ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 31, Number 1, Spring 2001

EXISTENCE OF THREE SOLUTIONS TO INTEGRAL AND DISCRETE EQUATIONS VIA THE LEGGETT WILLIAMS FIXED POINT THEOREM

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ABSTRACT. Criteria are developed for the existence of three nonnegative solutions to integral and discrete equations. The strategy involves using the Leggett Williams fixed point theorem.

1. Introduction. In this paper we present results which guarantee the existence of three nonnegative solutions to integral and discrete equations. The results we establish are new since this is the first paper, to our knowledge, that discusses the existence of three nonnegative solutions to integral equations. In addition, the results in this paper contain almost all results in the recent papers [**3**–**6, 8, 9**] on the existence of three solutions to higher order differential and difference equations since we make full use of the properties of the concave functional on the cone. Indeed, if we assume the conditions in [**3**–**6, 8, 9**], then the conditions in this paper are trivially satisfied.

For the remainder of the introduction we present some preliminaries which will be needed in Sections 2 and 3. Let $E = (E, \|.\|)$ be a Banach space and $C \subset E$ a cone. By a concave nonnegative continuous functional ψ on C we mean a continuous mapping $\psi : C \to [0, \infty)$ with

$$\psi(\lambda x + (1-\lambda)y) \geq \lambda\psi(x) + (1-\lambda)\psi(y)$$
$$\text{for all } x, y \in C \quad \text{and } \lambda \in [0,1].$$

Let $K, L, r > 0$ be constants with C and ψ as defined above. We let

$$C_K = \{y \in C : \|y\| < K\}$$

and

$$C(\psi, r, L) = \{y \in C : \psi(y) \geq r \text{ and } \|y\| \leq L\}.$$

Received by the editors on August 2, 1999, and in revised form on January 27, 2000.

We now state the Leggett Wiliams fixed point theorem [**5, 6**].

Theorem 1.1. *Let $E = (E, \|.\|)$ be a Banach space, $C \subset E$ a cone of E and $R > 0$ a constant. Suppose a concave nonnegative continuous functional ψ exists on C with $\psi(y) \le \|y\|$ for $y \in \overline{C_R}$, and let $A : \overline{C_R} \to \overline{C_R}$ be a continuous, compact map. Assume there are numbers r, L and K with $0 < r < L < K \le R$ such that*

(H1) $\{y \in C(\psi, L, K) : \psi(y) > L\} \ne \varnothing$ *and* $\psi(Ay) > L$ *for all* $y \in C(\psi, L, K)$;

(H2) $\|Ay\| < r$ *for all* $y \in \overline{C_r}$;

(H3) $\psi(Ay) > L$ *for all* $y \in C(\psi, L, R)$ *with* $\|Ay\| > K$.

Then A has at least three fixed points y_1, y_2 and y_3 in $\overline{C_R}$. Furthermore, we have

$$y_1 \in C_r, \quad y_2 \in \{y \in C(\psi, L, R) : \psi(y) > L\}$$

and

$$y_3 \in \overline{C_R} \setminus (C(\psi, L, R) \cup \overline{C_r}).$$

2. Integral equations. In this section we discuss the integral equation

$$y(t) = h(t) + \int_0^1 k(t,s) f(y(s))\, ds \quad \text{for } t \in [0,1]. \tag{2.1}$$

The following conditions will be assumed:

$$f : [0,\infty) \longrightarrow [0,\infty) \quad \text{is continuous and nondecreasing} \tag{2.2}$$

$$\begin{gathered} k_t(s) = k(t,s) \in L^1[0,1] \quad \text{with } k_t \ge 0 \\ \text{a.e. on } [0,1], \quad \text{for each } t \in [0,1] \end{gathered} \tag{2.3}$$

$$\text{the map} \quad t \longmapsto k_t \quad \text{is continuous from } [0,1] \text{ to } L^1[0,1] \tag{2.4}$$

$$\begin{gathered} \exists\, r > 0 \quad \text{with } |h|_0 + f(r) \sup_{t\in[0,1]} \int_0^1 k(t,s)\, ds < r \\ \left(\text{here} \quad |h|_0 = \sup_{t\in[0,1]} |h(t)|\right) \end{gathered} \tag{2.5}$$

(2.6)

$$\begin{cases} \exists M,\ 0 < M < 1,\ \kappa \in L^1[0,1] \text{ and an interval } [a,b] \subseteq [0,1],\ a < b, \\ \text{such that } k(t,s) \geq M\kappa(s) \geq 0 \text{ for } t \in [a,b] \text{ and a.e. } s \in [0,1] \end{cases}$$

$$k(t,s) \leq \kappa(s), \quad t \in [0,1], \quad \text{a.e. } s \in [0,1] \tag{2.7}$$

$$\begin{cases} h \in C[0,1] \text{ with } h(t) \geq 0 \text{ for } t \in [0,1] \\ \text{and } \min_{t\in[a,b]} h(t) \geq M|h|_0 \end{cases} \tag{2.8}$$

$$\exists L > r \quad \text{with} \min_{t\in[a,b]} \left[h(t) + f(L) \int_a^b k(t,s)\, ds \right] > L \tag{2.9}$$

and

$$\exists R \geq LM^{-1} \quad \text{with } |h|_0 + f(R) \sup_{t\in[0,1]} \int_0^1 k(t,s)\, ds \leq R. \tag{2.10}$$

Theorem 2.1. *Suppose* (2.2)–(2.10) *hold. Then* (2.1) *has three nonnegative solutions* y_1, y_2 *and* y_3 *in* $C[0,1]$ *with*

$$|y_1|_0 < r, \quad y_2(t) > L \quad \textit{for } t \in [a,b]$$

and

$$|y_3|_0 > r \quad \textit{with} \min_{t\in[a,b]} y_3(t) < L.$$

Proof. Let

$$E = (C[0,1], |.|_0) \quad \text{and} \quad C = \{u \in C[0,1] : u(t) \geq 0 \text{ for } t \in [0,1]\}.$$

Now let $A : C \to C$ be defined by

$$Ay(t) = h(t) + \int_0^1 k(t,s) f(y(s))\, ds \quad \text{for } t \in [0,1]; \tag{2.11}$$

here $y \in C$. It is immediate (see (2.2), (2.3) and (2.4)) from the results in [**7**] that

$$A : C \longrightarrow C \quad \text{is continuous and completely continuous.}$$

For $y \in C$ let

$$\psi(y) = \min_{t \in [a,b]} y(t).$$

Next choose and *fix* K so that

$$LM^{-1} \le K \le R; \tag{2.12}$$

this is possible since $R \ge LM^{-1}$. Let

$$C_r = \{y \in C : |y|_0 < r\}, \qquad C_R = \{y \in C : |y|_0 < R\}$$

and

$$C(\psi, L, K) = \{y \in C : \psi(y) \ge L \text{ and } |y|_0 \le K\},$$
$$C(\psi, L, R) = \{y \in C : \psi(y) \ge L \text{ and } |y|_0 \le R\}.$$

First notice condition (H2) of Theorem 1.1 holds since, for $y \in \overline{C_r}$, we have from (2.2), (2.5) and (2.11) that

$$|Ay|_0 \le |h|_0 + f(r) \sup_{t \in [0,1]} \int_0^1 k(t,s)\, ds < r.$$

Also $A : \overline{C_R} \to \overline{C_R}$ since if $y \in \overline{C_R}$,

$$|Ay|_0 \le |h|_0 + f(R) \sup_{t \in [0,1]} \int_0^1 k(t,s)\, ds \le R.$$

Next we show (H1) of Theorem 1.1 holds. First notice if

$$u(t) = \frac{L+K}{2} \quad \text{for } t \in [0,1]$$

then $u \in \{y \in C(\psi, L, K) : \psi(y) > L\}$. Also if $y \in C(\psi, L, K)$ then $\psi(y) = \min_{t \in [a,b]} y(t) \ge L$ and $|y|_0 \le K$, so

$$y(t) \in [L, K] \quad \text{for } t \in [a,b].$$

This together with (2.9) yields

$$\begin{aligned}\psi(Ay) &= \min_{t\in[a,b]}\left(h(t)+\int_0^1 k(t,s)f(y(s))\,ds\right)\\ &\geq \min_{t\in[a,b]}\left(h(t)+\int_a^b k(t,s)f(y(s))\,ds\right)\\ &\geq \min_{t\in[a,b]}\left(h(t)+f(L)\int_a^b k(t,s)\,ds\right) > L,\end{aligned}$$

so condition (H1) of Theorem 1.1 is satisfied. Finally, to see that (H3) of Theorem 1.1 holds, let $y \in C(\psi, L, R)$ with $|Ay|_0 > K$. First notice (2.7) and (2.11) imply

$$|Ay|_0 \leq |h|_0 + \int_0^1 \kappa(s)f(y(s))\,ds$$

and this together with (2.6), (2.8) and (2.12) yields

$$\begin{aligned}\psi(Ay) &= \min_{t\in[a,b]}\left(h(t)+\int_0^1 k(t,s)f(y(s))\,ds\right)\\ &\geq M|h|_0 + M\int_0^1 \kappa(s)f(y(s))\,ds\\ &\geq M|Ay|_0 > MK \geq L.\end{aligned}$$

Thus condition (H3) of Theorem 1.1 holds. Now apply Theorem 1.1. □

Remark 2.1. Notice (2.3) and (2.4) can be replaced by any conditions which guarantee that the map $A : C \to C$ is continuous and completely continuous.

To illustrate how Theorem 2.1 can be applied to nth ($n \geq 2$) order boundary value problems, we consider the Lidstone boundary value problem

$$\begin{cases}(-1)^n y^{(2n)} = \phi(t)f(y) & t\in[0,1],\\ y^{(2i)}(0)=0,\ y^{(2i)}(1)=0 & 0\leq i\leq n-1.\end{cases} \tag{2.13}$$

The Green's function $g_n(t,s)$ for the boundary value problem

$$\begin{cases} y^{(2n)} = 0 & \text{on } [0,1] \\ y^{(2i)}(0) = 0,\ y^{(2i)}(1) = 0 & 0 \le i \le n-1 \end{cases} \tag{2.14}$$

satisfied (see [**1, 9**]),

$$(-1)^n g_n(t,s) \le \frac{1}{6^{n-1}}\, s(1-s) \quad \text{for } (t,s) \in [0,1]\times[0,1] \tag{2.15}$$

and

$$(-1)^n g_n(t,s) \ge \frac{1}{4^n}\left(\frac{3}{32}\right)^{n-1} s(1-s) \quad \text{for } (t,s) \in \left[\frac{1}{4},\frac{3}{4}\right]\times[0,1]. \tag{2.16}$$

Theorem 2.2. *Assume the following conditions hold*:

$$f:[0,\infty) \longrightarrow [0,\infty) \quad \textit{is continuous and nondecreasing} \tag{2.17}$$

$$\phi \in C(0,1) \quad \textit{with } \phi > 0 \quad \textit{on } (0,1) \quad \textit{and} \quad \int_0^1 t(1-t)\phi(t)\,dt < \infty \tag{2.18}$$

$$\begin{cases} \lim_{t\to 0^+} t^2(1-t)\phi(t) = 0 & \textit{if } \int_0^1 (1-t)\phi(t)\,dt = \infty \\ \textit{and} & \\ \lim_{t\to 1^-} t(1-t)^2\phi(t) = 0 & \textit{if } \int_0^1 t\phi(t)\,dt = \infty \end{cases} \tag{2.19}$$

$$\exists r > 0 \quad \textit{with } f(r) \sup_{t\in[0,1]} \int_0^1 (-1)^n g_n(t,s)\phi(s)\,ds < r \tag{2.20}$$

$$\exists L > r \quad \textit{with } f(L) \min_{t\in[(1/4),(3/4)]} \int_{1/4}^{3/4} (-1)^n g_n(t,s)\phi(s)\,ds > L \tag{2.21}$$

and

$$\exists R \geq L\left[4^n \left(\frac{32}{3}\right)^{n-1} \frac{1}{6^{n-1}}\right] \tag{2.22}$$

with

$$f(R) \sup_{t\in[0,1]} \int_0^1 (-1)^n g_n(t,s)\phi(s)\, ds \leq R.$$

Then (2.13) *has three nonnegative solutions* y_1, y_2 *and* y_3 *in* $C[0,1]$ *with*

$$|y_1|_0 < r, \quad y_2(t) > L \quad \textit{for } t \in \left[\frac{1}{4}, \frac{3}{4}\right]$$

and

$$|y_3|_0 > r \quad \textit{with} \min_{t\in[(1/4),(3/4)]} y_3(t) < L.$$

Proof. Let $A : C \to C$ (here C is as defined in Theorem 2.1) be defined by

$$Ay(t) = \int_0^1 (-1)^n g_n(t,s)\phi(s) f(y(s))\, ds \quad \text{for } y \in C.$$

The results in [**1**] guarantee that

$$A : C \longrightarrow C \quad \text{is continuous and completely continuous.}$$

Now apply Theorem 2.1 (with Remark 2.1) with

$$k(t,s) = (-1)^n g_n(t,s)\phi(s), \qquad h \equiv 0, \qquad a = \frac{1}{4}, \qquad b = \frac{3}{4}$$

and

$$\kappa(s) = \frac{1}{6^{n-1}}\, s(1-s), \qquad M = \frac{1}{4^n}\left(\frac{3}{32}\right)^{n-1} 6^{n-1}.$$

Notice (2.15) and (2.16) guarantee that (2.6) and (2.7) hold. □

3. Discrete equations. In this section we discuss the discrete equation

$$(3.1)\qquad y(i) = h(i) + \sum_{j=0}^{N} k(i,j) f(y(j)) \quad \text{for } i \in \{0,1,\dots,T\} = T^+;$$

here $N, T \in \mathbf{N} = \{1,2,\dots\}$ and $T \geq N$. Throughout this section we let $C(T^+, \mathbf{R})$ denote the class of maps w continuous on T^+ (discrete topology) with norm $|w|_0 = \sup_{i\in T^+} |w(i)|$. The following conditions will be assumed:

$$(3.2)\qquad f : [0,\infty) \longrightarrow [0,\infty) \quad \text{is continuous and nondecreasing}$$

$$(3.3)\qquad k(i,j) \geq 0 \quad \text{for } (i,j) \in T^+ \times N^+ \quad (\text{here } N^+ = \{0,1,\dots,N\})$$

$$(3.4)\qquad \exists r > 0 \quad \text{with } |h|_0 + f(r) \max_{i\in T^+} \sum_{j=0}^{N} k(i,j) < r$$

$$(3.5)\qquad \begin{cases} \exists M,\ 0 < M < 1,\ W \subseteq N^+ \text{ and } k_0 : T^+ \longrightarrow [0,\infty) \\ \text{with } k(i,j) \geq M k_0(j) \geq 0 \text{ for } (i,j) \in W \times N^+ \end{cases}$$

$$(3.6)\qquad k(i,j) \leq k_0(j) \quad \text{for } (i,j) \in T^+ \times N^+$$

$$(3.7)\qquad \begin{cases} h \in C(T^+, \mathbf{R}) \text{ with } h(i) \geq 0 \text{ for } i \in T^+ \\ \text{and } h(i) \geq M|h|_0 \text{ for } i \in W \end{cases}$$

(3.8)

$$\exists L > r \quad \text{with } \min_{i\in W} \left[h(i) + f(L) \sum_{j\in W} k(i,j) \right] > L$$

and

(3.9)

$$\exists R \geq L M^{-1} \quad \text{with } |h|_0 + f(R) \max_{i\in T^+} \sum_{j=0}^{N} k(i,j) \leq R.$$

Theorem 3.1. *Suppose (3.2)–(3.9) hold. Then (3.1) has three nonnegative solutions y_1, y_2 and y_3 in $C(T^+, \mathbf{R})$ with*

$$|y_1|_0 < r, \qquad y_2(i) > L \quad \textit{for } i \in W$$

and

$$|y_3|_0 > r \quad \textit{with } \min_{i \in W} y_3(i) < L.$$

Proof. Let

$$E = (C(T^+, \mathbf{R}), |.|_0)$$

and

$$C = \{y \in C(T^+, \mathbf{R}) : y(i) \geq 0 \text{ for } i \in T^+\}$$

and let $A : C \to C$ be given by

$$Ay(i) = h(i) + \sum_{j=0}^{N} k(i,j) f(y(j)) \quad \text{for } i \in T^+, \tag{3.10}$$

here $y \in C$. Now [**2**] guarantees that

$$A : C \longrightarrow C \quad \text{is continuous and completely continuous.}$$

For $y \in C$, let

$$\psi(y) = \min_{i \in W} y(i).$$

Next choose and *fix* K so that

$$LM^{-1} \leq K \leq R.$$

Let

$$C_r = \{y \in C : |y|_0 < r\}, \qquad C_R = \{y \in C : |y|_0 < R\}$$

and

$$C(\psi, L, K) = \{y \in C : \psi(y) \geq L \text{ and } |y|_0 \leq K\},$$
$$C(\psi, L, R) = \{y \in C : \psi(y) \geq L \text{ and } |y|_0 \leq R\}.$$

Now if $y \in \overline{C_r}$ then (3.4) and (3.10) imply

$$|Ay|_0 \leq |h|_0 + f(r) \left(\max_{i \in T^+} \sum_{j=0}^{N} k(i,j) \right) < r,$$

so condition (H2) of Theorem 1.1 holds. Similarly it is immediate (see (3.9)) that $A : \overline{C_R} \to \overline{C_R}$. If

$$u(i) = \frac{L+K}{2} \quad \text{for } i \in T^+,$$

then $u \in \{y \in C(\psi, L, K) : \psi(y) > L\}$. In addition, if $y \in C(\psi, L, K)$ then $\psi(y) \geq L$ and $|y|_0 \leq K$, so

$$y(i) \in [L, K] \quad \text{for } i \in W.$$

This together with (3.8) yields

$$\begin{aligned}
\psi(Ay) &= \min_{i \in W} \left(h(i) + \sum_{j=0}^{N} k(i,j) f(y(j)) \right) \\
&\geq \min_{i \in W} \left(h(i) + \sum_{j \in W} k(i,j) f(y(j)) \right) \\
&\geq \min_{i \in W} \left(h(i) + f(L) \sum_{j \in W} k(i,j) \right) > L,
\end{aligned}$$

so condition (H1) of Theorem 1.1 is satisfied. Now let $y \in C(\psi, L, R)$ with $|Ay|_0 > K$. Notice

$$|Ay|_0 \leq |h|_0 + \sum_{j=0}^{N} k_0(j) f(y(j))$$

and this together with (3.5) yields

$$\begin{aligned}
\psi(Ay) &= \min_{i \in W} \left(h(i) + \sum_{j=0}^{N} k(i,j) f(y(j)) \right) \\
&\geq M|h|_0 + M \sum_{j=0}^{N} k_0(j) f(y(j)) \\
&\geq M|Ay|_0 > MK \geq L.
\end{aligned}$$

Thus condition (H3) of Theorem 1.1 holds. Now apply Theorem 1.1. □

Consider the (n,p) discrete problem $(n \geq 2,\ p \geq 1)$,

$$(3.11)\qquad \begin{cases} \Delta^n y(k) + f(k, y(k)) = 0, & k \in \{0, 1, \dots, N\} = N^+ \\ \Delta^i y(0) = 0, & 0 \leq i \leq n-2, \\ \Delta^p y(N+n-p) = 0, & 1 \leq p \leq n-1 \quad \text{is fixed;} \end{cases}$$

here $N \in \{1, 2, \dots\}$. Recall [**2, 3**] the Green's function $G(i,j)$ for the problem

$$(3.12)\qquad \begin{cases} -\Delta^n y(k) = 0 & \text{on } N^+ \\ \Delta^i y(0) = 0, & 0 \leq i \leq n-2 \\ \Delta^p y(N+n-p) = 0, & 1 \leq p \leq n-1 \quad \text{is fixed} \end{cases}$$

satisfies (here $T = N + n$),

$$(3.13)\qquad G(i,j) \leq \frac{(N+n)^{(n-1)}}{(n-1)!} \frac{(N+n-p-1-j)^{(n-p-1)}}{(N+n-p)^{(n-p-1)}}$$

for $(i,j) \in T^+ \times N^+$, and

$$(3.14)\qquad G(i,j) \geq \left[1 - \frac{N^{(p)}}{(N+1)^{(p)}}\right] \frac{(N+n-p-1-j)^{(n-p-1)}}{(N+n-p)^{(n-p-1)}}$$

for $(i,j) \in W \times N^+$; here $W = \{n-1, n, \dots, N+n-p\}$ and $t^{(m)} = t(t-1)\dots(t-m+1)$.

Theorem 3.2. *Let* $T = N + n$, $W = \{n-1, n, \dots, N+n-p\}$ *and assume the following conditions hold*:

$$(3.15)\qquad f : [0,\infty) \longrightarrow [0,\infty) \quad \textit{is continuous and nondecreasing}$$

$$(3.16)\qquad \exists r > 0 \quad \textit{with } f(r) \max_{i \in T^+} \sum_{j=0}^{N} G(i,j) < r$$

$$(3.17)\qquad \exists L > r \quad \textit{with } f(L) \min_{i \in W} \sum_{j \in W} G(i,j) > L$$

and

$$(3.18)\qquad \exists R \geq L \left[\left(1 - \frac{N^{(p)}}{(N+1)^{(p)}}\right) \frac{(n-1)!}{(N+n)^{(n-1)}}\right]^{-1}$$

with

$$f(R)\max_{i\in T^+}\sum_{j=0}^{N}G(i,j)\le R.$$

Then (3.11) *has three nonnegative solutions* y_1, y_2 *and* y_3 *in* $C(T^+,\mathbf{R})$ *with*

$$|y_1|_0 < r, y_2(i) > L \quad \textit{for } i\in W$$

and

$$|y_3|_0 > r \quad \textit{with } \min_{i\in W} y_3(i) < L.$$

Proof. Let $A : C \to C$ (here C is as defined in Theorem 3.1) be defined by

$$Ay(i)=\sum_{j=0}^{N}G(i,j)f(y(j)) \quad \text{for } i\in T^+,$$

here $y\in C$. We will now apply Theorem 3.1 with

$$k(i,j)=G(i,j), \qquad h\equiv 0, \qquad T=N+n,$$
$$W=\{n-1,\dots,N+n-p\},$$

together with

$$k_0(j)=\frac{(N+n)^{(n-1)}}{(n-1)!}\,\frac{(N+n-p-1-j)^{(n-p-1)}}{(N+n-p)^{(n-p-1)}}$$

and

$$M=\left[1-\frac{N^{(p)}}{(N+1)^{(p)}}\right]\frac{(n-1)!}{(N+n)^{(n-1)}}.$$

Notice (3.13) and (3.14) guarantee that (3.5) and (3.6) hold. □

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