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GEOMETRICAL PROPERTIES OF THE PRODUCT OF A C^*-ALGEBRA

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0. Introduction. The study of the geometry of norm-unital complex Banach algebras at their units [**5**], [**6**] takes its first impetus from the celebrated Bohnenblust-Karlin theorem [**3**] asserting that the unit of such an algebra A is a vertex of the closed unit ball of A. As observed in [**5**, pp. 33–34], the Bohnenblust-Karlin paper contains a stronger result, namely that, for such an algebra A, the inequality $n(A, \mathbf{1}) \geq (1/e)$ holds. Here $\mathbf{1}$ denotes the unit of A, and $n(A, \mathbf{1})$ is a suitably defined nonnegative real number which depends only on the Banach space of A and the norm-one distinguished element $\mathbf{1}$. As the main result, we prove in this paper that the product of every nonzero C^*-algebra A is a vertex of the closed unit ball of the Banach space $\Pi(A)$ of all continuous bilinear mappings from $A \times A$ into A. As in the above mentioned case, the vertex property follows from stronger "numerical" conditions. Indeed, if A is a nonzero C^*-algebra, and if p_A denotes the product of A, then $n(\Pi(A), p_A)$ is equal to 1 or 1/2 depending on whether or not A is commutative (Theorem 1.1). We note that our main result improves the recent one in [**24**, Corollary 2.7] asserting that the product of every nonzero C^*-algebra A is an extreme point of the closed unit ball of $\Pi(A)$.

In Section 2 we show that the main result remains true for the so-called alternative C^*-algebras (Theorem 2.5). Alternative C^*-algebras are defined by means of the Gelfand-Naimark abstract system of axioms but relaxing the familiar requirement of associativity to that of alternativity. Alternative C^*-algebras arise in a natural way in functional analysis. Indeed, Gelfand-Naimark axioms on a general nonassociative unital algebra imply the alternativity [**22**, Theorem 14] (see also [**9**]) and the existence of alternative C^*-algebras failing to be associative is well known (see [**17**, Example 13] and [**8**, Theorem 3.7]). Alternative C^*-algebras are studied in detail in [**20**] and [**8**] and have shown

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useful in the structure theory of JB^*-triples [**14**]. Most tools applied in the extension of the main result to the alternative setting are actually proved for the so-called noncommutative JB^*-algebras [**20**]. We note that the class of noncommutative JB^*-algebras contains that of alternative C^*-algebras. Moreover, as in the case of these last algebras, noncommutative JB^*-algebras have a natural birth. Namely, if a norm-unital complete normed nonassociative complex algebra A is subjected to the geometric Vidav condition characterizing C^*-algebras in the associative context [**4**, Theorem 38.14], then A is a noncommutative JB^*-algebra [**23**].

Finally, in Section 3, we raise the question if alternative C^*-algebras are the unique noncommutative JB^*-algebras A whose products are vertices of the closed unit ball of $\Pi(A)$. In relation to this question, we exhibit an example showing that, if the vertex property is relaxed to the extreme point property, then the answer is negative (Example 3.2).

1. The main result. Let X be a normed space. We denote by S_X and B_X the unit sphere and the closed unit ball, respectively, of X. $L(X)$ will denote the normed algebra of all bounded linear operators on X, and I_X will stand for the identity operator on X. Each continuous bilinear mapping from $X \times X$ into X will be called a *product* on X. Each product f on X has a natural norm $\|f\|$ given by

$$\|f\| := \sup\{\|f(x,y)\| : x, y \in X, \|x\| \leq 1, \|y \leq 1\}.$$

We denote by $\Pi(X)$ the normed space of all products on X. For every product f on a normed space X, $f^{***} : X^{**} \times X^{**} \to X^{**}$ will stand for the third Arens transpose of f [**2**].

Now let u be a norm-one element in the normed space X. The set of *states* of X relative to u, $D(X,u)$, is defined as the nonempty, convex and weak*-compact subset of X^* given by

$$D(X,u) := \{\phi \in B_{X^*} : \phi(u) = 1\}.$$

For x in X, the *numerical range* of x relative to u, $V(X,u,x)$ is given by

$$V(X,u,x) := \{\phi(x) : \phi \in D(X,u)\}.$$

Thanks to the Hahn-Banach theorem, numerical ranges are contracted (respectively, preserved) under linear contractions (respectively, isometries), preserving "distinguished" elements. Indeed, if Y is another normed space, if v is a norm-one element in Y, and if F is a linear contraction (respectively, isometry) from X into Y with $F(u) = v$, then for all x in X we have

$$V(Y, v, F(x)) \subseteq V(X, u, x) \quad \text{respectively, } V(Y, v, F(x)) = V(X, u, x).$$

We say that u is a *vertex* of B_X if the conditions $x \in X$ and $\phi(x) = 0$ for all ϕ in $D(X, u)$ imply $x = 0$. It is well known and easy to see that the vertex property for u implies that u is an extreme point of B_X. For x in X, we define the *numerical radius* of x relative to u, $v(X, u, x)$, by

$$v(X, u, x) := \text{Max}\,\{|\rho| : \rho \in V(X, u, x)\}.$$

The *numerical index* of X relative to u, $n(X, u)$, is the number given by

$$n(X, u) := \text{Max}\,\{r \geq 0 : r\|x\| \leq v(X, u, x) \quad \text{for all} \quad x \text{ in } X\}.$$

We note that $0 \leq n(X, u) \leq 1$ and that the condition $n(X, u) > 0$ implies that u is a vertex of B_X. Note also that, if Y is a subspace of X containing u, then $n(Y, u) \geq n(X, u)$. According to a result of Crabb, Duncan and McGregor [**10**, Theorem 3], if A is a nonzero C^*-algebra with a unit $\mathbf{1}$, then $n(A, \mathbf{1})$ is equal to 1 or 1/2 depending on whether or not A is commutative.

Let A be a complex algebra. The *unital hull* A_1 of A is defined by $A_1 := A$, if A has a unit, and otherwise by $A_1 := \mathbf{C1} \oplus A$ with product

$$(\lambda\mathbf{1} + x)(\mu\mathbf{1} + y) := \lambda\mu\mathbf{1} + (\lambda y + \mu x + xy).$$

In any case, A_1 is a unital complex algebra containing A as an ideal. For z in A_1, we denote by T_z the linear operator on A defined by $T_z(x) := zx$ for every x in A. We note that, if the algebra A is normed, then, for z in A_1, the operator T_z is continuous. We also note that, if $*$ is an algebra-involution on A, then it extends uniquely to an algebra-involution, also denoted by $*$, on A_1, which is given by $(\lambda\mathbf{1}+x)^* := \bar{\lambda}\mathbf{1}+x^*$. It is known that, if the algebra A is a C^*-algebra, then A_1, with the involution above and the norm $\|.\|$ given by $\|z\| := \|T_z\|$ for z in A_1 becomes a C^*-algebra containing A isometrically (see for instance Lemma 12.19 in [**4**] and its proof).

For every normed algebra A, p_A will denote the natural product of A and, when A is endowed with an involution, A_{sa} will stand for the self-adjoint part of A. Also, if E is any set, then $B(E,A)$ will mean the normed algebra of all bounded functions from E into A (with point-wise operations and the supremum norm). For background on C^*-algebras, the reader is referred to [**11**] and [**25**].

Theorem 1.1. *Let A be a nonzero C^*-algebra. Then $n(\Pi(A), p_A)$ is equal to 1 or 1/2 depending on whether or not A is commutative.*

Proof. Let δ denote either 1 or 1/2 depending on whether or not A is commutative. Consider the chain of linear mappings

$$A_1 \xrightarrow{F_1} L(A) \xrightarrow{F_2} \Pi(A) \xrightarrow{F_3} \Pi(A^{**}) \xrightarrow{F_4} B((A^{**})_{sa} \times (A^{**})_{sa}, A^{**}),$$

where $F_1(z) := T_z$ for every z in A_1, $F_2(T)(x,y) := T(xy)$ for every T in $L(A)$ and all x,y in A, $F_3(f) := f^{***}$ for every f in $\Pi(A)$, and

$$F_4(g)(h,k) := e^{-ih} g(e^{ih}, e^{ik}) e^{-ik}$$

for every g in $\Pi(A^{**})$ and all h,k in $(A^{**})_{sa}$. We know that F_1 and F_3 are isometries. But the same is true for F_2 (because A has an approximate unit bounded by one) and also for F_4 (thanks to the Russo-Dye-Palmer theorem [**4**, Theorem 38.13]). Moreover, we have $F_1(\mathbf{1}) = I_A$, $F_2(I_A) = p_A$, $F_3(p_A) = (p_A)^{***}$, which is nothing but the natural C^*-product $p_{A^{**}}$ of A^{**}, and $F_4(P_{A^{**}}) = \Box$, where $\Box$ denotes the constant mapping equal to the unit of A^{**} on $(A^{**})_{sa} \times (A^{**})_{sa}$. Since A_1 and $B((A^{**})_{sa} \times (A^{**})_{sa}, A^{**})$ are C^*-algebras with units $\mathbf{1}$ and $\Box$, respectively, and they are commutative if and only if A is, it follows that

$$\begin{aligned}\delta &= n(A_1, \mathbf{1}) \geq n(L(A), I_A) \geq n(\Pi(A), p_A) \\ &\geq n(\Pi(A^{**}), p_{A^{**}}) \geq n(B((A^{**})_{sa} \times (A^{**})_{sa}, A^{**}), \Box) = \delta. \qquad \square\end{aligned}$$

The method of proof in the above theorem leads to other interesting consequences. As a first application the next corollary shows how, for a unital C^*-algebra A, numerical ranges in $\Pi(A)$ relative to p_A can be computed in terms of numerical ranges in A relative to its unit.

Corollary 1.2. *Let A be a nonzero C^*-algebra with a unit $\mathbf{1}$. Denote by U the set of all unitary elements in A. Then, for every f in $\Pi(A)$, we have*

$$\begin{aligned} V(\Pi(A), p_A, f) &= \overline{\mathrm{co}}\,[\cup\{V(A, \mathbf{1}, u^* f(u,v)v^*) : u, v \in U\}] \\ &= \overline{\mathrm{co}}\,[\cup\{V(A, \mathbf{1}, e^{-ih} f(e^{ih}, e^{ik})e^{-ik}) : h, k \in A_{sa}\}], \end{aligned}$$

where $\overline{\mathrm{co}}$ denotes closed convex hull.

Proof. The inclusion

$$\begin{aligned} &\overline{\mathrm{co}}\,[\cup\{V(A, \mathbf{1}, e^{-ih} f(e^{ih}, e^{ik})e^{-ik}) : h, k \in A_{sa}\}] \\ &\qquad\qquad \subseteq \overline{\mathrm{co}}\,[\cup\{V(A, \mathbf{1}, u^* f(u,v)v^*) : u, v \in U\}] \end{aligned}$$

is clear, and the one

$$\overline{\mathrm{co}}\,[\cup\{V(A, \mathbf{1}, u^* f(u,v)v^*) : u, v \in U\}] \subseteq V(\Pi(A), p_A, f)$$

follows since, for u, v in U, the mapping $f \to u^* f(u,v)v^*$ from $\Pi(A)$ to A is a linear contraction sending p_A to $\mathbf{1}$. Put $E := A_{sa} \times A_{sa}$ and, for f in $\Pi(A)$, let $\hat{f}$ be the element in $B(E, A)$ defined by

$$\hat{f}(h,k) := e^{-ih} f(e^{ih}, e^{ik})e^{-ik}.$$

As seen in the proof of Theorem 1.1, the linear mapping $f \to \hat{f}$ from $\Pi(A)$ to $B(E, A)$ is an isometry sending p_A to the unit $\square$ of $B(E, A)$. Therefore

$$V(\Pi(A), p_A, f) = V(B(E, A), \square, \hat{f}).$$

But, by [**22**, Proposition 3], we have

$$V(B(E, A), \square, \hat{f}) = \overline{\mathrm{co}}\,[\cup\{V(A, \mathbf{1}, \hat{f}(h,k)) : (h,k) \in E\}].$$

It follows that

$$V(\Pi(A), p_A, f) = \overline{\mathrm{co}}\,[\cup\{V(A, \mathbf{1}, e^{-ih} f(e^{ih}, e^{ik})e^{-ik}) : h, k \in A_{sa}\}]. \qquad \square$$

Now we handle other consequences of the method of proof of Theorem 1.1. Let X be a nonzero normed space. The *duality mapping* of X

is the set-valued function $v \to D(X, v)$ from S_X into $\mathcal{P}(X^*)$. Following [**13**] we say that the *duality mapping of X is norm-to-norm upper semi-continuous* at a point u of S_X if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $D(X, v) \subseteq D(X, u) + \varepsilon B_X$ whenever v is in S_X and $\|v - u\| \leq \delta$. We note that the requirement $n(X, u) = 1$ implies that the duality mapping of X is upper semi-continuous at u whereas, for every $0 < \rho < 1$, upper semi-continuity of the duality mapping of X at u and the requirement $n(X, u) = \rho$ are independent conditions [**1**, pp. 134–135].

Corollary 1.3. *Let A be a nonzero C^*-algebra. Then the duality mapping of $\Pi(A)$ is norm-to-norm upper semi-continuous at p_A.*

Proof. With the notation in the proof of Theorem 1.1, the mapping $F_4 \circ F_3$ identifies $\Pi(A)$ with a subspace of the norm-unital Banach algebra $B((A^{**})_{sa} \times (A^{**})_{sa}, A^{**})$ in such a way that p_A converts into the unit of that algebra. Then the result follows from [**19**, Proposition 4.5] and the well-known hereditary character of the upper semi-continuity of the duality mapping. $\square$

Again let X be a nonzero normed space (no distinguished norm-one element is chosen). The *normed space numerical index,* $N(X)$ of X is defined by $N(X) := n(L(X), I_X)$. The proof of Theorem 1.1 directly gives the result of T. By Huruya [**15**], if A is a nonzero C^*-algebra, then $N(A)$ is equal to 1 or 1/2 depending on whether or not A is commutative. As a methodological remark, we point out that a part of our proof can be useful to clarify the first step in Huruya's original argument. In our opinion that first step (establishing the inequality $N(A) \leq 1/2$ for every noncommutative C^*-algebra A) is easily understood only by passing through the unit hull A_1 of A in the way we have done.

We conclude this section with a variant of Huruya's theorem. For an element x in an algebra A, we denote by L_x the operator of left multiplication by x on A.

Proposition 1.4. *Let A be a nonzero W^*-algebra, and let A_* denote the predual of A. Then $N(A_*)$ is equal to* 1 *or* 1/2 *depending on whether* *or not A is commutative.*

Proof. First note that, for every normed space X, the inequality $N(X^*) \leq N(X)$ holds [**6**, Lemma 32.7]. Assume that the W^*-algebra A is commutative. Then we have $1 = N(A) \leq N(A_*) \leq 1$. Now assume that A is not commutative. Then $1/2 = N(A) \leq N(A_*)$. On the other hand, by [**18**, Appendix 3], there exists x in S_A with $x^2 = 0$. By [**7**, p. 214], for such an x we have $v(A, \mathbf{1}, x) = 1/2$ (here $\mathbf{1}$ stands for the unit of A). Since the mapping $y \to L_y$ from A to $L(A)$ is a linear isometry sending $\mathbf{1}$ to I_A, we derive $v(L(A), I_A, L_x) = 1/2$. By the separate weak*-continuity of the product of A, there exists S in $L(A_*)$ such that $L_x = S^*$, and hence we obtain $v(L(A_*), I_{A^*}, S) = 1/2$. It follows that $N(A_*) = n(L(A_*), I_{A_*}) \leq 1/2$. □

2. Extending the results to alternative C^*-algebras. In this section we show that Theorem 1.1 remains true if the assumption of associativity for the C^*-algebra A is relaxed to that of alternativity.

Alternative algebras are defined as those (not necessarily associative) algebras A satisfying $x^2y = x(xy)$ and $yx^2 = (yx)x$ for all x, y in A. By Artin's theorem [**26**, p. 29], an algebra A is alternative, (if and) only if, for all x, y in A, the subalgebra of A generated by $\{x, y\}$ is associative. As a consequence, alternative algebras are *power-associative* (i.e., all one-generated subalgebras are associative). Let A be an alternative algebra with a unit $\mathbf{1}$. An element x in A is said to be *invertible* if there exists y in A such that $xy = yx = \mathbf{1}$. If this is the case, then the element y above is uniquely determined by x, is called the *inverse* of x, is denoted by x^{-1} and satisfies $x(x^{-1}z) = x^{-1}(xz) = z$ and $(zx^{-1})x = (zx^{-1})x = z$ for every z in A (see for instance [**26**, p. 38]).

A complete normed complex alternative algebra A with (conjugate-linear) algebra-involution $*$ satisfying $\|x^*x\| = \|x\|^2$ for all x in A is called an *alternative C^*-algebra*. Let A be an alternative C^*-algebra with a unit $\mathbf{1}$. An element u in A is said to be *unitary* if the equalities $uu^* = u^*u = \mathbf{1}$ hold (equivalently, if u is invertible in A with $u^{-1} = u^*$). Such an element u satisfies $\|u\| = \|u^*\| = 1$ and $u(u^*z) = u^*(uz) = z$ and $(zu^*)u = (zu)u^* = z$ for every z in A, so that the mappings $z \to uz$ and $z \to zu$ from A to A become surjective linear isometries. Distinguished unitary elements of A are those of the form e^{ix} where

x is in the self-adjoint part A_{sa} of A, and it is easily shown (see for instance [**8**, Theorem 2.10]) that the verbatim translation of the Russo-Dye-Palmer theorem holds for A. Indeed, the equalities

$$B_A = \overline{\text{co}}\,\{u : u \text{ unitary in } A\} = \overline{\text{co}}\,\{e^{ih} : h \in A_{sa}\}$$

are true.

Most remaining facts of the theory of (associative) C^*-algebras applied in the proof of Theorem 1.1 are also known in the wider setting of alternative C^*-algebras. For instance, if A is an alternative C^*-algebra, then the bidual A^{**} of A, with product equal to the third Arens transpose of the product of A, and involution equal to the second transpose of the involution of A, becomes an alternative C^*-algebra with a unit [**20**, Corollary 1.9]. Other results needed for our purpose are directly derivable from the theory of noncommutative JB^*-algebras [**20**] and the fact that alternative C^*-algebras are noncommutative JB^*-algebras. Following [**26**, p. 141], we define *noncommutative Jordan* algebras as those algebras A satisfying the *Jordan identity* $(xy)x^2 = x(yx^2)$ and the *flexibility condition* $(xy)x = x(yx)$. Noncommutative Jordan algebras are also power associative [**26**, p. 141]. As a consequence of Artin's theorem, alternative algebras are noncommutative Jordan algebras. For an element x in a noncommutative Jordan algebra A, we denote by U_x the mapping $y \to x(xy+yx)-x^2y$ from A to A. By a *noncommutative JB^*-algebra* we mean a complete normed noncommutative Jordan complex algebra (say A) with algebra involution $*$ satisfying $\|U_x(x^*)\| = \|x\|^3$ for every x in A. Now alternative C^*-algebras are nothing but those noncommutative JB^*-algebras which are alternative [**20**, Proposition 1.3]. As a first application, the extension for alternative C^*-algebras of the Crabb-Duncan-McGregor result in [**10**, Theorem 1] will follow from the next proposition.

Proposition 2.1 [**22**, Theorem 26] (see also [**16**, Theorem 4]). *Let A be a nonzero, noncommutative JB^*-algebra with a unit $\mathbf{1}$. Then $n(A,\mathbf{1})$ is equal to 1 or $1/2$ depending on whether or not A is associative and commutative.*

Since commutative alternative complex algebras are associative [**29**, Corollary 7.1.2], we obtain

Corollary 2.2. *Let A be a nonzero alternative C^*-algebra with a unit $\mathbf{1}$. Then $n(A, \mathbf{1})$ is equal to 1 or 1/2 depending on whether or not A is commutative.*

Now the only remaining auxiliary tools for the proof of the alternative extension of Theorem 1.1 are the following two lemmas.

Lemma 2.3. *Let A be a nonzero, noncommutative JB^*-algebra. Then A_1, endowed with the unique algebra involution extending the one of A and the norm $\|.\|$ given by $\|z\| := \|T_z\|$ for all z in A_1, is a noncommutative JB^*-algebra containing A isometrically.*

Proof. By [**20**, Theorem 1.7], the bidual A^{**} of A, with product equal to the third Arens transpose of the product of A, and involution equal to the second transpose of the involution of A, becomes a noncommutative JB^*-algebra with a unit (say $\mathbf{1}$). Then we can see A_1 as the norm-closed $*$-invariant subalgebra of A^{**} consisting of those elements z in A^{**} which can be written in the form $\lambda\mathbf{1} + x$ for some λ in $\mathbf{C}$ and x in A. In this way A_1 is a noncommutative JB^*-algebra containing A isometrically so that, to conclude the proof, it is enough to show that the equality $\|z\| = \|T_z\|$ holds for every z in A_1 (here $\|z\|$ means the norm of z as an element of A^{**}). Let z be in A_1. Then $(T_z)^{**} : A^{**} \to A^{**}$ and the operator of left multiplication by z on A^{**}, say $L_z^{A^{**}}$, coincide on A and are weak*-continuous (the second one, by [**20**, Theorem 3.5]). It follows from the weak*-density of A in A^{**} that $(T_z)^{**} = L_z^{A^{**}}$, and hence

$$\|T_z\| = \|(T_z)^{**}\| = \|L_z^{A^{**}}\|.$$

Since $\|z\| = \|L_z^{A^{**}}\|$, because A^{**} is a norm-unital normed algebra, we obtain $\|z\| = \|T_z\|$, as required. □

Lemma 2.4. *Let A be a noncommutative JB^*-algebra and x an element of A. Then x belongs to the norm-closure of xB_A.*

Proof. Take a net $\{y_\lambda\}$ in B_A convergent to the unit $\mathbf{1}$ of A^{**} in the weak* topology of A^{**}. Then, by [**20**, Theorem 3.5], $\{xy_\lambda\}$ converges to x in that topology. Since the net $\{xy_\lambda\}$ lies in A and x belongs to

A, it follows that $\{xy_\lambda\}$ converges to x in the weak topology of A, and therefore x actually belongs to the weak closure of xB_A in A. Finally apply that xB_A is convex. $\square$

Now we are ready to prove the extension of Theorem 1.1 to alternative C^*-algebras.

Theorem 2.5. *Let A be a nonzero alternative C^*-algebra. Then $n(\Pi(A), p_A)$ is equal to 1 or 1/2 depending on whether or not A is commutative.*

Proof. If A is commutative, then by Theorem 1.1 we have $n(\Pi(A), p_A) = 1$. Assume that A is not commutative. We know that A^{**}, with product equal to the third Arens transpose of the product of A, and involution equal to the second transpose of the involution of A, becomes an alternative C^*-algebra with a unit $\mathbf{1}$. Moreover, since the unital hull of an alternative algebra is an alternative algebra too, it follows from Lemma 2.3 that A_1 is an alternative C^*-algebra in such a way that the mapping $z \to T_z$ from A_1 to $L(A)$ is an isometry. Now, consider the chain of linear mappings

$$A_1 \xrightarrow{F_1} L(A) \xrightarrow{F_2} \Pi(A) \xrightarrow{F_3} \Pi(A^{**}) \xrightarrow{F_4} B((A^{**})_{sa} \times (A^{**})_{sa}, A^{**}),$$

where F_1, F_2 and F_3 are defined verbatim as in the proof of Theorem 1.1, whereas F_4 is determined by the equality

$$F_4(g)(h,k) := e^{-ih}(g(e^{ih}, e^{ik})e^{-ik})$$

for every g in $\Pi(A^{**})$ and all h, k in $(A^{**})_{sa}$. At this time, the isometric character of F_1 and F_3 is not in doubt. But F_2 and F_4 are also isometries. Indeed, apply Lemma 2.4 for the case of F_2 and, concerning F_4, keep in mind the extended Russo-Dye-Palmer theorem together with the fact that left and right multiplications by unitary elements on A^{**} are isometries. On the other hand, the equalities $F_1(\mathbf{1}) = I_A$, $F_2(I_A) = p_A$, $F_3(p_A) = p_{A^{**}}$ are clear, whereas $F_4(p_{A^{**}}) = \square$ follows from Artin's theorem. Since the unital alternative C^*-algebras A_1 and $B((A^{**})_{sa} \times (A^{**})_{sa}, A^{**})$ are not commutative, it follows from

Corollary 2.2 that

$$\begin{aligned}\frac{1}{2} &= n(A_1, \mathbf{1}) \geq n(L(A), I_A) \geq n(\Pi(A), p_A) \\ &\geq n(\Pi(A^{**}), p_{A^{**}}) \geq n(B((A^{**})_{sa} \times (A^{**})_{sa}, A^{**}), \square) = \frac{1}{2}. \qquad \square\end{aligned}$$

It is worth mentioning that Corollary 1.3 remains true, without any change in its formulation and proof, in the wider setting of alternative C^*-algebras. Concerning Corollary 1.2, it needs a light retouching of formulation in order to remain valid in our new context. Namely, if A is a nonzero alternative C^*-algebra with a unit $\mathbf{1}$, and if U denotes the set of all unitary elements in A, then, for every f in $\Pi(A)$, we have

$$\begin{aligned}V(\Pi(A), p_A, f) &= \overline{\text{co}}\,[\cup\{V(A, \mathbf{1}, u^*(f(u,v)v^*)) : u, v \in U\}] \\ &= \overline{\text{co}}\,[\cup\{V(A, \mathbf{1}, e^{-ih}(f(e^{ih}, e^{ik})e^{-ik})) : h, k \in A_{sa}\}].\end{aligned}$$

Let us also note that the proof of Theorem 2.5 contains an extended Huruya theorem. Indeed, if A is an alternative C^*-algebra, then $N(A)$ is equal to 1 or 1/2 depending on whether or not A is commutative. In fact, the next proposition provides us with a more general result. For a complex normed space X and a norm-one element u in X, we write

$$H(X, u) := \{x \in X : V(X, u, x) \subseteq \mathbf{R}\}.$$

Proposition 2.6. *Let A be a nonzero noncommutative JB^*-algebra. Then $N(A)$ is equal to 1 or 1/2 depending on whether or not A is associative and commutative.*

Proof. Assume that A is associative and commutative. Then A is a commutative C^*-algebra and, therefore, we have $N(A) = 1$. Now assume that A fails to be associative or commutative. By the already applied Theorem 1.7 in [**20**], A^{**} is a unital noncommutative JB^*-algebra in a natural way. Moreover, by Lemma 2.3, A_1 is a noncommutative JB^*-algebra in such a way that the mapping $z \to T_z$ from A_1 to $L(A)$ becomes an isometry. Consider the chain of linear mappings

$$A_1 \xrightarrow{G_1} L(A) \xrightarrow{G_2} L(A^{**}) \xrightarrow{G_3} B((A^{**})_{sa}, A^{**}),$$

where $G_1(z) := T_z$ for every z in A_1, $G_2(T) := T^{**}$ for every T in $L(A)$, and

$$G_3(S)(h) := e^{-iL_h} S(e^{ih})$$

for every S in $L(A^{**})$ and all h in $(A^{**})_{sa}$. We know that G_1 and G_2 are isometries. To see that the same is true for G_3, apply the Russo-Dye-Palmer-type theorem for noncommutative JB^*-algebras [**27**, Corollary 2.4], and note that, if h is in $(A^{**})_{sa}$, then h belongs to $H(A^{**}, \mathbf{1})$ [**28**, Theorem 7(a)], so L_h belongs to $H(L(A^{**}), I_{-A^{**}})$ and so e^{-iL_h} is an isometry [**5**, Lemma 5.2]. Since we have $G_1(\mathbf{1}) = I_A$, $G_2(I_A) = I_{A^{**}}$, and $G_3(I_{A^{**}}) = \Box$ (where $\Box$ denotes the constant mapping equal to the unit of A^{**} on $(A^{**})_{sa}$), it follows from Proposition 2.1 that

$$\begin{aligned}\frac{1}{2} &= n(A_1, \mathbf{1}) \geq n(L(A), I_A) \geq n(L(A^{**}), I_{A^{**}}) \\ &\geq n(B((A^{**})_{sa}, A^{**}), \Box) = \frac{1}{2}. \qquad \square\end{aligned}$$

Remark 2.7. (i) Proposition 2.6 was formulated in [**16**, Theorem 5] as a direct consequence of [**20**, Theorem 1.7], the particular case of that proposition for unital algebras [**22**, Corollary 33], and the claim in [**12**] that, for every normed space X, the equality $N(X^*) = N(X)$ holds. As a matter of fact, the proof of the claim in [**12**] never appeared, and the question if for an arbitrary normed space X the equality $N(X^*) = N(X)$ holds remains an open problem among people interested in the field.

(ii) Let A be a noncommutative JB^*-algebra. Since A^{**} is a noncommutative JB^*-algebra and A^{**} is associative and commutative if (and only if) the same is true for A [**20**, Theorem 1.7], it follows from Proposition 2.6 that the equality $N(X^*) = N(X)$ holds for X equal to either A or A^*.

Noncommutative JBW^*-algebras are defined as those noncommutative JB^*-algebras which are dual Banach spaces. If A is a noncommutative JBW^*-algebra, then A has a unit, the predual of A is unique, and the product of A is separately weak*-continuous [**20**, p. 104]. Our concluding result in this section shows that the equality $N(X^*) = N(X)$ also holds for X equal to the predual A_* of every noncommutative JBW^*-algebra A. Replacing Huruya's classical theorem, [**18**, Appendix 3, Theorem B] and [**7**, Corollary 2, p. 214] by Proposition 2.6, [**16**, Theorem 1] and [**22**, Proposition 29], respectively, the proof is almost the same as that of Proposition 1.4, and therefore it is omitted.

Proposition 2.8. *Let A be a nonzero noncommutative JBW^*-algebra. Then $N(A_*)$ is equal to* 1 *or* 1/2 *depending on whether or not A is associative and commutative.*

3. Discussing the results. Keeping in mind that most auxiliary results applied to prove Theorem 2.5 are also valid in the setting of noncommutative JB^*-algebras, one can suspect in a first instance that, if A is a nonzero noncommutative JB^*-algebra, then $n(\Pi(A), p_A)$ is equal to 1 or 1/2 depending on whether or not A is associative and commutative. As a matter of fact, this suspicion is very far from being right. Indeed, it is very easy to provide us with noncommutative JB^*-algebras A whose products p_A are not extreme points of $B_{\Pi(A)}$, so that p_A cannot be a vertex of $B_{\Pi(A)}$, and hence $n(\Pi(A), p_A) = 0$. For instance, if B is a C^*-algebra which fails to be commutative, if λ is a real number with $0 < \lambda < 1$, and if we replace the product xy of B by the one

$$(x, y) \longrightarrow \lambda xy + (1 - \lambda)yx,$$

then we obtain a noncommutative JB^*-algebra (say A) whose product is not an extreme point of $B_{\Pi(A)}$. With $\lambda = 1/2$ in the above construction we even obtain a (commutative) JB^*-algebra with such a pathology. Since, on the other hand, we do not know noncommutative JB^*-algebras A whose products have the vertex property in $\Pi(A)$ other than alternative C^*-algebras, we dare to formulate the following conjecture.

Conjecture 3.1. *A noncommutative JB^*-algebra A is alternative if* (*and only if*) *the product of A is a vertex of the closed unit ball of* $\Pi(A)$.

Concerning the above conjecture, the only remarkable fact that we know up to date is that, if we relax the vertex property to the extreme point property, then the answer is negative. This is shown by the next example. Following [**26**, p. 50], we say that a complex algebra A is quadratic if A has a unit $\mathbf{1}$, $A \neq \mathbf{C1}$ and x^2 belongs to the linear hull of $\{\mathbf{1}, x\}$ for all x in A. A *Vidav* (in short, V-) algebra is a norm-unital complete normed complex algebra A satisfying $A = H(A, \mathbf{1}) \oplus iH(A, 1)$ where $\mathbf{1}$ stands for the unit of A. For such an algebra A, the so-called *natural involution* of A, given by $(x + iy)^* := x - iy$, $x, y \in H(A, \mathbf{1})$, is an algebra-involution [**23**, Theorem 1].

Example 3.2. Let A be the (commutative) JB^*-algebra whose Banach space is the $*$-invariant subspace of the C^*-algebra $M_2(\mathbf{C})$ given by

$$\left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \alpha \end{pmatrix} : \alpha, \beta, \gamma \in \mathbf{C} \right\},$$

and whose product is the one $\circ$ defined by

$$x \circ y := \frac{1}{2}(xy + yx)$$

for all x, y in A. Since A is commutative and fails to be associative, it follows that A is not alternative. We will prove that p_A is an extreme point of $B_{\Pi(A)}$. To this end we take f, g in $B_{\Pi(A)}$ and $0 < \lambda < 1$ such that $\lambda f + (1 - \lambda) g = p_A$, and we proceed to show that $f = p_A$, say. If $\mathbf{1}$ denotes the unit of A then, by [**20**, Lemma 1.5], we have $f(x, \mathbf{1}) = f(\mathbf{1}, x) = x$ for every x in A. In this way $\mathbf{1}$ is a unit for the complete normed complex algebras (say B and B^+) consisting of the Banach space of A and the products f and

$$f^+ : (x, y) \longrightarrow \frac{1}{2}(f(x, y) + f(y, x)),$$

respectively. Since A is a V-algebra and Vidav's requirement involves only the Banach space and the unit, B and B^+ are also V-algebras whose natural (automatically algebra-) involutions coincide with the JB^*-involution of A. Since A and B^+ are commutative V-algebras, and the mapping $F : x \to x$ from A to B^+ is a surjective linear isometry preserving the units, the argument in the proof of the implication (i) $\Rightarrow$ (ii) in [**17**, Lemma 6] shows that F is an algebra isomorphism, so f^+ coincides with p_A, and hence $B^+ = A$. Now, by [**17**, Theorem 8] and [**20**, Proposition 1.2], B is a noncommutative JB^*-algebra and, since A is quadratic and squares in B and B^+ coincide, B is quadratic too.

Then, as a consequence of [**21**, Theorem 3.2], there exist a real Hilbert space $(E, (.|.))$ and an anti-commutative product $\wedge$ on E satisfying $(x|x \wedge y) = 0$ for all x, y in E such that, if we consider the real algebra C consisting of the vector space $\mathbf{R}\mathbf{1} \oplus E$ and the product

$$(\lambda \mathbf{1} + x)(\mu \mathbf{1} + y) := (\lambda\mu - (x|y))\mathbf{1} + (\lambda y + \mu x + x \wedge y),$$

then, as a complex algebra, B is nothing but the complexification of C. Since A is three-dimensional over $\mathbf{C}$, E must be two-dimensional over $\mathbf{R}$. Let $\{u, v\}$ be a basis of E. Then we have $(u|u \wedge v) = (v|u \wedge v) = 0$, so $u \wedge v = 0$ and hence $\wedge$ is identically zero on E. Therefore, B is commutative, so that $B = B^+$. Since we know that $B^+ = A$, we obtain $B = A$. This means $f = p_A$ as required. $\square$

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