive norm and $\mathrm{Dim}\,(Y) \geq 2$ and for every closed subspace M of any norm-unital complex Banach algebra $\mathcal{A}$ containing the unit of $\mathcal{A}$, we have $d(Y, M) \geq C$ where $d(\cdot, \cdot)$ means the Banach-Mazur distance.*

Proof. Assume the result is not true. Then, for each n in $\mathbf{N}$, we can find a complex Banach space Y_n with almost transitive norm, norm-one elements u_n, v_n in Y_n with $\|u_n + \mathbf{C}v_n\| \geq 1$, and a norm-unital complex Banach algebra $\mathcal{A}_n$ together with a bicontinuous injective linear mapping $\phi_n : Y_n \to \mathcal{A}_n$ satisfying $\mathbf{1}_n \in \Phi_n(Y_n)$, where $\mathbf{1}_n$ denotes the unit of $\mathcal{A}_n$, $\|\Phi_n\| = 1$, and $\|\Phi_n^{-1}\| < 1 + (1/n)$. If we take an ultrafilter $\mathcal{U}$ in $\mathbf{N}$ refining the Fréchet filter, then the mapping

$\Phi : (x_n) \to (\Phi_n(x_n))$ from the complex Banach space $(Y_n)_{\mathcal{U}}$ to the norm-unital complex Banach algebra $(\mathcal{A}_n)_{\mathcal{U}}$ is a linear isometry with $\mathbf{1} \in \Phi((Y_n)_{\mathcal{U}})$, where $\mathbf{1}$ stands for the unit of $(\mathcal{A}_n)_{\mathcal{U}}$. Moreover, the fact that Y_n has almost transitive norm for all n in $\mathbf{N}$ implies that the norm of $(Y_n)_{\mathcal{U}}$ is (almost) transitive; this is folklore, see, for instance, [**11**, Remark p. 479]. It follows from the comment before the proposition that $(Y_n)_{\mathcal{U}}$ is one-dimensional. But this is a contradiction because (u_n) and (v_n) are norm-one elements in $(Y_n)_{\mathcal{U}}$ satisfying $\|(u_n)+\mathbf{C}(v_n)\| \geq 1$. $\square$

A bounded linear operator T on X is called Hermitian if, for all λ in $\mathbf{R}$, $\exp(i\lambda T)$ is an isometry.

Proposition 4.2. *Let e be a norm-one element in X. Then $\mathbf{C}e$ is the range of a Hermitian projection on X if and only if e is an isometric reflection vector in $X_{\mathbf{R}}$.*

Proof. Assume that e is a norm-one element in X such that $\mathbf{C}e$ is the range of a Hermitian projection π on X. Then there exists e^* in X^* satisfying $\pi(x) = e^*(x)e$ for all x in X, hence $e^*(e) = 1$. For x in X, we may choose λ in $\mathbf{R}$ with $\overline{e^*(x)} = (\sin(\lambda/2) - i\cos(\lambda/2))|e^*(x)|$, so that

$$\begin{aligned}\|x\| &= \|\exp(i\lambda\pi)(x)\| = \|x + (\exp(i\lambda) - 1)\pi(x)\| \\ &= \|x - 2\sin(\lambda/2)(\sin(\lambda/2) - i\cos(\lambda/2))\pi(x)\| \\ &= \|x - 2\sin(\lambda/2)(\sin(\lambda/2) - i\cos(\lambda/2))e^*(x)e\| \\ &= \|x - 2\mathrm{Re}\,(e^*(x))e\|.\end{aligned}$$

Therefore the equality $\|x - 2\mathrm{Re}\,(e^*(x))e\| = \|x\|$ holds for all x in X, hence e is an isometric reflection vector in $X_{\mathbf{R}}$. Now assume the e is an isometric reflection vector in $X_{\mathbf{R}}$. Then there exists a norm-one element e^* in X^* satisfying $e^*(e) = 1$ and $\|x - 2\mathrm{Re}\,(e^*(x))e\| = \|x\|$ for all x in X. For every x in X and λ in $\mathbf{R}$, we take μ in $\mathbf{C}$ with $|\mu| = 1$ and

$$\mu|e^*(x)| = \overline{e^*(x)}(\sin(\lambda/2) + i\cos(\lambda/2)),$$

so that

$$\begin{aligned}\|x\| = \|\mu x\| &= \|\mu x - 2\mathrm{Re}\,(e^*(\mu x))e\| \\ &= \|x - 2\bar{\mu}\mathrm{Re}\,(\mu e^*(x))e\| \\ &= \|x - 2\sin(\lambda/2)(\sin(\lambda/2) - i\cos(\lambda/2))e^*(x)e\|.\end{aligned}$$

Therefore the mapping $\pi : x \to e^*(x)e$ is a Hermitian projection on X with range equal to $\mathbf{C}e$. □

The next corollary is the complex variant of Theorem 2.2.

Corollary 4.3. *The following assertions are equivalent*:

(i) *The set of all elements* e *in* $S(X)$ *such that* $\mathbf{C}e$ *is an* L^2*-summand of* X *is not rare in* $S(X)$.

(ii) *The set of all elements* e *in* $S(X)$ *such that* $\mathbf{C}e$ *is the range of a Hermitian projection on* X *is not rare in* $S(X)$.

(iii) X *is a Hilbert space.*

Proof. (i) implies (ii) because every L^2-projection on a complex Banach space is Hermitian, (ii) implies (iii) in view of Proposition 4.2 and Theorem 2.2 and (iii) clearly implies (i). □

The above result improves the one of Berkson [**3**], see also [**16**, Corollary 4.4] showing that, if for every e in $S(X)$, $\mathbf{C}e$ is the range of a Hermitian projection on X, then X is a Hilbert space.

Now note that, for our complex Banach space X, $\mathcal{G}_0(X_{\mathbf{R}})$ is always nontrivial. More precisely, $\mathcal{G}_0(X)$ contains all multiplications by unimodular complex numbers and, from the inclusion $\mathcal{G}(X) \subseteq \mathcal{G}(X_{\mathbf{R}})$ we obtain that $\mathcal{G}_0(X)$ is contained in $\mathcal{G}_0(X_{\mathbf{R}})$. Therefore the next corollary follows immediately from Proposition 4.2 and Theorem 3.4, with $E = X_{\mathbf{R}}$.

Corollary 4.4. *Assume that there exists a Hermitian projection on* X *with one-dimensional range and that there is a* $\delta > 0$ *such that* $\overline{\mathrm{co}}\,\mathcal{G}(X)(x) \supseteq \delta B(X)$ *for all* x *in* $S(X)$. *Then* X *is isomorphic to a Hilbert space. More precisely, there exist a natural number* $n \leq \delta^{-2}$

and pairwise isomorphic (complex) Hilbert subspaces $H_1, \dots, H_n$ *of* X *satisfying*:

(i) $X = \oplus_{i=1}^n H_i$.

(ii) *For* $i = 1, \dots, n$ *and* F *in* $\mathcal{G}(X)$, *there is* $j = 1, \dots, n$ *with* $F(H_i) = H_j$.

(iii) *For* $i = 1, \dots, n$, H_i *is invariant under* $\mathcal{G}_0(x)$.

The above corollary is the natural complex variant of Theorem 3.4. However, in deriving it from that theorem, the reader can have observed that the corollary remains true if the assumption that there is a $\delta > 0$ with $\overline{\text{co}}\, \mathcal{G}(X)(x) \supseteq \delta B(X)$ for all x in $S(X)$ is relaxed to the one that there is $\delta > 0$ with $\overline{\text{co}}\, \mathcal{G}(X_{\mathbf{R}})(x) \supseteq \delta B(X)$ for all x in $S(X)$. Moreover, property (ii) above can be replaced by the stronger one that, for $i = 1, \dots, n$ and F in $\mathcal{G}(X_{\mathbf{R}})$, there is $j = 1, \dots, n$ with $F(H_i) = H_j$.

The next corollary is derived from the preceding one in a similar way as Corollaries 3.5 and 3.6 were obtained from Theorem 3.4.

Corollary 4.5. *Assume that there exists a Hermitian projection on* X *with one-dimensional range and that either there is a* $\delta > 2^{-1/2}$ *such that* $\overline{\text{co}}\, \mathcal{G}(X)(x) \supseteq \delta B(X)$ *for all* x *in* $S(X)$ *or there is a* $\delta > 0$ *such that* $\overline{\text{co}}\, \mathcal{G}(X)(x) \supseteq \delta B(X)$ *for all* x *in* $S(X)$ *and the norm of* X *is maximal. Then* X *is a Hilbert space.*

Corollary 4.5 contains the Kalton-Wood result asserting that, if the norm of X is convex transitive and if there is a Hermitian projection on X with one-dimensional range, then X is a Hilbert space [**16**, Theorem 6.4]. As we noticed in the appropriate place, we adapted with some refinements the Kalton-Wood argument to cover a part of the proof of Theorem 3.4. This has allowed us to obtain the improvement of the Kalton-Wood theorem provided by Corollary 4.4. In fact, similar refinements can be made in the original Kalton-Wood proof to directly obtain Corollary 4.4, and hence Corollary 4.5. The actual interest of our approach relies on the fact that, via Proposition 4.2, the Kalton-Wood theorem is nothing but Corollary 3.7 when applied to the real Banach spaces underlying the complex ones.

Remark 4.6. For $\delta < 2^{-1/2}$, let $p = p(\delta)$ be as in the proof of Proposition 3.10 and, for every nonzero complex Hilbert space H, let X denote the complex Banach space $H \times H$ under the norm $\|(\eta, \xi)\| := (\|\eta\|^p + \|\xi\|^p)^{1/p}$. Then X is non-Hilbert, there exist Hermitian projections on X with one-dimensional range, and, arguing as in the proof of Proposition 3.10, we see that the inclusion $\overline{\mathrm{co}}\, \mathcal{G}(X)(x) \supseteq \delta B(X)$ holds for all x in $S(X)$.

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