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MULTIPLIER OPERATORS ON WEIGHTED FUNCTION SPACES

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ABSTRACT. We prove that certain multiplier operators are bounded in weighted Hardy spaces provided that the multipliers are defined in Herz spaces.

Suppose $0 \leq \alpha < \infty$, $1 \leq a < \infty$ and $0 < p < \infty$. Let $K_a^{\alpha,p}$ be the Herz spaces which consists of all functions $f \in L^a(R^n)$ with the norm or quasi-norm

$$\|f\|_{K_a^{\alpha,p}} = \|f\|_{L^a(R^n)} + \bigg(\sum_{k=-\infty}^{\infty} \bigg(\bigg(\bigg(\int_{A_k} |f|^a \bigg)^{1/a} 2^{k\alpha} \bigg)^p \bigg)^{1/p},$$

where $A_k = \{2^k \leq |x| < 2^{k+1}\}$, $-\infty < k < \infty$. The spaces $K_a^{\alpha,p}$ were first introduced by Herz [**3**]. These spaces are relative to both Hardy spaces [**4**] and multiplier operators [**1**].

Suppose $\eta \in C_0^\infty(R^n)$ with $0 \leq \eta \leq 1$, $\eta(\xi) = 1$ on $1/2 \leq |\xi| \leq 2$ and $\operatorname{supp} \eta \subset \{1/4 \leq |\xi| \leq 4\}$. Let m be a function on R^n and denote $m_\delta(\xi) = m(\delta\xi)\eta(\xi)$ where $\delta > 0$. The multiplier operator Tf is defined by

$$\widehat{Tf}(\xi) = m(\xi) f(\xi).$$

In [**1**], Baernstein and Sawyer studied the boundedness of the multiplier operators on the Hardy spaces where the multiplier m is defined on the Herz spaces, more precisely, they proved that if m satisfies

$$\sup_{\delta > 0} \|\hat{m}_\delta\|_X < \infty$$

where $X = K_1^{n(1/p-1),p}$ for $0 < p < 1$ and X is a certain subspace of $K_1^{0,1}$ for $p = 1$, then the operator Tf is bounded on H^p spaces,

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$0 < p \leq 1$. The hypotheses of the multipliers on the Herz spaces is sharp in the sense that X cannot be replaced by any larger space of Herz spaces and also it is easy to check that any multiplier of Hörmander types of order N is in the Herz spaces $K_2^{N,2}$, i.e.,

$$\sup_{\delta>0} \|\hat{m}_\delta\|_{K_2^{N,2}} < \infty.$$

Recently, Onneweer and Quek extended the Baernstein and Sawyer's results to the case of mixed-norm type. Instead of assuming m is in the Herz spaces, they obtained the H^p boundedness of Tf by assuming that m is in some mixed-norm type space, see [**5, 6**].

The main purpose of this paper is to study the boundedness of multiplier operators on the weighted Hardy spaces where the multiplier is defined on the Hertz spaces.

Let D_q be the collection of positive measures with doubling conditions, that is, $\mu \in D_q$, $q > 0$, if μ is a positive measure on R^n such that there exists a constant $C > 0$ with the property that for all $t \geq 1$ and $r > 0$ we have

$$\mu(\{|x - x_0| \leq \mathrm{tr}\}) \leq Ct^q \mu(\{|x - x_0| \leq r\})$$

where the constant C is independent of $x_0 \in R^n$. Suppose W is a positive function. A tempered distribution f in R^n is in the weighted Hardy spaces $H_W^p(R^n)$ if

$$\|f\|_{H_W^p(R^n)} = \left(\int_{R^n} \Big(\sup_{(y,t)\in\Gamma(x)} |\Phi_t * f(y)| \Big)^p W(x)\, dx \right)^{1/p} < \infty$$

where Φ is a smooth function with $\int \Phi \neq 0$, $\Phi_t(x) = t^{-n}\Phi(t^{-1}x)$ and $\Gamma(x) = \{(y,t) \mid |x - y| < t\}$.

A function $a(x)$ is called an N-atom on $H_W^p(R^n)$ if

(i) the $a(x)$ is supported on a ball B;

(ii) $\|a\|_\infty \leq CW(B)^{-1/p}$ where $W(B) = \int_B W(x)\, dx$;

(iii) $\int x^\alpha a(x)\, dx = 0$ for $|\alpha| \leq N$.

Theorem A [**7**]. *Suppose $W(x)$ is a positive function and define a measure $d\mu(x) = W(x)\, dx$. Assume that $\mu \in D_q$ for some $q > 0$, that*

$0 < p \leq 1$ and that $\tilde{N} = [qn/p - 1]$. Then for every $f \in H^p_W(R^n)$ there is a sequence $\{a_k\}$ of N-atoms, $N \geq \tilde{N}$, such that

$$f = \sum \lambda_k a_k \quad \textit{and} \quad \|f\|^p_{H^p_W(R^n)} \approx \sum |\lambda_k|^p.$$

We will write $W \in RH_r$ if W satisfies the reverse Hölder condition of degree r, $\infty > r > 1$, that is, there is a constant $C > 0$ such that

$$\left(\frac{1}{|B|}\int_B (W(x)^r\,dx\right)^{1/r} \leq C\frac{1}{|B|}\int_B W(x)\,dx$$

for all balls B on R^n.

Definition. W is called an A_q weight if, for $\infty > q > 1$,

$$\left(\frac{1}{|B|}\int_B W(x)\,dx\right)\left(\frac{1}{|B|}\int_B W(x)^{-1/(q-1)}\,dx\right)^{q-1} \leq c$$

for every ball B in R^n, for $q = 1$

$$MW(x) \leq W(x) \quad \text{a.e.}$$

where M is the classical Hardy-Littlewood maximal function, and

$$A_\infty = \bigcup_{q \geq 1} A_q.$$

Remark. 1) We denote $W \in D_{nq}$ if $\mu \in D_{nq}$, where $d\mu$ is defined by $W(x)\,dx$ for $x \in R^n$.

2) Suppose W is an A_q weight, $\infty > q \geq 1$. Then $W \in D_{nq}$, see [**2**, p. 396].

In this paper we establish the following theorem.

Theorem. *Assume $W \in D_{nq} \cap A_\infty \cap RH_r$ for some $q \geq 1$, $\infty > r > 1$. Then T maps $H^p_W(R^n)$ boundedly to $H^p_W(R^n)$, provided*

$$\sup_{\delta > 0} \|\hat{m}_\delta\|_{K_1^{n(q/p+1/r-1),p}} < \infty$$

where $0 < p \leq 1$.

Remark. 1) Suppose the hypothesis in the above theorem

$$W \in D_{nq} \cap A_\infty \cap RH_r,$$

is replaced by $W \in D_{nq} \cap RH_r$. Then the multiplier operator T maps $H^p_W(R^n)$ boundedly to $L^p_W(R^n)$, $0 < p \leq 1$. That the extra hypothesis $W \in A_\infty$ stands in the theorem is due to the characterization of the weighted Hardy space in terms of singular integrals, see [**8**, p. 87].

2) If one defines RH_r, where $r = \infty$, as usual, by

$$\|W(\cdot)\chi_Q\|_\infty \leq \frac{1}{|Q|}\int_Q W(x)\,dx$$

for every ball Q, then it is easy to see that $W(x)$ is a constant function almost everywhere. Therefore, the boundedness of T on the weighted Hardy spaces is exactly the same as the boundedness of T on the Hardy spaces. These results can be found in [**1**].

Next we write a theorem which is a special case of Theorem 3b in [**1**, p. 21] which we will need in our proof. Here we should remark that the proof of the theorem depends on the ideas in paper [**1**].

Theorem B [**1**]. *For every fixed* $\varepsilon > 0$, *if*

$$\sup_{\delta>0} \|\hat{m}_\delta\|_{K_1^{\varepsilon,1}} < \infty, \tag{1}$$

then T *maps* $H^1(R^n)$ *boundedly to* $H^1(R^n)$.

On the other hand, it is clearly seen that

$$\|m\|_\infty \leq \int |\hat{m}(x)|\,dx \leq \sup_{\delta>0} \|\hat{m}_\delta\|_{K_1^{\varepsilon,1}}.$$

Hence, by Plancherel's theorem, T is a bounded operator on $L^2(R^n)$. Applying the interpolation theorem between the L^2 boundedness and Theorem B, one has the following:

Theorem C. *Suppose m is a function satisfying* (1). *Then* $\|Tf\|_{L^p(R^n)} \le C\|f\|_{L^p(R^n)}$, *for* $1 < p < \infty$.

Take and fix a function $\phi \in C_0^\infty(R)$, $\operatorname{supp}\phi \subset \{1/2 < |x| < 2\}$ and $0 \le \phi(x) \le 1$ such that $\sum_{j=-\infty}^{\infty} \phi(2^{-j}\xi) = 1$ for every $\xi \ne 0$. Then it is clear that $\phi(\xi)\eta(\xi) = \phi(\xi)$ and

$$m(\xi)\hat{a}(\xi) = \sum_{j=-\infty}^{\infty} m(\xi)\eta(2^{-j}\xi)\hat{a}(\xi)\phi(2^{-j}\xi)$$
$$\equiv \sum_{j=-\infty}^{\infty} m_j(2^{-j}\xi)\hat{a}_j(2^{-j}\xi)$$

where $m_j(\xi) = m(2^j\xi)\eta(\xi)$ and $\hat{a}_j(\xi) = \hat{a}(2^j\xi)\phi(\xi)$. Hence,

$$Ta(x) = \sum_{j=-\infty}^{\infty} 2^{jn}(K_j * a_j)(2^j x), \tag{2}$$

where $\hat{K}_j(\xi) = m_j(\xi)$.

Lemma 1. *Suppose $a(x)$ is an N-atom on $H^p_W(R^n)$ and is supported on the unit ball with center at the origin. For every $\beta > 0$, then*

(i) $\quad |a_j(x)| \le C_\beta 2^{j(N+1)} W(B)^{-1/p}(1+|x|)^{-\beta}, \quad j \le 1;$

(ii) $\quad |a_j(x)| \le C_\beta W(B)^{-1/p}\{2^{-jn}\chi|_{|x|\le 2^{j+1}} + |x|^{-\beta}\chi_{|x|>2^{j+1}}\}$

where χ denotes a characteristic function.

Proof. For (i), by the moments of atom a, we write

$$|a_j(x)| = \left| \int \hat{\phi}(x - 2^j z)a(z)\,dz \right|$$
$$\le C \sum_{|\gamma|=N+1} \int_0^1 \int_{|z|\le 1} |(\partial^\gamma \hat{\phi})(x - 2^j zt)||2^j z|^{N+1}|a(z)|\,dz\,dt$$
$$\le C_\beta 2^{j(N+1)} W(B)^{-1/p}(1+|x|)^{-\beta} \quad \text{for } j \le 1.$$

For (ii), let us write

$$|a_j(x)| = \left|\int_{|z|\le 1} \hat{\phi}(x-2^j z)a(z)\,dz\right|.$$

By the smoothness of $\hat{\phi}$, if $|x| \ge 2^{j+1}$, one has

$$|a_j(x)| \le C_\beta |x|^{-\beta} W(B)^{-1/p}.$$

On the other hand, for every $x \in R^n$,

$$|a_j(x)| = 2^{-jn}\left|\int \hat{\phi}(x-u)a(2^{-j}u)\,du\right| \le C2^{-jn}W(B)^{-1/p}.$$

Proof of Theorem. Let a be an N-atom on $H^p_W(R^n)$ where the support of a is a ball B. It suffices to assume the center of B is at the origin. Let us assume the radius of B is 1. We will remove this assumption later. By a characterization of the weighted Hardy spaces in terms of singular integrals (see [**8**, p. 87]), we need only to estimate the $L^p(R^n)$ quasi-norm of Ta since the multiplier operator of T commutes with those singular integrals. Therefore, by Theorem A, we have to show $\|Ta\|_{L^p_W(R^n)}$ is uniformly bounded for every N-atom a. Let us write

$$\|Ta\|^p_{L^p_W(R^n)} = \|Ta\|^p_{L^p_W(|x|\le 2)} + \|Ta\|^p_{L^p_W(|x|>2)}.$$

For the first integral on the righthand side, we have

$$\begin{aligned}\|Ta\|^p_{L^p_W(|x|\le 2)} &= \int_{|x|\le 2} |Ta(x)|^p W(x)\,dx \\ &\le \left(\int_{|x|\le 2} |Ta|^{pr'}\,dx\right)^{1/r'} \left(\int_{|x|\le 2} (W(x)^r\,dx)\right)^{1/r} \\ &\equiv \Omega.\end{aligned}$$

If $pr' > 1$, applying Theorem C and the hypothesis $W \in RH_r$, one has

$$\begin{aligned}\Omega &\le C\|a\|^p_{pr'}\left(\int_{|x|\le 2} (W(x)^r\,dx)\right)^{1/r} \\ &\le C\|a\|^p_{pr'}\int_{|x|\le 2} W(x)\,dx \\ &\le CW(B)^{-1}W(B) \le C.\end{aligned}$$

On the other hand, if $pr' \le 1$, then one can use Hölder's inequality to raise the exponent of pr', that is,

$$\Omega \le C\bigg(\int_{|x|\le 2} |Ta|^{pr's}\,dx\bigg)^{1/(sr')} W(B),$$

where $pr's > 1$. Therefore,

$$\|Ta\|^p_{L^p_W(|x|\le 2)} \le C.$$

Next we compute

$$\|Ta\|^p_{L^p_W(|x|>2)}.$$

Let us denote $A_k = \{2^k \le |x| < 2^{k+1}\}$ and write

$$\begin{aligned}\int_{|x|>2} |Ta|^p W(x)\,dx &= \sum_{k=1}^{\infty} \int_{A_k} |Ta|^p W(x)\,dx \\ &\le \sum_{k=1}^{\infty} \bigg(\int_{A_k} |Ta| W\bigg)^p \bigg(\int_{A_k} W\bigg)^{1-p} \\ &= \sum_{k=1}^{\infty} \bigg(\int_{A_k} |Ta| W\bigg)^p W(A_k)^{1-p}.\end{aligned}$$

First we decompose the integral on the last equality and use (2).

$$\begin{aligned}\int_{A_k} |Ta| W &\le \sum_{j=-\infty}^{\infty} 2^{jn} \int_{A_k} |K_j * a_j(2^j x)| W(x)\,dx \\ &= \sum_{j=-\infty}^{\infty} \int_{A_{k+j}} |(K_j * a_j)(x)| W(2^{-j}x)\,dx \\ &= \sum_{j=-\infty}^{-k+1} \cdots + \sum_{j=-k+2}^{0} \cdots + \sum_{j=1}^{\infty} \cdots \equiv I + II + III.\end{aligned}$$

Therefore,

$$
\begin{aligned}
&\int_{|x|>2} |Ta|^p W(x)\,dx \\
&\qquad \le C\sum_{k=1}^{\infty} I^p W(A_k)^{1-p} && (3)\\
&\qquad\quad + C\sum_{k=1}^{\infty} II^p W(A_k)^{1-p} && (4)\\
&\qquad\quad + C\sum_{k=1}^{\infty} III^p W(A_k)^{1-p}. && (5)
\end{aligned}
$$

By an easy observation, one has $K_1^{\alpha,p} \subset L^1$ if $\alpha \ge 0$, $p \le 1$. By (i) of Lemma 1,

$$
\begin{aligned}
\|K_j * a_j\|_\infty &\le \|K_j\|_{L^1}\|a_j\|_\infty \\
&\le C\|K_j\|_{K_1^{n(q/p+1/r-1),p}} 2^{j(N+1)} W(B)^{-1/p}.
\end{aligned}
$$

We write

$$
\begin{aligned}
I &= \sum_{j=-\infty}^{-k+1} \int_{A_{k+j}} |(K_j * a_j)(x)| W(2^{-j}x)\,dx \\
&\le C\sum_{j=-\infty}^{-k+1} 2^{j(N+1)} W(B)^{-1/p} W(A_k) 2^{jn} \\
&\le C\sum_{j=-\infty}^{-k+1} 2^{j(N+1+n)} W(B)^{-1/p} 2^{knq} W(B) \\
&\le C 2^{knq} W(B)^{1-1/p} \sum_{j=-\infty}^{-k+1} 2^{j(N+1+n)} \\
&= C 2^{k(nq-N-1-n)} W(B)^{1-1/p}.
\end{aligned}
$$

Therefore,

$$\begin{aligned}(3) &= \sum_{k=1}^{\infty} I^p W(A_k)^{1-p} \\ &\leq \sum_{k=1}^{\infty} 2^{kp(nq-N-1-n)} W(B)^{p-1} 2^{knq(1-p)} W(B)^{1-p} \\ &\leq C\end{aligned}$$

if N is chosen large enough such that

$$N > (nq - np - p)/p. \tag{6}$$

Next, for the term (4),

$$\begin{aligned}(4) &= \sum_{k=1}^{\infty} II^p W(A_k)^{1-p} \\ &\leq \sum_{k=1}^{\infty} \left(\sum_{j=-k+2}^{0} \int_{A_{k+j}} |(K_j * a_j)(x)| W(2^{-j}x)\, dx \right)^p W(A_k)^{1-p} \\ &\leq \sum_{j=-\infty}^{1} \sum_{k=-j+2}^{\infty} \left(\int_{A_{k+j}} |(K_j * a_j)(x)| W(2^{-j}x)\, dx \right)^p W(A_k)^{1-p} \\ &= \sum_{j=-\infty}^{1} \sum_{l=2}^{\infty} \left(\int_{A_l} |(K_j * a_j)(x)| W(2^{-j}x)\, dx \right)^p W(A_{l-j})^{1-p}.\end{aligned}$$

Here we decompose the convolution function $K_j * a_j$ into three parts.

$$\begin{aligned}K_j * a_j(x) &= \sum_{i=-\infty}^{\infty} \int_{A_i} a_j(x-y) K_j(y)\, dy \\ &= \sum_{i=-\infty}^{l-2} \cdots + \sum_{i=l-1}^{l+1} \cdots + \sum_{i=l+2}^{\infty} \cdots \\ &\equiv II_A + II_B + II_C.\end{aligned} \tag{7}$$

Hence, (4) is dominated by

$$\sum_{j=-\infty}^{1}\sum_{l=2}^{\infty}\left(\int_{A_l}|II_A|W(2^{-j}x)\,dx\right)^p W(A_{l-j})^{1-p} \tag{8}$$

$$+\sum_{j=-\infty}^{1}\sum_{l=2}^{\infty}\left(\int_{A_l}|II_B|W(2^{-j}x)\,dx\right)^p W(A_{l-j})^{1-p} \tag{9}$$

$$+\sum_{j=-\infty}^{1}\sum_{l=2}^{\infty}\left(\int_{A_l}|II_C|W(2^{-j}x)\,dx\right)^p W(A_{l-j})^{1-p}. \tag{10}$$

For $l \geq 2$, if $i \leq l-2$ or $i \geq l+2$, $y \in A_i$, $x \in A_l$, then $|x-y| \geq 2^{l-1}$. Therefore, by (i) of Lemma 1, II_A is bounded by

$$\begin{aligned}\sum_{i=-\infty}^{l-2}\int_{A_i}&|a_j(x-y)|\,|K_j(y)|\,dy\\ &\leq C\sum_{i=-\infty}^{l-2}W(B)^{-1/p}2^{j(N+1)}2^{-l\beta}\int_{A_i}|K_j(y)|\,dy\\ &\leq C2^{j(N+1)}2^{-l\beta}W(B)^{-1/p}\|\hat{m}_j\|^p_{K_1^{n(q/p+1/r-1),p}}\\ &\leq C2^{j(N+1)}2^{-l\beta}W(B)^{-1/p}.\end{aligned}$$

This shows (8) is bounded by

$$\begin{aligned}\sum_{j=-\infty}^{1}\sum_{l=2}^{\infty}&2^{jp(N+1)}2^{-l\beta p}W(B)^{-1}\left(\int_{A_l}W(2^{-j}x)\,dx\right)^p W(A_{l-j})^{1-p}\\ &\leq \sum_{j=-\infty}^{1}\sum_{l=2}^{\infty}W(B)^{-1}2^{-l\beta p}2^{j(N+n+1)p}W(A_{l-j})\\ &\leq \sum_{j=-\infty}^{1}\sum_{l=2}^{\infty}2^{l(-\beta p+nq)}2^{j[(N+n+1)p-nq]}\leq C,\end{aligned}$$

if β is chosen large enough and N satisfies (6).

Following the same proof as in the proof of (8), one can show that (10) is uniformly bounded. Denote $\tilde{A}_l = A_{l-1}\cup A_l\cup A_{l+1}$. Hence, (9) can be written as

(11)
$$\sum_{j=-\infty}^{1}\sum_{l=2}^{\infty}\left(\int_{A_l}\left|\int_{\tilde{A}_l} a_j(x-y)K_j(y)\,dy\right|W(2^{-j}x)\,dx\right)^p W(A_{l-j})^{1-p}.$$

The double integrals in the above formula (11) are bounded by

$$(12)\qquad \begin{aligned}\int_{A_l}\int_{\tilde{A}_l} &|a_j(x-y)K_j(y)|\,dy W(2^{-j}x)\,dx\\ &= \int_{\tilde{A}_l}\int_{A_l}|a_j(x-y)|W(2^{-j}x)\,dx|K_j(y)|\,dy.\end{aligned}$$

Applying Hölder's inequality, (12) is not bigger than

$$\int_{\tilde{A}_l}\left[\left(\int_{A_l}|a_j(x-y)|^{r'}\,dx\right)^{1/r'}\left(\int_{A_l}|W(2^{-j}x)|^r\,dx\right)^{1/r}\right]|K_j(y)|\,dy.$$

For $j\le 1$, employing (i) of Lemma 1 and $W\in RH_r$, the above formula is bounded by

$$\begin{aligned}&W(B)^{-1/p}2^{j(N+1)}2^{jn/r}2^{(l-j)n/r}\\ &\quad\cdot\left(\frac{1}{|A_{l-j}|}\int_{A_{l-j}}|W(y)|^r\,dy\right)^{1/r}\int_{\tilde{A}_l}|K_j(y)|\,dy\\ &\quad\le CW(B)^{-1/p}2^{j(N+1)}2^{ln/r}\frac{1}{|A_{l-j}|}\int_{A_{l-j}}|W(y)|\,dy\int_{\tilde{A}_l}|K_j(y)|\,dy\\ &\quad\le CW(B)^{-1/p}2^{j(N+1)}2^{ln/r}2^{(j-1)n}W(A_{l-j})\int_{\tilde{A}_l}|K_j(y)|\,dy\\ &\quad\le CW(B)^{1-1/p}2^{j(N+1+n-nq)}2^{l(n/r-n+nq)}\int_{\tilde{A}_l}|K_j(y)|\,dy.\end{aligned}$$

Therefore,

$$(9) \le C \sum_{j=-\infty}^{1} \sum_{l=2}^{\infty} W(B)^{p-1} 2^{jp(N+1+n-nq)} 2^{lp(n/r-n+nq)} 2^{(l-j)nq(1-p)} \cdot W(B)^{1-p} \left(\int_{\tilde{A}_l} |K_j(y)|\, dy \right)^p$$
$$= C \sum_{j=-\infty}^{1} \sum_{l=2}^{\infty} 2^{j[p(N+n+1)-nq]} 2^{nlp(1/r-1+q/p)} \left(\int_{\tilde{A}_l} |K_j(y)|\, dy \right)^p$$
$$\le C \|\hat{m}_j\|^p_{K_1^{n(q/p+1/r-1),p}},$$

if N satisfies (6). Combining all of those estimates, we show (4) is uniformly bounded.

Finally, we need to estimate (5), i.e., $\sum_{k=1}^{\infty} III^p W(A_k)^{1-p}$. We write (5) as

$$(13) \quad \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \Big(\int_{A_{k+j}} |(K_j * a_j)(x)| W(2^{-j}x)\, dx \Big)^p W(A_k)^{1-p}$$
$$= \sum_{j=1}^{\infty} \sum_{l=j+1}^{\infty} \left(\int_{A_l} |(K_j * a_j)(x)| W(2^{-j}x)\, dx \right)^p W(A_{l-j})^{1-p}.$$

From (7) and (13), we see

$$(5) = \sum_{k=1}^{\infty} III^p W(A_k)^{1-p}$$
$$(14) \qquad \le \sum_{j=1}^{\infty} \sum_{l=j+1}^{\infty} \Big(\int_{A_l} |II_A| W(2^{-j}x)\, dx)^p W(A_{l-j})^{1-p}$$
$$(15) \qquad + \sum_{j=1}^{\infty} \sum_{l=j+1}^{\infty} \left(\int_{A_l} |II_B| W(2^{-j}x)\, dx \right)^p W(A_{l-j})^{1-p}$$
$$(16) \qquad + \sum_{j=1}^{\infty} \sum_{l=j+1}^{\infty} \left(\int_{A_l} |II_C| W(2^{-j}x)\, dx \right)^p W(A_{l-j})^{1-p}.$$

We sketch the proof of (14) and (16) since they are similar to the proof of (8) and (10), respectively. In the proof of (8) and (10), we used inequality (i) of Lemma 1. For the proof of (14) and (16), we will apply inequality (ii) of Lemma 1.

For $l \geq j+1$ if $i \leq l-2$, or $i \geq l+2$, $y \in A_i$, $x \in A_l$, then $|x-y| > 2^l \geq 2^{j+1}$. Therefore, applying (ii) of Lemma 1,

$$|a_j(x-y)| \leq C_\beta W(B)^{-1/p} 2^{-l\beta}$$

if $|x-y| \geq 2^{j+1}$. Hence, if $l \geq j+1$, then

$$\begin{aligned}
&|II_A| + |II_C| \\
&\quad \leq \sum_{i=-\infty}^{l-2} \int_{A_i} |a_j(x-y)||K_j(y)|\,dy + \sum_{i=l+2}^{\infty} \int_{A_i} |a_j(x-y)||K_j(y)|\,dy \\
&\quad \leq C \int_{A_i} |K_j(y)|\,dy W(B)^{-1/p} 2^{-l\beta} \\
&\quad \leq C \|\hat{m}_j\|^p_{K_1^{n(q/p+1/r-1),p}} W(B)^{-1/p} 2^{-l\beta}.
\end{aligned}$$

Let us write

$$\begin{aligned}
&(14) + (16) \\
&\quad \leq C \sum_{j=1}^{\infty} \sum_{j+1}^{\infty} \left(\int_{A_l} (|II_A| + |II_C|) W(2^{-j}x)\,dx \right)^p W(A_{l-j})^{1-p} \\
&\quad \leq C \sum_{j=1}^{\infty} \sum_{l=j+1}^{\infty} W(B)^{-1} 2^{lp\beta} \left(\int_{A_i} W(2^{-j}x)\,dx \right)^p W(A_{l-j})^{1-p} \\
&\quad \leq C \sum_{j=1}^{\infty} \sum_{l=j+1}^{\infty} W(B)^{-1} 2^{-lp\beta} 2^{jnp} W(A_{l-j}) \\
&\quad \leq C \sum_{j=1}^{\infty} \sum_{l=j+1}^{\infty} 2^{-l(p\beta - nq)} 2^{-j(nq-np)} \leq C
\end{aligned}$$

if β is large.

For estimating (15), we write

$$\begin{aligned}
(15) &= \sum_{j=1}^{\infty}\sum_{l=j+1}^{\infty}\left(\int_{A_l}|II_B|W(2^{-j}x)\,dx\right)^p W(A_{l-j})^{1-p}\\
&\le \sum_{j=1}^{\infty}\sum_{l=j+1}^{\infty}\left(\int_{A_l}\int_{\tilde{A}_l}|K_j(y)||a_j(x-y)|\,dyW(2^{-j}x)\,dx\right)^p\\
&\qquad\qquad\qquad\qquad\qquad\qquad\cdot W(A_{l-j})^{1-p}\\
&\le \sum_{j=1}^{\infty}\sum_{l=j+1}^{\infty}\left(\left(\left(\int_{\tilde{A}_l}\left(\int_{A_l}|a_j(x-y)|^{r'}\,dx\right)^{1/r'}\right.\right.\right.\\
&\qquad\left.\left.\left.\cdot\left(\int_{A_l}|W(2^{-j}x)|^r\,dx\right)^{1/r}\right)|K_j(y)|\,dy\right)^p W(A_{l-j})^{1-p}.
\end{aligned}$$

By applying (ii) of Lemma 1 with large β and the fact that $W \in RH_r$, one has

$$\begin{aligned}
(15) &\le \sum_{j=1}^{\infty}\sum_{l=j+1}^{\infty}\left(\int_{\tilde{A}_l}|K_j(y)|\,dyW(B)^{-1/p}(2^{jn/r'-jn}+2^{-j(\beta-n)/r'})\right.\\
&\qquad\qquad\qquad\left.\cdot\left(\int_{A_l}|W(2^{-j}x)|^r\,dx\right)^{1/r}\right)^p W(A_{l-j})^{1-p}\\
&\le C\sum_{j=1}^{\infty}\sum_{l=j+1}^{\infty}W(B)^{p-1}2^{lp(n/r-n+nq)}2^{-jp(n/r-n+nq)}2^{(l-j)nq(1-p)}\\
&\qquad\qquad\qquad\qquad\cdot\left(\int_{\tilde{A}_l}|K_j(y)|\,dy\right)^p W(B)^{1-p}\\
&\le C\sum_{j=1}^{\infty}\sum_{l=j+1}^{\infty}\left(\int_{\tilde{A}_l}|K_j(y)|\,dy2^{ln(q/p+1/r-1)}\right)^p 2^{jn(p-p/r-q)}\\
&\le C\|\hat{m}_j\|^p_{K_1^{n(q/p+1/r-1),p}},
\end{aligned}$$

since $q/p+1/r-1>0$.

So far we have proved that if $W \in D_{nq}\cap RH_r$, then

$$\|Ta\|_{L^p_W(R^n)} \le C$$

for any N-atom, a, with support in a unit ball. To remove the assumption "the radius of ball is 1," suppose an N-atom a is supported on a ball B_s with radius s, center at origin, and let $a_s(x) = a(sx)$. Note that the support of a_s is a unit ball, B_1, and $\hat{a}_s(\xi) = s^{-n}\hat{a}(s^{-1}\xi)$. Let us write

$$
\begin{aligned}
Ta(x) &= \int m(\xi)\hat{a}(\xi)e^{ix\cdot\xi}\,d\xi \\
&= \int m(s^{-1}\xi)\hat{a}_s(\xi)e^{is^{-1}x\cdot\xi}\,d\xi \qquad (17)\\
&\equiv T^s a_s(s^{-1}x).
\end{aligned}
$$

From the hypothesis of the multiplier $m(\xi)$,

$$\sup_{\delta>0}\|\hat{m}_\delta\|_{K_1^n(q/p+1/r-1),p} < \infty,$$

one concludes that if the support of an N-atom b is in a unit ball, then

$$\|T^s b\|_{L^p_W(R^n)} \le C$$

where C is independent on s.

On the other hand, write $W_s(x) = W(sx)$. Since $W \in D_{nq}$ it is clear to see $W_s \in D_{nq}$ and W_s also satisfy the reverse Hölder condition RH_r if W does. More precisely, if

$$\left(\frac{1}{|B|}\int_B (W(x)^r\,dx\right)^{1/r} \le C\,\frac{1}{|B|}\int_B W(x)\,dx$$

for all balls B on R^n with center at origin, then

$$\left(\frac{1}{|B|}\int_B |W_s(x)|^r\,dx\right)^{1/r} \le C\,\frac{1}{|B|}\int_B W_s(x)\,dx$$

for all balls B on R^n with center origin where C again is independent on s. Hence, if b is an N-atom with respect to weight W_s and with support on a unit ball, then

$$\|T^s b\|_{L^p_{W_s}(R^n)} \le C. \qquad (18)$$

Recall that the support of $a_s(x)$ is in a unit ball B_1 and B_s is a ball with center 0 and radius s. Since

$$\|a_s(x)\|_\infty = \|a(sx)\|_\infty \le CW(B_s)^{-1/p}$$

and

$$\begin{aligned} W(B_s) &= \int_{B_S} W(x)\,dx = s^n \int_{B_1} W(sy)\,dy \\ &= s^n \int_{B_1} W_s(y)\,dy = s^n W_s(B_1), \end{aligned}$$

we have

$$\|a_s(x)\|_\infty \le Cs^{-n/p} W_s(B_1)^{-1/p}.$$

It implies

$$\|s^{n/p} a_s(x)\|_\infty \le CW_s(B_1)^{-1/p}$$

where $s^{n/p}a_s(x)$ is an N-atom with respect to weight W_s and with support B_1.

Therefore, applying (17) and (18),

$$\begin{aligned} \int |Ta(x)|^p W(x)\,dx &= \int |T^s a_s(s^{-1}x)|^p W(x)\,dx \\ &= s^n \int |T^s a_s(x)|^p W_s(x)\,dx \\ &= \int |T^s s^{n/p} a_s(x)|^p W_s(x)\,dx \le C. \end{aligned}$$

This completes the proof of the Theorem.

Conclusion. Comparing the main theorem in this paper with Theorems 3a and 3b in [**1**] raises several questions which we are not presently able to answer.

(1) The results in the main theorem with $W(x) = 1$ in this paper are weaker than those in Theorem 3a in [**1**]. Does a sharper result than the one implied by the theorem hold in the nonweighted case, i.e., Theorem 3a in [**1**]?

(2) There is a clear difference between the results for the H^p multipliers for $0 < p < 1$ and for $p = 1$ in the paper [**1**]. Does such a difference also exist in the case of the weighted Hardy spaces in this paper?

(3) The authors, Baernstein and Sawyer, were able to prove the sharpness of their results in [**1**]. Does a comparable sharpness result hold for the theorem in this paper?

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