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ZARISKI-FINITE MODULES

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ABSTRACT. Let R be a commutative ring with identity, and let M be an R-module. We examine the situation where the Zariski space of M is finitely generated. In case R is a Noetherian UFD and M is finitely generated and nontorsion, we prove that this condition is equivalent to M being isomorphic to a direct sum of R and a finite R-module. We further describe in this case the structure of all but finitely many prime submodules of M.

Zariski spaces of modules were introduced in [**8**], and they provide a demonstration that the study of prime submodules is significantly more rich and complex than is the study of prime ideals. A Zariski space is determined by the following. Let R be a commutative ring with identity, and let M be a unital R-module. Note that the Zariski topology on $\operatorname{spec} R$, denoted $\zeta(\mathcal{R})$, is a semiring, where "addition" is given by intersection and "multiplication" is given by union. In a like manner, we let $\zeta(M)$ be the collection of all varieties of submodules of M, and we observe that although $\zeta(M)$ is not in general a topology itself, it is a semimodule over $\zeta(R)$, where "addition" and "scalar multiplication" are given by

$$V(L) + V(N) = V(L) \cap V(N) = V(L + N)$$

and

$$V(\mathfrak{a})V(N) = V(\mathfrak{a}N)$$

for all submodules L and N of M and for all ideals $\mathfrak{a}$ of R.

This note concerns the generating of the semimodule $\zeta(M)$ as "linear combinations" of its elements, particularly finitely many of them. For a basic study of semimodules, see, for example, [**3**].

Now in the case that M is a finitely generated, noncyclic, faithful multiplication module over R, note that although the structure of M is

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in some ways a bit more complicated than is the structure of the ring itself, the collection of prime submodules of M is essentially identical to the collection of prime ideals of R. This indeed is borne out in the fact that $\zeta(M)$ is (semimodule) isomorphic to $\zeta(R)$. However, one should dismiss forever any notion one might have that so long as R is Noetherian, say, and M is finitely generated, then $\zeta(M)$ must itself be finitely generated. Indeed, this notion fails rather spectacularly, even in the very simple case $R = \mathbf{Z}$ and $M = R \oplus R$. It turns out that if S is any generating set of $\zeta(M)$, then $|S| \geq |R|$ [**10**, Theorem 1.3].

The question remains as to which modules M have the property that $\zeta(M)$ is finitely generated as a $\zeta(R)$-semimodule. Such a module is said to be *Zariski-finite*. Our main result in this paper is that if R is an infinite Noetherian UFD and M is a finitely generated R-module, then M is Zariski-finite if and only if $M \cong R \oplus T$, where T is a finite R-module. One particular advantage of this characterization is that it tells us in this case exactly what all but finitely many of the prime submodules look like. And for the remaining primes, we still know quite a lot. The same cannot be said, for example, merely in knowing that there is a one-to-one correspondence between the $\mathfrak{p}$-prime submodules of M and the proper subspaces of the $k(\mathfrak{p})$-vector space $k(\mathfrak{p}) \otimes_R M$ where $k(\mathfrak{p})$ denotes the residue field of $\mathfrak{p}$.

Before moving on to our study, we remark that although rarely is a Zariski space of a module finitely generated in an algebraic sense, things are not quite so pathological as they might seem. In a subsequent paper, "Subtractive bases of Zariski spaces," we demonstrate a topological means of generating these spaces, in such a way that if M is a free R-module generated by n elements (n is a positive integer), then $\zeta(M)$ can likewise be generated by n elements.

1. Special cases. Throughout this paper R will denote a commutative ring with identity and M a unital R-module. A submodule P of M is called *prime* if $P \neq M$ and whenever $rm \in P$, $r \in R$ and $m \in M$, then $m \in P$ or $rM \subseteq P$. For the basic properties of prime submodules, see [**1**], [**4**], [**6**] or [**12**], for example. The (possibly empty) collection of prime submodules of M will be denoted by $\operatorname{spec} M$. For any prime ideal $\mathfrak{p}$ of R, we define $\operatorname{spec}_{\mathfrak{p}} M = \{P \in \operatorname{spec} M : (P : M) = \mathfrak{p}\}$.

Let N be any submodule of M. The *variety* of N is given by

$V(N) = \{P \in \operatorname{spec} M : N \subseteq P\}$, and the collection of all such varieties is denoted by $\zeta(M)$. For a study of the $\zeta(R)$-semimodule structure on $\zeta(M)$, see [**8**]. Now let $N_1, \dots, N_k$ be submodules of M. Then $\langle V(N_1), \dots, V(N_k)\rangle$ will denote the $\zeta(R)$-subsemimodule of $\zeta(M)$ generated by $V(N_1), \dots, V(N_k)$, i.e., $\langle V(N_1), \dots, V(N_k)\rangle$ consists of all varieties of the form $V(\mathfrak{a}_1 N_1 + \cdots + \mathfrak{a}_k N_k)$, where $\mathfrak{a}_1, \dots, \mathfrak{a}_k$ are ideals of R. The module M is called *Zariski-finite* if $\zeta(M) = \langle V(N_1), \dots, V(N_k)\rangle$ for some positive integer k and submodules $N_1, \dots, N_k$ of M, and in case $k = 1$, M is called *Zariski-cyclic*. On the other hand, if a positive integer k and submodules $N_1, \dots, N_k$ of M exist such that for every prime submodule P of M, we have $V(P) \in \langle V(N_1), \dots, V(N_k)\rangle$, then we say that M is *weakly Zariski-finite*. Finally we call the module M *Zariski-bounded* if there exists a positive integer n such that $|\operatorname{spec}_{\mathfrak{p}} M| \le n$ for every prime ideal $\mathfrak{p}$ of R.

Lemma 1.1. *For any R-module M, M is Zariski-finite implies that M is weakly Zariski-finite, which implies that M is Zariski-bounded.*

Proof. The first implication is obvious. The second appears in [**9**, Lemma 2.5]. $\square$

The next result is easily proven.

Lemma 1.2. *If M is Zariski-finite* (*respectively, weakly Zariski-finite, Zariski-bounded*)*, then any homomorphic image of M is likewise Zariski-finite* (*respectively, weakly Zariski-finite, Zariski-bounded*).

Recall that M is a multiplication module if for every submodule N of M, $N = (N : M)M$.

Theorem 1.3. *If M is finitely generated, then the following are equivalent.*

1) *M is a multiplication module.*
2) *M is Zariski-cyclic.*
3) $|\operatorname{spec}_{\mathfrak{p}} M| \le 1$ *for every prime ideal $\mathfrak{p}$ of R.*

4) $|\mathrm{spec}_{\mathfrak{m}} M| \leq 1$ *for every maximal ideal* $\mathfrak{m}$ *of* R.

Proof. 1) $\Rightarrow$ 2). By [**9**, Theorem 2.1].

2) $\Rightarrow$ 3). By [**7**, Theorem 3.5].

3) $\Rightarrow$ 4). Clear.

4) $\Rightarrow$ 1). Since for each maximal ideal $\mathfrak{m}$ of R, there is at most one $\mathfrak{m}$-prime submodule of M, then $M/\mathfrak{m}M$ is a cyclic $R/\mathfrak{m}$-module. By [**2**, Corollary 1.5], M is a multiplication module. $\square$

Theorem 1.3 fails for modules which are not finitely generated, as the following examples show.

Example 1.1. (i) There exists a $\mathbf{Z}$-module M such that $|\mathrm{spec}_{\mathfrak{p}} M| = 1$ for every maximal ideal $\mathfrak{p}$ of $\mathbf{Z}$ but $\mathrm{spec}_0 M$ is infinite.

(ii) There exists a $\mathbf{Z}$-module M such that $|\mathrm{spec}_{\mathfrak{p}} M| = 1$ for every prime ideal $\mathfrak{p}$ of $\mathbf{Z}$ but M is not a multiplication module.

Proof. (i) Let the $\mathbf{Z}$-module N be a direct sum of an infinite number of copies of $\mathbf{Q}$ and let $M = \mathbf{Z} \oplus N$. For any maximal ideal $\mathfrak{p}$ of $\mathbf{Z}$, $\mathfrak{p}M = \mathfrak{p} \oplus N$ and $M/\mathfrak{p}M \cong \mathbf{Z}/\mathfrak{p}$, so that $\mathrm{spec}_{\mathfrak{p}} M = \{\mathfrak{p} \oplus N\}$. However, for any direct summand K of N, M/K is torsion-free so that $K \in \mathrm{spec}_0 M$.

(ii) Let L be any nonzero torsion divisible $\mathbf{Z}$-module, and let $M = \mathbf{Z} \oplus L$. If $\mathfrak{p}$ is any maximal ideal of $\mathbf{Z}$, then $\mathrm{spec}_{\mathfrak{p}} M = \{\mathfrak{p} \oplus L\}$ by the proof of (i). Clearly $\mathrm{spec}_0 M = \{0 \oplus L\}$. If H is the submodule $\mathbf{Z} \oplus 0$ of M, then $(H : M) = 0$, and hence $H \neq (H : M)M$, i.e., M is not a multiplication module. $\square$

In [**9**, Theorem 4.7] we proved that for a Dedekind domain R and a finitely generated R-module M, then Zariski-finite, weakly Zariski-finite and Zariski-bounded are all equivalent. We are now able to show that the same can be said if R is any Noetherian one-dimensional domain and M is any finitely generated R-module. But first we record the following result [**11**, Theorem 1.8].

Lemma 1.4. *Let R be a Noetherian ring, and let M be a finitely generated R-module. If M is Zariski-bounded, then there are at most finitely many prime ideals $\mathfrak{p}$ of R such that $|\operatorname{spec}_{\mathfrak{p}} M| > 1$.*

If R is any ring and M is any R-module, then for each maximal ideal $\mathfrak{p}$ of R such that $|\operatorname{spec}_{\mathfrak{p}} M| = 1$, we have $\operatorname{spec}_{\mathfrak{p}} M = \{\mathfrak{p}M\}$. Now if, in addition, R is a one-dimensional Noetherian domain and M is finitely generated, then it follows that there are only finitely many prime submodules P of M which are not of the form $P = (P : M)M$.

Theorem 1.5. *Let R be a Noetherian one-dimensional domain, and let M be a finitely generated R-module. Then the following statements are equivalent.*

1) *M is Zariski-finite.*

2) *M is weakly Zariski-finite.*

3) *M is Zariski-bounded.*

4) *M has only finitely many prime submodules P not of the form $P = (P : M)M$.*

Proof. 1) $\Rightarrow$ 2) $\Rightarrow$ 3). By Lemma 1.1.

3) $\Rightarrow$ 4). By the above remarks.

4) $\Rightarrow$ 1). By [**9**, Lemma 4.5]. $\square$

We shall see that, at least in some cases, those primes P not of the form $P = (P : M)M$ are not the only kinds of prime submodules useful in determining whether a module is Zariski-finite. We now turn our attention to the case where M is torsion-free.

2. Torsion-free modules. Let R be an integral domain, and let M be a torsion-free R-module. A prime submodule P of M is said to be *directional* if $V(P) \neq V((P : M)M)$. We remark that if P is a directional prime, then some nonzero submodule A of M must exist such that $V(P) = V((P : M)M + A)$, for example, $A = P$.

Lemma 2.1. *Let R be an infinite domain, and let M be a finitely* *generated, torsion-free Zariski-bounded R-module. Let $\mathfrak{p}$ be a height 1 prime ideal of R, and let $P \in \operatorname{spec}_{\mathfrak{p}} M$. If P is a directional prime and A is any submodule of M such that $V(P) = V(\mathfrak{p}M + A)$, then P is a minimal prime to A.*

Proof. Note that $A \neq 0$ and that $\operatorname{spec}_0 M = \{0\}$ [**11**, Lemma 1.1]. Now $P \supseteq A$; hence P contains some minimal prime to A, say Q. Since $\mathfrak{p} \supseteq (Q : M) \neq 0$ and since $\mathfrak{p}$ is height one, we must have $\mathfrak{p} = (Q : M)$. It follows that $Q \in V(\mathfrak{p}M + A) = V(P)$, and thus $P = Q$. $\square$

Let R be any ring and M any R-module. For a prime submodule P of M, we define the *R-height* of P to be the height of $(P : M)$. Now consider the case where $N_1, \dots, N_k$ are submodules of M and P is a $\mathfrak{p}$-prime submodule of M such that $V(P) \in \langle V(N_1), \dots, V(N_k)\rangle$. Then there exist ideals $\mathfrak{a}_1, \dots, \mathfrak{a}_k$ of R such that $V(P) = V(\mathfrak{a}_1 N_1 + \cdots + \mathfrak{a}_k N_k)$. Since, for each i, $1 \leq i \leq k$, we have $P \supseteq \mathfrak{a}_i N_i$, then either $\mathfrak{a}_i \subseteq \mathfrak{p}$ or $N_i \subseteq P$. Let $I = \{i : 1 \leq i \leq k \text{ and } \mathfrak{a}_i \subseteq \mathfrak{p}\}$, and let $J = \{i : 1 \leq i \leq k \text{ and } \mathfrak{a}_i \not\subseteq \mathfrak{p}\}$. Then $\mathfrak{a}_1 N_1 + \cdots + \mathfrak{a}_k N_k \subseteq \mathfrak{p}M + \sum_{j \in J} N_j \subseteq P$, and thus $V(P) \subseteq V(\mathfrak{p}M + \sum_{j\in J} N_j) \subseteq V(\mathfrak{a}_1 N_1 + \cdots + \mathfrak{a}_k N_k) = V(P)$. In particular, note that $V(P) = V(\mathfrak{p}M + \sum_{j \in J} N_j)$.

Lemma 2.2. *Let R be a Noetherian domain, and let M be a finitely generated R-module. If M is torsion-free and weakly Zariski-finite, then M has only finitely many directional primes of R-height 1.*

Proof. There exist nonzero submodules $N_1, \dots, N_k$ of M such that for every $P \in \operatorname{spec} M$, $V(P) \in \langle V(N_1), \dots, V(N_k)\rangle$. From the above remarks, for every $P \in \operatorname{spec} M$ such that P is directional, then $V(P) = V((P : M)M + \sum_{i \in J} N_i)$ for some nonempty subset $J \subseteq \{1, \dots, k\}$. If $\operatorname{ht}(P : M) = 1$, then by Lemma 2.1, P must be a minimal prime to $\sum_{i\in J} N_i$. However, there are only finitely many such sums, and each such sum has only finitely many minimal primes [**12**, Theorem 4.2]. $\square$

Lemma 2.3. *Let R be a Noetherian domain, and let M be a finitely generated torsion-free R module. If M has only finitely many directional primes, then M is Zariski-finite.*

Proof. Let $Q_1, \dots, Q_k$ be the directional primes of M. We will show that $\zeta(M)$ is generated by varieties of the form $V(\cap_{i\in J} Q_i)$, where $J \subseteq \{1, \dots, k\}$. To simplify matters, assume that if $J = \varnothing$, then $\cap_{i\in J} Q_i = M$.

Let N be a submodule of M. There are only finitely many minimal primes to N, say $P_1, \dots, P_r$, which are not directional, and $\{Q_i\}_{i\in J}$, where $J \subseteq \{1, \dots, k\}$. Let $\mathfrak{p}_i = (P_i : M)$, $1 \le i \le r$. We claim that

$$V(N) = \begin{cases} V(\cap_{i\in J} Q_i) & r = 0, \\ V((\prod_{i=1}^r \mathfrak{p}_i)(\cap_{i\in J} Q_i)) & r > 0. \end{cases}$$

The case $r = 0$ is obvious. Now if $P \in V(N)$, then P must contain some minimal prime to N, but note that $(\prod_{i=1}^r \mathfrak{p}_i)(\cap_{i\in J} Q_i)$ is contained in every minimal prime to N. On the other hand, if $P \in V((\prod_{i=1}^r \mathfrak{p}_i)(\cap_{i\in J} Q_i))$, then either $P \supseteq \cap_{i\in J} Q_i \supseteq N$ or $P \supseteq (\prod_{i=1}^r \mathfrak{p}_i)M$. The latter implies that $P \supseteq \mathfrak{p}_i M$ for some i, $1 \le i \le r$, thus $P \in V(\mathfrak{p}_i M) = V(P_i)$. In either case, $P \in V(N)$. The result now follows. □

Theorem 2.4. *Let R be a Noetherian two-dimensional domain, and let M be a finitely generated torsion-free R-module. Then the following statements are equivalent.*

1) *M is Zariski-finite.*

2) *M is weakly Zariski-finite.*

3) *M has only finitely many directional primes.*

Proof. 1) $\Rightarrow$ 2). By Lemma 1.1.

2) $\Rightarrow$ 3). By Lemma 2.2, M has only finitely many directional primes of R-height 1. Now, since M is Zariski-bounded, there are only finitely many directional primes of R-height 2, by [**11**, Theorem 1.8]. Finally, 0 is the only 0-prime submodule of M [**11**, Lemma 1.1].

3) $\Rightarrow$ 1). By Lemma 2.3. □

Conspicuous by its absence in the preceding theorem is any mention of Zariski-bounded. But the next example shows that Zariski-bounded cannot be included in the above list. Before proceeding with the example, however, we need some more notation. For any prime ideal $\mathfrak{p}$ of R, let $M(\mathfrak{p}) = \{m \in M : cm \in \mathfrak{p}M$ for some $c \in R \backslash \mathfrak{p}\}$. In [**12**] it was shown that either $M(\mathfrak{p}) = M$ or $M(\mathfrak{p}) \in \operatorname{spec}_{\mathfrak{p}} M$ and that, in particular, if M is finitely generated, then $M(\mathfrak{p}) \in \operatorname{spec}_{\mathfrak{p}} M$ provided $\mathfrak{p} \supseteq \operatorname{ann} M$.

Example 2.1. There exists a finitely generated, torsion-free module M over a two-dimensional Noetherian domain R such that M is Zariski-bounded, but M is not Zariski-finite.

Proof. Let $R = \mathbf{Z}[x]$, let $F = R \oplus R$, and let $y = (x, x+2) \in F$. Note that $M = F/Ry$ is torsion-free and indeed M is Zariski-bounded by [**11**, Proposition 2.2]. Note also that $\mathfrak{m} = Rx + R2$ is the only prime ideal $\mathfrak{p}$ of R such that $Ry \subseteq \mathfrak{p}F$.

Now, for every positive integer n, let $\mathfrak{p}_n = R(x+2n)$. Then $\mathfrak{p}_n \in \operatorname{spec} R$ and $\mathfrak{p}_n = ((\mathfrak{p}_n F + Ry) : F)$ [**6**, Corollary 3.4]. However, $x(n, n-1) = -(0, x+2n) + n(x, x+2) \in \mathfrak{p}_n F + Ry$, but $x \notin \mathfrak{p}_n$ and $(n, n-1) \notin \mathfrak{p}_n F + Ry$. To see this, let $\varphi : F \to M$ be the natural epimorphism, and note that $\varphi((n, n-1)) \in M(\mathfrak{p}_n)$. In addition, we see that $\varphi((n, n-1)) \notin \mathfrak{m}M$, since $(n, n-1) \notin \mathfrak{m}F = \mathfrak{m}F + Ry$. Therefore, $V(M(\mathfrak{p}_n)) \neq V(\mathfrak{p}_n M)$, and thus M has infinitely many directional primes. Now apply Theorem 2.4. $\square$

The authors do not know at present whether Theorem 2.4 can be generalized to arbitrary dimensions. It would be helpful to know, in the context of Lemma 2.2, if M actually has no directional primes of R-height 1. This would imply, at least if R has Krull dimension, that the only directional primes P of M would have to be such that $(P : M)$ is maximal, and there could only be finitely many of these. However, the authors have not as yet resolved this matter, except in the case that R is a Noetherian UFD (Corollary 3.3.)

We can say a bit more about directional primes, however, if R is any Noetherian domain and M is a finitely generated torsion-free Zariski-bounded module. Given any directional prime $P \in \operatorname{spec}_{\mathfrak{p}} M$, then there exists some maximal ideal $\mathfrak{m}$ of R such that $\mathfrak{p} \subseteq \mathfrak{m}$ but $P \nsubseteq \mathfrak{m}M$. This is obvious, of course, if $\mathfrak{p}$ is maximal. If $\mathfrak{p}$ is not maximal, then there must exist some prime ideal $\mathfrak{q}$ of R and some $\mathfrak{q}$-prime submodule Q of

M such that $\mathfrak{p}M \subseteq Q$ but $P \nsubseteq Q$. Now since $\mathfrak{p} \subseteq \mathfrak{q}$, then P must be contained in some $\mathfrak{q}$-prime submodule of M [**6**, Theorem 3.3]. But if $\mathfrak{q}$ is not maximal, then $\operatorname{spec}_{\mathfrak{q}} M = \{Q\}$, since M is Zariski-bounded [**11**, Corollary 1.2]. Indeed, it now follows that for every prime ideal $\mathfrak{q}'$ of R such that $\mathfrak{p} \subsetneq \mathfrak{q}' \subsetneq \mathfrak{m}$, then $M(\mathfrak{q}')$ is likewise a directional prime.

Before moving on, we remark that every torsion-free Zariski-bounded module over an infinite domain is uniform [**11**], where a nonzero module M is *uniform* if every pair of nontrivial submodules of M intersect nontrivially. Compare the next result with [**9**, Corollary 4.2] which says that M is isomorphic to an ideal of R, provided that R is a Dedekind domain and M is a finitely generated, torsion-free R-module such that $\operatorname{spec}_0 M$ is finite.

Lemma 2.5. *Let R be a domain, and let M be a finitely generated torsion-free R-module. If M is uniform, then M is isomorphic to some ideal of R.*

Proof. We have $\operatorname{spec}_0 M = \{0\}$ by [**11**, Lemma 1.1]. Now let F be a finitely generated free R-module such that $M \cong F/P$ for some submodule P of F. It follows that P is a maximal element of $\operatorname{spec}_0 F$. By [**6**, Lemma 3.5], P is the kernel of some $\varphi \in F^*$. Then we have $M \cong F/P \cong \varphi(F) \triangleleft R$. $\square$

The question as to exactly which ideals of a domain R are Zariski-bounded is an intriguing one, and one to which the authors do not at present know the answer. However, the next result says quite a lot about which ideals in a Noetherian domain can be weakly Zariski-finite.

Lemma 2.6. *Let R be an infinite Noetherian domain, and let $\mathfrak{a}$ be an ideal of R such that $\mathfrak{a}$ is not contained in any height 1 prime ideal of R. If $\mathfrak{a}$ (as an R-module) is weakly Zariski-finite, then $\mathfrak{a} = R$.*

Proof. Suppose that $\mathfrak{a}$ is proper, and let $\mathfrak{m}$ be a maximal ideal of R containing $\mathfrak{a}$. Now $\mathfrak{m}\mathfrak{a} \neq \mathfrak{a}$, so choose $x_1 \in \mathfrak{a} \backslash \mathfrak{m}\mathfrak{a}$. Since $x_1 \in \mathfrak{m}$ there exists $\mathfrak{p}_1 \subseteq \mathfrak{m}$ such that $\mathfrak{p}_1$ is a minimal prime to Rx_1, and thus $\mathfrak{p}_1$ is height 1 by the principal ideal theorem. Since $\mathfrak{a} \nsubseteq \mathfrak{p}_1$, then $\mathfrak{p}_1 \cap \mathfrak{a} \in \operatorname{spec}_{\mathfrak{p}_1} \mathfrak{a}$, and since $x_1 \in (\mathfrak{p}_1 \cap \mathfrak{a}) \backslash \mathfrak{m}\mathfrak{a}$, then $\mathfrak{p}_1 \cap \mathfrak{a}$ is a directional prime of $\mathfrak{a}$. Now $\mathfrak{m}\mathfrak{a} \not\subseteq \mathfrak{p}_1 \cap \mathfrak{a}$, else $\mathfrak{m} \subseteq \mathfrak{p}_1$. So choose $x_2 \in \mathfrak{m}\mathfrak{a}\backslash(\mathfrak{p}_1 \cap \mathfrak{a})$. Then $x_1 + x_2 \in \mathfrak{a}\backslash(\mathfrak{m}\mathfrak{a} \cup (\mathfrak{p}_1 \cap \mathfrak{a}))$. As before, there exists $\mathfrak{p}_2 \subseteq \mathfrak{m}$ which is a height 1 prime minimal to $R(x_1 + x_2)$. Note that $\mathfrak{p}_1 \neq \mathfrak{p}_2$, and that $\mathfrak{p}_2 \cap \mathfrak{a}$ is likewise directional.

Let $k \geq 2$ be given. Our induction hypothesis is that, for each i, $2 \leq i \leq k$, there exist $x_i \in \mathfrak{m}\mathfrak{a}\backslash(\mathfrak{p}_1 \cap \mathfrak{a})$ and $\mathfrak{p}_i \subseteq \mathfrak{m}$ with $\mathfrak{p}_i$ a minimal prime to $x_1 + x_i$ such that for each j, $1 < j < i$,

$$\begin{aligned} &\text{if } x_1 \notin \mathfrak{p}_j, \text{ then } x_i \in \mathfrak{p}_j, \text{ and} \\ &\text{if } x_1 \in \mathfrak{p}_j, \text{ then } x_i \notin \mathfrak{p}_j. \end{aligned}$$

Note that for each pair j, i such that $1 \leq j < i \leq k$, we have $\mathfrak{p}_j \neq \mathfrak{p}_i$ since $x_1 + x_i \in \mathfrak{p}_i\backslash\mathfrak{p}_j$. Now let $S = \{i \leq k : x_1 \in \mathfrak{p}_i\}$, and let $T = \{i \leq k : x_1 \notin \mathfrak{p}_i\}$. It is apparent that $\mathfrak{m}\mathfrak{a} \cap (\cap_{i \in T}(\mathfrak{p}_i \cap \mathfrak{a})) \not\subseteq \cup_{i \in S}\mathfrak{p}_i$. Hence we may choose $x_{k+1} \in \mathfrak{m}\mathfrak{a} \cap (\cap_{i \in T}(\mathfrak{p}_i \cap \mathfrak{a}))\backslash \cup_{i \in S} \mathfrak{p}_i$, and thus there exists $\mathfrak{p}_{k+1} \subseteq \mathfrak{m}$ which is a height 1 prime minimal to $x_1 + x_{k+1}$ and is distinct from $\mathfrak{p}_i$ for all i, $1 \leq i \leq k$. Note that $\mathfrak{p}_{k+1} \cap \mathfrak{a}$ is directional. It follows that there are infinitely many directional primes of R-height 1 in $\mathfrak{a}$, which contradicts Lemma 2.2. $\square$

3. Nontorsion modules. Since directional primes are defined in terms of their varieties, there is conceivably a considerable number of possible variations for the structure of a nondirectional prime. It would be useful to know what (at least all but finitely many of) the prime submodules of a module actually look like. Our main result indeed tells us precisely that, in the given setting.

Let R be a domain, and let M be a nontorsion R-module with $T = \operatorname{tor} M$. For a $\mathfrak{p}$-prime submodule P of M, we say that P is *tame* if $P = \mathfrak{p}M$, P is *semi-tame* if P is not tame and $P = \mathfrak{p}M + T$ and p is *wild* if P is neither tame nor semi-tame. Recall from [**12**] that a $\mathfrak{p}$-prime submodule is *virtually maximal* if $\mathfrak{p}$ is a maximal ideal of R.

Lemma 3.1. *Let R be a Noetherian domain, and let M be a finitely generated nontorsion R-module. If M has only finitely many wild primes, each of which is virtually maximal, then M is Zariski-finite.*

Proof. Let $T = \operatorname{tor} M$. In a similar fashion to Lemma 2.3 we claim that $\zeta(M)$ is generated by varieties of the form $V(\cap Q)$ and $V(T\cap(\cap Q))$, where the Q's are the wild primes of M.

Let N be a submodule of M, and let $P_1, \dots, P_r$ be the minimal primes to N which are tame, $P_{r+1}, \dots, P_s$ the minimal primes to N which are semi-tame, and $Q_1, \dots, Q_t$ the minimal primes to N which are wild. For each i, $1 \le i \le s$, let $\mathfrak{p}_i = (P_i : M)$ and for each i, $1 \le i \le t$, let $\mathfrak{q}_i = (Q_i : M)$. We claim that

$$V(N) = \begin{cases} V((\prod_{i=1}^{s} \mathfrak{p}_i)(\cap_{i=1}^{t} Q_i)) & s = r, \\ V((\prod_{i=1}^{s} \mathfrak{p}_i)(\cap_{i=1}^{t} Q_i) + (\prod_{i=1}^{r} \mathfrak{p}_i)(T \cap (\cap_{i=1}^{t} Q_i))) & s > r, \end{cases}$$

with the understanding that if $t = 0$, then $\cap_{i=1}^{t} Q_i = M$ and, if $r = 0$, then $\prod_{i=1}^{r} \mathfrak{p}_i = R$. We demonstrate the case where $s > r > 0$ and $t > 0$.

Let $L = (\prod_{i=1}^{s} \mathfrak{p}_i)(\cap_{i=1}^{t} Q_i) + (\prod_{i=1}^{r} \mathfrak{p}_i)(T \cap (\cap_{i=1}^{t} Q_i))$. Now if $P \in V(N)$, then P must contain some minimal prime to N, and it is an easy check that L is contained in every minimal prime to N, noting that for every $i, r+1 \le i \le s$, we have $T \subseteq P_i$. Conversely, suppose that $P \in V(L)$. Note first that $P \supseteq \cap_{i=1}^{t} Q_i$ implies $P \supseteq N$, so suppose that $P \not\supseteq \cap_{i=1}^{t} Q_i$. Then $(P : M) \supseteq \prod_{i=1}^{s} \mathfrak{p}_i$. If $\mathfrak{p}_i \subseteq (P : M)$ for some i, $1 \le i \le r$, we again have $P \supseteq N$, so suppose further that $(P : M) \not\supseteq \mathfrak{p}_i$ for all i, $1 \le i \le r$. Then there exists some j, $r+1 \le j \le s$, such that $\mathfrak{p}_j \subseteq (P : M)$. Now turning our attention to the second summand of L, we see that P must contain $T \cap (\cap_{i=1}^{t} Q_i)$. If $P \supseteq T$, then $P \supseteq \mathfrak{p}_j M + T = P_j \supseteq N$, and we are done. If, on the other hand, $P \not\supseteq T$, then $(P : M) \supseteq \mathfrak{q}_n$ for some $1 \le n \le t$ [**5**, Lemma 2]. But $\mathfrak{q}_n$ is a maximal ideal, so that $\mathfrak{p}_j \subseteq (P : M) = \mathfrak{q}_n$. It follows that $\mathfrak{p}_j M + (T \cap Q_n) = (\mathfrak{p}_j M + T) \cap Q_n = P_j \cap Q_n \supseteq N$. Finally, observe that for $k \ne n$, $1 \le k \le t$, we have $\mathfrak{q}_k \ne \mathfrak{q}_n$. This is so because for each prime ideal $\mathfrak{p}$ of R, there can be at most one $\mathfrak{p}$-prime submodule of R which is minimal to N. Hence $(P : M) \not\supseteq \cap_{i \ne n} \mathfrak{q}_i$, and so $P \supseteq T \cap Q_n$. It now follows that $P \supseteq N$, and we are done. $\square$

Theorem 3.2. *Let R be a Noetherian* UFD, *and let M be a nontorsion, finitely generated R-module. Then the following statements are equivalent.*

1) *M is Zariski-finite.*

2) *M is weakly Zariski-finite.*

3) *$M \cong R \oplus T$ for some finite R-module T.*

4) *M has only finitely many wild primes, each of which is virtually maximal.*

Proof. Clearly the result holds if R is finite, so suppose that R is infinite.

1) $\Rightarrow$ 2). By Lemma 1.1.

2) $\Rightarrow$ 3). Let $T = \operatorname{tor} M$. Note that, by Lemma 1.2, M/T is torsion-free and weakly Zariski-finite, and thus by Lemma 1.1 and Lemma 2.5 (and the paragraph preceding it) we have $M/T \cong \mathfrak{a}$ for some ideal $\mathfrak{a}$ of R. With g the gcd of the (finitely many) generators of $\mathfrak{a}$, we let $\mathfrak{a}' = \{r \in R : gr \in \mathfrak{a}\}$. Note that $\mathfrak{a}' \cong \mathfrak{a} \cong M/T$. As R is a UFD, it is apparent that $\mathfrak{a}'$ is not contained in any height 1 prime ideal of R. By Lemma 2.6, $M/T \cong \mathfrak{a}' = R$. It now follows that $M \cong R \oplus T$.

3) $\Rightarrow$ 4). See the closing remarks of [**11**].

4) $\Rightarrow$ 1). By Lemma 3.1. $\square$

Corollary 3.3. *Let R be a Noetherian* UFD*, and let M be a torsion-free finitely generated R-module. Then M is Zariski-finite if and only if $M \cong R$.*

Proof. $\Leftarrow$. This is obvious.

$\Rightarrow$. By Theorem 3.2, $M \cong R \oplus T$ for some finite R-module T. But clearly $T = \operatorname{tor} M = 0$. $\square$

The UFD condition is critical in the above theorem and corollary. For example, if R is a Dedekind domain, then every ideal of R is Zariski-finite [**9**, Lemma 4.6].

An interesting question still remains as to whether Zariski-finite is equivalent to weakly Zariski-finite, at least in the Noetherian setting. The authors at present know of no examples which suggest that they are not equivalent.

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