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# A WEAK EFFECTIVE ROTH'S THEOREM OVER FUNCTION FIELDS

MIN RU

**1. Introduction.** The correspondence between Diophantine approximation and Nevanlinna theory, observed by Osgood and Vojta [**2**], [**7**], has motivated many recent works. Furthermore, the Diophantine approximation over function fields also has recently attracted attention because of its correspondence to Nevanlinna theory with moving targets. In [**8**], Julie Wang obtained an effective version of Roth's theorem over function fields by adapting the method of Steinmetz in proving Nevanlinna's conjecture with slowly moving targets in Nevanlinna theory. We note that the Thue-Siegel-Roth theorem over function fields was proved by Uchiyama [**6**] in 1961, with a proof similar to the one for number fields, hence is ineffective. To state Wang's result we recall some definitions. Let $C$ be an irreducible nonsingular algebraic curve of genus $g$ over an algebraically closed field $k$ of characteristic zero. Let $K$ be the function field of $C$. For a nonzero element $f \in K$, we define the height as $h(f) = \sum_{P \in C} -\min\{0, v_P(f)\}$, where $v_P(f)$ is the order of $f$ at the point $P$ of $C$. Let $t$ be a nonconstant function in $K$; we denote by, for every $y \in K$, $y' = (dy/dt)$. Julie Wang's result is stated as follows:

**Theorem [8].** *Let $S$ be a finite set of points in $C$. Suppose that $t, a_1, \dots, a_q$ are $S$-units and that $f$ is a nonzero element of $K$. Let $L(r)$ be the vector space over $k$ spanned by $a_1^{n_1} \dots a_q^{n_1}$ with $n_1, \dots, n_q \geq 0$ and $n_1 + \dots + n_q = r$. Let $\beta_1, \dots, \beta_n$ be a base of $L(r)$ and $b_1, \dots, b_m$ a base of $L(r+1)$. If $f\beta_1, \dots, f\beta_n, \beta_1, \dots, b_m$ are linearly independent*

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*over $k$, then*

$$\sum_{P\in S}\sum_{j=1}^{q}\max\{0, v_P(f-a_j)\}$$
$$\leq \frac{m+n}{n}h(f)+\frac{(m+n-1)(m+n)}{n}(2g-2+2\#S+h(t))+(q-1)^2\sum_{i=1}^{q}h(a_i).$$

Note that, in the above inequality, although the coefficient $((m+n)/n) \to 2$ as $r \to \infty$, the coefficient $((m+n-1)(m+n)/n)$ actually tends to $\infty$ as $r \to \infty$. Furthermore, the term $(2g-2+2\#S+h(t))$; thus the coefficient $((m+n-1)(m+n)/n)$ plays an important role for the general function field $K$. The purpose of this paper is, among others, to improve on this term. We shall establish a new version of effective Roth's theorem over function fields that the coefficient before the term $(2g-2+2\#S+h(t))$ is 1, as conjectured. However, unfortunately, the price paid is where the coefficient before the term $h(f)$ is increased to $[q/2]+1$ rather than 2. Therefore, our theorem can be viewed as a supplement to Julie Wang's theorem. Also our theorem may be called an effective Thue's theorem over function fields, since it really corresponds to Thue's theorem in a number field. The following is the statement.

**Theorem.** *Let $C$ be an irreducible nonsingular algebraic curve of genus $g$ over an algebraically closed field $k$ of characteristic zero. Let $K$ be the function field of $C$. Let $S$ be a finite set of points in $C$. Let $a_1,\ldots,a_q$ be distinct elements in $K$ and $f$ a nonzero element in $K$. Suppose that $t$ is an $S$-integer. Then either*

$$\sum_{P\in S}\sum_{j=1}^{q}\max\{0, v_P(f-a_j)\}$$
$$\leq ([q/2]+1)h(f)+\min\{2g-2+2\#S+h(t), 3g\}$$
$$+(q+1)^2\sum_{i=1}^{q}(h(a_i)+h(a_i'))+(q-1)^2\sum_{i=1}^{q}h(a_i)$$

*or*

$$h(f)\leq (q+1)^3\sum_{j=1}^{q}(h(a_j)+h(a_j'))+\min\{2g-2+\#S+h(t), 3g\}.$$

**2. Proof of the theorem.** To prove the theorem, we shall construct a nonzero differential polynomial of the form $Q(y,y') = \phi_1(y)y' - \phi_2(y)$ over $K$ such that $Q(a_j, a_j') = 0$, $1 \le j \le q$, where $\phi_1$ and $\phi_2$ are polynomials in $y$ of degree at most $[q/2-1]$ and $[q/2+1]$. Since the number of coefficients which can occur in $\phi_1$ and $\phi_2$ is at least $(q/2-1+1/2)+(q/2+1+1/2) = q+1$, and the conditions of $Q(a_j,a_j') = 0$, $1 \le j \le q$, are only $q$ linear homogeneous conditions, such nontrivial $Q(y,y')$ can be constructed. $Q(y,y')$ has the form $Q(y,y') = (m_0 y^{d-2} + \cdots + m_{d-2})y' - (n_0 y^d + \cdots + n_d)$, where $d \le [q/2]+1$. To prove the theorem, we first consider the case that $Q(f,f') \ne 0$. We have the following lemma.

**Lemma 1.** *Assume that $m_0 \ne 0$ and $Q(f,f') \ne 0$. Then*

$$\begin{aligned}\sum_{P\in S}\sum_{j=1}^{q} &\max\{0, v_P(f-a_j)\} \\ &\le ([q/2]+1)h(f) + \#S + \sum_{P\in C}\max\left\{0, v\left(\frac{dt}{dt_P}\right)\right\} \\ &\quad + (q+1)^2\sum_{i=1}^{q}(h(a_i)+h(a_i')) + (q-1)^2\sum_{i=1}^{q}h(a_i).\end{aligned}$$

*Proof of Lemma* 1. For a vector $\mathbf{x} = (x_0, \ldots, x_m)$, we define

$$v_P(\mathbf{x}) = \min\{v_P(x_0), \ldots, v_P(x_m)\}.$$

For the differential polynomial $Q(y,y')$, we then define $v_P(Q) = v_P(\mathbf{x})$ where the components of $\mathbf{x}$ are the coefficients of $Q$ and define the height $h(Q) = -\sum_{P\in C} v_P(\mathbf{x})$. That is,

$$h(Q) = -\sum_{P\in C}\min\{v_P(m_0), \ldots, v_P(m_{d-2}), v_P(n_0), \ldots, v_P(n_d)\}.$$

Without loss of generality, we may assume that $m_0 = 1$; thus, we always have $v_P(Q) \le 0$. Since the coefficients of $Q(y,y')$ are determined by solving linear equations $(m_0 a_j^{d-2} + \cdots + m_{d-2})a_j' - (n_0 a_j^d + \cdots + n_d) = 0$, $1 \le j \le q$,

$$h(Q) \le dq\left(\sum_{j=1}^{q}(h(a_j)+h(a_j'))\right). \tag{2.1}$$

Let $S$ be a nonempty set of finitely many points in $C$. Let $T = \{P \in S \mid v_P(f - a_j) > 0 \text{ for some } j\}$ and $\alpha_P = \max_{i \neq j} \max\{0, v_P(a_i - a_j)\}$. Since $a_i - a_j = (f - a_i) - (f - a_j)$, we have

$$v_P(a_i - a_j) \geq \min\{v_P(f - a_i), v_P(f - a_j)\}.$$

Thus, for every $P \in T$,

$$\sum_{j=1}^{q} v_P(f - a_j) \leq v_P(f - a_{\mu(P)}) + \sum_{j \neq \mu(P)} \max\{0, v_P(a_j - a_{\mu(P)})\},$$

where $\mu(P) \in \{1, 2, \ldots, q\}$ such that $v_P(f - a_{\mu(P)}) = \max_{1 \leq j \leq q} v_P(f - a_j)$. Hence, for every $P \in T$,

$$\sum_{j=1}^{q} v_P(f - a_j) \leq \max_{1 \leq j \leq q} v_P(f - a_j) + (q - 1)\alpha_P. \tag{2.2}$$

For $P \in T$, and for the index $j$, $1 \leq j \leq q$, with $v_P(f - a_j) > 0$, since $Q(a_j, a_j') = 0$, by Taylor's expansion formula, we have the following finite sum

$$\begin{aligned} Q(y, y') = (y - a_j)Q_1(a_j, a_j') + (y' - a_j')Q_2(a_j, q_j') + \frac{1}{2}(y - a_j)^2 Q_{11}(a_j, a_j') \\ + (y - a_j)(y' - a_j')Q_{12}(a_j, a_j') + \frac{1}{3}(y - a_j)^3 Q_{111}(a_j, a_j') + \cdots, \end{aligned}$$

where $Q_1$, respectively $Q_2$, represents the partial derivative with respect to the first variable, respectively the second variable. Since $f' - a_j' = (d(f - a_j)/dt_P)(dt/dt_P)^{-1}$, where $t_P$ is a local parameter of a point $P \in C$ and $v_P(d(f - a_j)/dt_P) \geq v_P(f - a_j) - 1$, $v_P(Q(f, f')) \geq v_P(f - a_j) - 1 - \max\{0, v_P(dt/dt_P)\} + v_P(Q) + d \min\{0, v_P(a_j), v_P(a_j')\}$.

Thus, by (2.2),

$$
\begin{aligned}
\sum_{j=1}^{q} \sum_{P\in S} &\max\{0, v_P(f-a_j)\}\\
&= \sum_{P\in T} \sum_{j=1}^{q} v_P(f-a_j)\\
&\le \sum_{P\in T} v_P(Q(f,f')) + \#T + \sum_{P\in T} \max\left\{0, v_P\left(\frac{dt}{dt_P}\right)\right\} \qquad (2.3)\\
&\quad - \sum_{P\in T} v_P(Q) - d\sum_{P\in T} \sum_{1\le j\le q} \min\{0, v_P(a_j), v_P(a_j')\}\\
&\quad + (q-1)\sum_{P\in S} \alpha_P.
\end{aligned}
$$

Since $-d\sum_{P\in T}\sum_{1\le j\le q}\min\{0, v_P(a_j), v_P(a_j')\} \le d\sum_{1\le j\le q}(h(a_j)+h(a_j'))$, (2.3) becomes

$$
\begin{aligned}
\sum_{j=1}^{q} \sum_{P\in S} &\max\{0, v_P(f-a_j)\}\\
&\le \sum_{P\in T} v_P(Q(f,f')) + \#T\\
&\quad + \sum_{P\in T} \max\left\{0, v_P\left(\frac{dt}{dt_P}\right)\right\} - \sum_{P\in T} v_P(Q) \qquad (2.4)\\
&\quad + d\sum_{1\le j\le q} (h(a_j)+h(a_j')) + (q-1)\sum_{P\in S}\alpha_P.
\end{aligned}
$$

We now consider the case where $P \notin T$. Let $T_\infty = \{P \notin T \mid v_P(f) < 0\}$. For $P \in T_\infty$, $v(df/dt_P) \ge v_P(f) - 1 \ge 2v_P(f)$. Since

$$f' = \frac{df}{dt_P}\left(\frac{dt}{dt_P}\right)^{-1}$$

and $Q(f,f') = (m_0 f^{d-2} + \cdots + m_{d-2})f' - (n_0 f^d + \cdots + n_d)$,

$$v_P(Q(f,f')) \ge d v_P(f) + v_P(Q) - \max\left\{0, v_P\left(\frac{dt}{dt_P}\right)\right\}, \qquad (2.5)$$

for $P \in T_\infty$. On the other hand, if $P \notin T_\infty$, then $v_P(df/dt_P) \geq 0$. Thus,

$$v_P(Q(f,f')) \geq -\max\left\{0, v_P\left(\frac{dt}{dt_P}\right)\right\} + v_P(Q), \tag{2.6}$$

for $P \notin T_\infty$. Combining (2.4), (2.5) and (2.6),

$$\begin{aligned}\sum_{j=1}^{q}\sum_{P\in S}\max\{0, v_P(f-a_j)\} + \sum_{P\in T_\infty} dv_P(f) \\ \leq \sum_{P\in C} v_P(Q(f,f')) \\ + \sum_{P\in C}\max\left\{0, v_P\left(\frac{dt}{dt_P}\right)\right\} - \sum_{P\in C} v_P(Q) + \#T \\ + d\sum_{1\leq j\leq q}(h(a_j)+h(a_j')) + (q-1)\sum_{P\in S}\alpha_P.\end{aligned}$$

Since

$$-\sum_{P\in T_\infty} dv_P(f) \leq dh(f),$$

$\sum_{P\in C} v_P(Q(f,f')) = 0$ by the sum formula and $h(Q) = -\sum_{P\in C} v_P(Q)$, we have

$$\begin{aligned}\sum_{j=1}^{q}\sum_{P\in S}\max\{0, v_P(f-a_j)\} \\ \leq dh(f) + \#S + \sum_{P\in C}\max\left\{0, v\left(\frac{dt}{dt_P}\right)\right\} + h(Q) \\ + d\sum_{1\leq j\leq q}(h(a_j)+h(a_j')) + (q-1)\sum_{P\in S}\alpha_P.\end{aligned}$$

Using $d \leq [q/2]+1$, as well as (2.1) and

$$\begin{aligned}\sum_{P\in S}\alpha_P &= \sum_{P\in S}\max_{i\neq j}\{0, v_P(a_i-a_j)\} \\ &\leq \sum_{i\neq j} h(a_i-a_j) \leq (q-1)\sum_{i=1}^{q} h(a_i),\end{aligned}$$

we have

$$\begin{aligned}\sum_{j=1}^{q}\sum_{P\in S}&\max\{0, v_P(f-a_j)\}\\ &\le ([q/2]+1)h(f) + \#S + \sum_{P\in C}\max\left\{0, v\left(\frac{dt}{dt_P}\right)\right\}\\ &\quad + (q+1)^2\sum_{j=1}^{q}(h(a_j)+h(a_j')) + (q-1)^2\sum_{i=1}^{q}h(a_i).\end{aligned}$$

This finishes the proof of Lemma 1. □

**Lemma 2.** *If $t$ is an $S$-integer, then*

$$\sum_{P\in C}\max\left\{0, v_P\left(\frac{dt}{dt_P}\right)\right\} \le 2g-2+\#S+h(t).$$

*Proof of Lemma* 2. By Riemann-Roch's theorem, noticing that $t$ is an $S$-integer,

$$\begin{aligned}2g-2 &= \sum_{P\in C} v_P\left(\frac{dt}{dt_P}\right)\\ &= \sum_{v_P(t)<0} v_P\left(\frac{dt}{dt_P}\right) + \sum_{P\in C}\max\left\{0, v_P\left(\frac{dt}{dt_P}\right)\right\}\\ &\ge \sum_{v_P(t)<0}(v_P(t)-1) + \sum_{P\in C}\max\left\{0, v_P\left(\frac{dt}{dt_P}\right)\right\}\\ &\ge -h(t) - \#S + \sum_{P\in C}\max\left\{0, v_P\left(\frac{dt}{dt_P}\right)\right\}.\end{aligned}$$

Hence

$$\sum_{P\in C}\max\left\{0, v_P\left(\frac{dt}{dt_P}\right)\right\} \le 2g-2+\#S+h(t). \qquad \square$$

**Lemma 3.** *An element $t \in K$ exists such that $t \notin k$ and*

$$\sum_{P \in C} \max\left\{0, v_P\left(\frac{dt}{dt_P}\right)\right\} \le 3g.$$

*Proof of Lemma* 3. By Riemann's theorem, fix a point $P_0 \in C$, the elements $t \in K$ having $v_{P_0}(t) \ge -g-1$, $v_P(t) \ge 0$ if $P \ne P_0$ form a vector space of dimension greater than or equal to 2. Choosing such an element $t \notin k$, let $t_P$ be a local parameter of $P \in C$ by Riemann-Roch,

$$\begin{aligned} 2g-2 = \sum_{P \in C} v_P\left(\frac{dt}{dt_P}\right) &= v_{P_0}\left(\frac{dt}{dt_P}\right) + \sum_{P \in C} \max\left\{0, v_P\left(\frac{dt}{dt_P}\right)\right\} \\ &\ge -g-2 + \sum_{P \in C} \max\left\{0, v_P\left(\frac{dt}{dt_P}\right)\right\}. \end{aligned}$$

So

$$\sum_{P \in C} \max\left\{0, v_P\left(\frac{dt}{dt_P}\right)\right\} \le 3g.$$

Combining Lemmas 1, 2 and 3, we have obtained the following statement: If $m_0 \ne 0$ and $Q(f, f') \ne 0$, then

$$\begin{aligned} &\sum_{j=1}^{q} \sum_{P \in S} \max\{0, v_P(f - a_j)\} \\ &\quad \le ([q/2]+1)h(f) + \min\{2g-2+2\#S + h(t), 3g\} \\ &\qquad + (q+1)^2 \sum_{j=1}^{q} (h(a_j) + h(a_j')) + (q-1)^2 \sum_{i=1}^{q} h(a_i). \end{aligned} \tag{2.7}$$

Our next step is to deal with the case $Q(f, f') = 0$. In this case we have

$$(m_0 f^{d-2} + \cdots + m_{d-2}) f' - (n_0 f^d + \cdots + n_d) = 0.$$

We write $M(X) = m_0 X^{d-2} + \cdots + m_{d-2}$, $N(X) = n_0 X^d + \cdots + n_d$. Fix a point $P \in C$ at the moment. We first consider the case where

$$v_P(M(f)) \ge v_P(Q). \tag{2.8}$$

Since $f$ satisfies the equation

$$m_0 f^{d-2} + \cdots + m_{d-3} f + (m_{d-2} - M(f)) = 0,$$

whose coefficients have valuation $\geq v_P(Q)$, it follows from the Gauss lemma that

$$v_P(f) \geq v_P(Q) - v_P(m_0). \tag{2.9}$$

Since $f' = (df/dt_P)(dt/dt_P)^{-1}$, observing that $v_P(Q) - v_P(m_0) \leq 0$,

$$v_P(f') \geq 2(v_P(Q) - v_P(m_0)) - \max\left\{0, v_P\left(\frac{dt}{dt_P}\right)\right\}.$$

The differential equation $Q(f, f') = 0$ implies that

$$v_P(N(f)) \geq v_P(M(f)) + 2(v_P(Q) - v_P(m_0))) - \max\left\{0, v_P\left(\frac{dt}{dt_P}\right)\right\}. \tag{2.10}$$

The resultant $R$ of $M(X), N(X)$ may be written as

$$R = M(X)V(X) + N(X)W(X), \tag{2.11}$$

where $V(X), W(X)$ are certain polynomials defined in terms of determinants. In particular, $V, W$ are of respective degrees $\leq d-1$, $d-3$, and

$$v_P(V), v_P(W) \geq (2d-3)v_P(Q).$$

Now since $v_P(m_0 f) \geq v_P(Q)$ by (2.9), it follows, using the Gauss lemma again, that

$$v_P(m_0^{d-1} V(f)) \geq (2d - 3 + d - 1)v_P(Q),$$

and also,

$$v_P(m_0^{d-3} W(f)) \geq (2d - 3 + d - 3)v_P(Q).$$

Thus, (2.10) and (2.11) yield

$$v_P(m_0^{d-1} R) \geq v_P(M(f)) + (3d-4)v_P(Q) - \max\left\{0, v_P\left(\frac{dt}{dt_P}\right)\right\},$$

whence

$$v_P(M(f)) \leq -(2d-3)v_P(Q) + (d-1)(v_P(m_0) - v_P(Q)) + v_P(R) + \max\left\{0, v_P\left(\frac{dt}{dt_P}\right)\right\}. \tag{2.12}$$

Now if (2.8) does not hold, then since $v_P(m_0) \geq v_P(Q)$ and since $v_P(R) \geq (2d-2)v_P(Q)$, (2.12) is still true.

Now we want to estimate the height $h(f)$ by using (2.12). If $v_P(m_0 f^{d-2}) < \min\{v_P(m_1 f^{d-3}), \dots, v_P(m_{d-2})\}$. Then $v_P(m_0 f^{d-2}) = v_P(M(f))$ and $(d-2)v_P(f) = v_P(M(f)) - v_P(m_0)$. If $v_P(m_0 f^{d-2}) \geq \min\{v_P(m_1 f^{d-3}), \dots, v_P(m_{d-2})\}$, then $v_P(m_0 f^{d-2}) \geq v_P(m_i f^{d-2-i})$ for some $i$, $1 \leq i \leq d-2$. Then $v_P(f^i) \geq v_P(m_i) - v_P(m_0) \geq v_P(Q) - v_P(m_0)$, whence $v_P(f) \geq v_P(Q) - v_P(m_0)$ and $(d-2)v_P(f) \geq (d-2)(v_P(Q) - v_P(m_0))$. So in any case, we have

$$\begin{aligned}(d-2)\min\{0, v_P(f)\} \geq \min\{&(d-2)(v_P(Q) - v_P(m_0)),\\ &v_P(M(f)) - v_P(m_0)\}.\end{aligned}$$

Using the sum formula for $m_0^{d-2}M(f)^{-1}$, we obtain

$$\begin{aligned}(d-2)h(f) &= -(d-2)\sum_{P\in C}\min\{0, v_P(f)\}\\ &\leq -\sum_{P\in C}\max\{(d-2)v_P(Q) - v_P(M(f)), (d-3)v_P(m_0)\}.\end{aligned}$$

Applying (2.12) and once again the sum formula, we get

$$\begin{aligned}(d-2)h(f) &\leq \sum_{P\in C}\max\Big\{-(3d-5)v_P(Q) + (d-1)\\ &\qquad\cdot (v_P(m_0) - v_P(Q)) + v(R)\\ &\qquad + \max\left\{0, v_P\left(\frac{dt}{dt_P}\right)\right\}, (3-d)v_P(m_0)\Big\}\\ &= \sum_{P\in C}\max\Big\{-(4d-6)v_P(Q) + \max\left\{0, v_P\left(\frac{dt}{dt_P}\right)\right\},\\ &\qquad - v_P(R) + (4-2d)v_P(m_0)\Big\}.\end{aligned}$$

Recall that, by Lemmas 2 and 3, $\sum_{P\in C}\max\{0, v_P(dt/dt_P)\} \le \min\{2g-2+\#S+h(t), 3g\}$ and note that

$$\begin{aligned}-v_P(R)+(4-2d)v_P(m_0) &\le -(2d-2)v_P(Q)+(4-2d)v_P(Q)\\ &= (6-4d)v_P(Q).\end{aligned}$$

It then follows that

$$\begin{aligned}(d-2)h(f) &\le \sum_{P\in C}\left((6-4d)v_P(Q)+\max\left\{0, v_P\left(\frac{dt}{dt_P}\right)\right\}\right)\\ &\le (4d-6)h(Q)+\min\{2g-2+\#S+h(t), 3g\},\end{aligned}$$

and, if $d-2>0$, then also, using (2.1),

$$\begin{aligned}h(f) &\le (d-2)^{-1}((4d-6)h(Q)+\min\{2g-2+\#S+h(t), 3g\})\\ &\le (q+1)^3\sum_{j=1}^{q}(h(a_j)+h(a_j')) + \min\{2g-2+\#S+h(t), 3g\}.\end{aligned}$$

If $d-2\le 0$, then either $M(X)=m_0\ne 0$ or $M(X)=0$. If $M(X)=m_0\ne 0$, then the differential equation $Q(y,y')=0$ is either linear or Ricatti. In the case that $Q(y,y')=0$ is linear, then any three solutions $z, z_1, z_2$ have

$$\frac{z-z_1}{z_2-z_1}=c$$

with $c\in k$, whence $v_P(z-z_1)=v_P(z_2-z_1)$. In particular, since $Q(f,f')=0$, $Q(a_j,a_j')=0$, we have, for $2\le j\le q$,

$$v_P(f-a_j)=v_P(a_1-a_j).$$

In this case (2.7) trivially holds. If the differential equation is Ricatti, then as is well known, any four solutions of the Ricatti differential equation have a constant cross ratio. Again, since $f, a_j$, $1\le j\le q$ are all the solutions,

$$\frac{f-a_j}{f-a_1}\Big/\frac{a_2-a_j}{a_2-a_1}=c,$$

with $c\in k$ so, for $3\le j\le q$,

$$v_P(f-a_j)+v_P(a_2-a_1)=v_P(f-a_1)+v_P(a_2-a_j).$$

Thus, in this case (2.7) also holds trivially. If $M(X) = 0$, then $Q(f, f') = 0$ implies that $n_d f^d + \cdots + n_0 = 0$, so by the Gauss lemma, $h(f) \leq h(Q)$. This finishes the proof of our theorem.

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Department of Mathematics, University of Houston, Houston, TX 77204
*E-mail address:* minru@math.uh.edu