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FACTORIZATION IN COMMUTATIVE RINGS WITH ZERO DIVISORS, III

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ABSTRACT. Let R be a commutative ring with identity. We continue our study of factorization in commutative rings with zero divisors. In Section 2 we consider inert extensions and atomicity. In Section 3 we characterize the atomic rings in which almost all atoms are prime. In Section 4 we investigate bounded factorization rings (BFR's) and U-BFR's, and in Section 5 we study finite factorization rings (FFR's).

1. Introduction. Throughout this paper, R will be a commutative ring with identity. This article is the third in a series of papers [**10**], [**11**] considering factorization in commutative rings with zero divisors. Here we concentrate on atomic rings, especially bounded factorization rings and finite factorization rings, which are defined below. We first review the various forms of irreducible elements introduced in [**10**].

For an integral domain R, a nonzero nonunit $a \in R$ is said to be irreducible or to be an atom if $a = bc$, $b, c \in R$, implies b or $c \in U(R)$, the group of units of R. It is easily checked that a is an atom $\Leftrightarrow$ (a) is a maximal (proper) principal ideal of $R \Leftrightarrow a = bc$ implies b or c is an associate of a. Now if R has zero divisors, these various characterizations of being irreducible no longer need to be equivalent. The following different forms of irreducibility are based on elements being associates. Let $a, b \in R$. Then a and b are *associates*, denoted $a \sim b$ if $a|b$ and $b|a$, i.e., $(a) = (b)$, a and b are *strong associates*, denoted $a \approx b$, if $a = ub$ for some $u \in U(R)$, and a and b are *very strong associates*, denoted $a \cong b$, if $a \sim b$ and either $a = 0$ or $a = cb$ implies $c \in U(R)$. Then a nonunit $a \in R$ (possibly with $a = 0$) is *irreducible* (respectively, *strongly irreducible, very strongly irreducible*), if $a = bc \Rightarrow a \sim b$ or $a \sim c$, respectively $a \approx b$ or $a \approx c$, $a \cong b$ or $a \cong c$. And a is m-*irreducible* if (a) is maximal in the set of proper principal ideals of R. A nonzero nonunit $a \in R$ is very strongly irreducible $\Leftrightarrow$ $a = bc$ implies b or $c \in U(R)$ [**10**, Theorem 2.5]. Now a is very

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strongly irreducible $\Rightarrow$ a is m-irreducible $\Rightarrow$ a is strongly irreducible $\Rightarrow$ a is irreducible (where in the first implication we assume $a \neq 0$). But examples given in [**10**] show that none of these implications can be reversed. As usual, a nonunit $p \in R$ is *prime* if (p) is a prime ideal of R. Finally, R is said to be *atomic* (respectively, *strongly atomic, very strongly atomic, m-atomic, p-atomic*), if each nonzero nonunit of R is a finite product of irreducible elements (respectively, strongly irreducible elements, very strongly irreducible elements, m-irreducible elements, prime elements). If R satisfies the ascending chain condition on principal ideals, ACCP, then R is atomic [**10**, Theorem 3.2]; if R is atomic, then R is a finite direct product of indecomposable rings [**10**, Theorem 3.3]; and a direct product $R = \prod_{\alpha \in \Lambda} R_\alpha$ of rings is atomic $\Leftrightarrow |\Lambda| < \infty$ and each R_α is atomic [**10**, Theorem 3.4.].

In Section 2 we consider (weakly) inert extensions (defined in Section 2) and atomicity. We show that if R satisfies any of the various forms of atomicity or ACCP, so does R_S, S a regular multiplicatively closed subset of R, in the case where $R \subset R_S$ is a weakly inert extension (e.g., if S is generated by regular primes).

In Section 3 we characterize the atomic rings, which we call generalized CK rings, with the property that almost all of their atoms are prime. We show that R is a generalized CK ring if and only if R is a finite direct product of finite local rings, SPIRs and generalized CK domains.

In Section 4 we study bounded factorization rings (BFR's) and U-BFR's. Recall that R is a BFR if, for each nonzero nonunit $a \in R$, a natural number $N(a)$ exists so that if $a = a_1 \cdots a_n$ where each a_i is nonunit, then $n \leq N(a)$. It is easily checked that a BFR satisfies ACCP and hence is atomic. Moreover, R is a BFR $\Leftrightarrow$ R is atomic and, for each nonzero nonunit $a \in R$, a natural number $N(a)$ exists so that if $a = a_1 \cdots a_n$ where each a_i is irreducible, then $n \leq N(a)$. Clearly a BFR can contain only trivial idempotents. In his study of unique factorization in commutative rings with zero divisors, Fletcher [**14**], [**15**] introduced the notion of a U-decomposition. For a nonunit $a \in R$, a U-decomposition of a is a decomposition $a = (a_1 \cdots a_n)(b_1 \cdots b_m)$ where each a_i, b_j is irreducible, $a_i(b_1 \cdots b_m) = (b_1 \cdots b_m)$ for $i = 1, \dots, n$ but $b_i(b_1 \cdots \hat{b}_i \cdots b_m) \neq (b_1 \cdots \hat{b}_i \cdots b_m)$. If we replace the condition that each a_i, b_j is irreducible by each a_i, b_j is a nonunit, we have what we call a U-factorization. We define a ring R to be a U-BFR if, for each nonzero nonunit $a \in R$, a natural number $N(a)$ exists so that for each U-factorization of a, $a = (a_1 \cdots a_n)(b_1 \cdots b_m)$, $m \leq N(a)$. Among other things, we show that a finite direct product of BFD's is a U-BFR and that any Noetherian ring is a U-BFR.

In Section 5 we consider finite factorization rings. Recall that a commutative ring R is a *finite factorization ring*, FFR, if each nonzero nonunit $a \in R$ has only a finite number of factorizations up to order and associates. We show that if (R, M) is a finite local ring with elements a and b such that Ra and $\mathrm{ann}\,(b)$ are not comparable, then $R[X]$ and $R[[X]]$ are not FFR's, but if (R, M) is an SPIR or $M^2 = 0$, then $R[[X]]$ is an FFR.

2. Inert extensions. Following Cohn [**13**], we say that an extension $A \subset B$ of commutative rings is a (*weakly*) *inert extension* if, whenever $(0 \neq xy \in A)$ $xy \in A$ for nonzero $x, y \in B$, then xu, $u^{-1}y \in A$ for some $u \in U(B)$. Clearly an inert extension is weakly inert, but not conversely, see Remark 2.2(b). Of course, if B is an integral domain, the two notions coincide. For the case of integral domains, factorization properties of inert extensions were investigated in [**4**]. We extend some of these results to commutative rings with zero divisors.

Proposition 2.1. *Let $A \subset B$ be a weakly inert extension of commutative rings. If $0 \neq a \in A$ is irreducible* (*respectively, strongly irreducible, very strongly irreducible, m-irreducible*)*, then as an element of B, either a is irreducible* (*respectively, strongly irreducible, very strongly irreducible, m-irreducible*) *or a is a unit.*

Proof. We may suppose that a is not a unit in B. If $a = xy$ in B, then $a = (xu)(u^{-1}y)$ where $xu, u^{-1}y \in A$ for some $u \in U(B)$. First suppose that a is irreducible in A. Then in A, $a \sim xu$ or $a \sim u^{-1}y$ and hence in B, $a \sim x$ or $a \sim y$. Thus a is irreducible in B. A similar proof holds for the case where a is strongly irreducible in A. Next suppose that a is very strongly irreducible in A. By [**10**, Theorem 2.5] xu or $u^{-1}y$ is a unit in A and so x or y is a unit in B. Thus by [**10**, Theorem 2.5] again, a is very strongly irreducible in B. A similar proof using [**10**, Theorem 2.12] shows that if a is m-irreducible in A, then a is also m-irreducible in B. $\square$

Remark 2.2. (a) Let $A \subset B$ be a weakly inert extension. Easy examples (for instance, use Proposition 2.3) show that even in the case where B is an integral domain, a can be irreducible in A but be a unit in B. However, if further, $U(B) \cap A = U(A)$, then $0 \neq a \in A$ satisfies any of the irreducibility conditions in $A \Leftrightarrow$ it satisfies the corresponding irreducibility condition in B.

(b) Note that in Proposition 2.1 it is necessary to assume $a \neq 0$. If we take $A = \mathbf{Z}_2$ and $B = \mathbf{Z}_2 \oplus \mathbf{Z}_2$ using the diagonal embedding, then $A \subset B$ is a weakly inert extension. Here $a = 0$ is irreducible, strongly irreducible, very strongly irreducible, m-irreducible and prime in A, but none of these in B. Also, note that while $A \subset B$ is a weakly inert extension, it is not an inert extension.

(c) In Proposition 2.1 we cannot add "a is prime." For $\mathbf{Z}_4 \subset \mathbf{Z}_4[u]$, where $u^2 = 0$, is a weakly inert extension (but not an inert extension), however, $\bar{2}$ is prime in $\mathbf{Z}_4$ but not in $\mathbf{Z}_4[u]$.

(d) Suppose that $A \subset B$ is a weakly inert extension where A is an integral domain. Then $A \subset B$ is an inert extension if and only if B is an integral domain. Hence if $A \subset B$ is an inert extension, we can allow $a = 0$ in Proposition 2.1 for a irreducible, strongly irreducible, very strongly irreducible, or prime, but not for a m-irreducible.

Let S be a regular multiplicative set of the commutative ring R. While in general $R \subset R_S$ need not be (weakly) inert (even when R is an integral domain, for example, take $R = k[X^2, X^3]$, k a field and $S = \{uX^n \mid u \in k^\times, n = 0 \text{ or } n \geq 2\}$), we next give a case where it is.

Proposition 2.3. *Let R be a commutative ring and S a multiplicative set of R generated by regular primes. Then $R \subset R_S$ is an inert extension.*

Proof. Suppose $xy \in R$ for some nonzero $x, y \in R_S$. We can write $x = a/s$, $y = b/t$ where $a, b \in R$, $s = p_1 \cdots p_n$, $t = q_1 \cdots q_m$ where $p_i, q_j \in S$ are primes and $p_i \nmid a$ and $q_j \nmid b$ in R. Then $ab = (xy)st = (xy)p_1 \cdots p_n q_1 \cdots q_m$. Hence $s|b$ and $t|a$ in R. Take $u = s/t$, a unit in R_S. Then xu, $u^{-1}y \in R$. Hence $R \subset R_S$ is an inert extension. □

However, $R \subset R_S$ can be an inert extension without S being generated by primes. For example, let $R = k[[Y^2, Y^3]]$ where k is a field, and let $S = \{uY^n \mid u \in U(R), n = 0 \text{ or } n \geq 2\}$. Then $R \subset R_S = k((Y))$ is inert, but S is not generated by primes. It is interesting to note that $R[X] \subset R_S[X]$ is not inert.

Proposition 2.4. *Let R be a commutative ring and S a regular multiplicative set of R. Suppose that $R \subset R_S$ is a weakly inert extension* (*e.g., S is generated by regular primes* (*Proposition* 2.3)). *If R is atomic* (*respectively, strongly atomic, very strongly atomic, m-atomic, p-atomic*), *then so is R_S. If R satisfies* ACCP*, so does R_S.*

Proof. Let r/s be a nonzero nonunit of R_S. Suppose that R is atomic. Then we can write $r = a_1 \cdots a_n$ where each a_i is an irreducible element of R. Then $r/s = (sa_1/s^2)(sa_2/s) \cdots (sa_n/s)$ where each factor is either irreducible or a unit in R_S by Proposition 2.1. Hence R_S is atomic. A similar proof holds for the other cases (in the case where R is p-atomic we use the fact that if p is a prime in R then p/s is either a prime or unit in R_S; here we do not need that $R \subset R_S$ is weakly inert). The proof that R satisfies ACCP implies R_S satisfies ACCP is identical to the domain case [**4**, Theorem 2.1]. $\square$

The notion of a splitting multiplicative set which played such an important role in [**4**] does not have a good analog for commutative rings with zero divisors. Recall that for an integral domain R, a saturated multiplicative set S of R is a *splitting multiplicative set* if for each $0 \neq x \in R$ we can write $x = as$ where $s \in S$ and $aR \cap tR = atR$ for all $t \in S$. Hence the only elements of S dividing a are units. In the case where S is generated by primes, S is a splitting set $\Leftrightarrow$ for each $0 \neq x \in R$ we can write $x = as$ where $s \in S$ and no prime in S divides a. Now suppose that R is a commutative ring with zero divisors and S is generated by regular primes. Let $p \in S$ be a regular prime. Then $\cap_{n=1}^{\infty} p^n R$ is a prime ideal. If $0 = \cap_{n=1}^{\infty} p^n R$, then R is an integral domain. So suppose $0 \neq \cap_{n=1}^{\infty} p^n R$. Let $0 \neq x \in \cap_{n=1}^{\infty} p^n R$. Then it is not possible to write $x = as$ where $s \in S$ and no prime in S divides a since $a \in \cap_{n=1}^{\infty} p^n R$.

We end this section with the following result.

Proposition 2.5. *Let $\{R_\gamma\}$ be a directed family of commutative rings with identity. Suppose that each $R_\alpha \subset R_\beta$ is a weakly inert extension. If each R_γ is atomic (respectively, strongly atomic, very strongly atomic, m-atomic, p-atomic), then so is $R = \cup R_\gamma$.*

Proof. Suppose that each R_γ is atomic. Let $0 \neq x \in R$ be a nonunit. Now $x \in R_\alpha$ for some α. Since R_α is atomic, we can write $x = x_1 \cdots x_n$ where each x_i is irreducible in R_α. It is easily checked that $R_\alpha \subset R$ is a weakly inert extension. Hence by Proposition 2.1 each x_i is either a unit or irreducible in R. Thus R is atomic. A similar proof holds for the strongly atomic, very strongly atomic and m-atomic cases. Suppose that each R_α is p-atomic. Let $0 \neq p \in R_\alpha$ be a prime. For $R_\alpha \subset R_\beta$, p is irreducible in R_α and hence irreducible or a unit in R_β (Proposition 2.1). Since R_β is p-atomic, p is either a prime or unit in R_β. If p is a unit in some R_β, then p is a unit in R. So suppose p is not a unit in R. Then p is a prime in each $R_\beta \supset R_\alpha$. So if $p|xy$ in R, then in some $R_\beta \supset R_\alpha$, $x, y \in R_\beta$ and $p|xy$ in R_β so $p|x$ or $p|y$ in R_β. Hence, $p|x$ or $p|y$ in R. Thus p is a prime in R. Hence a proof similar to the atomic case shows that R is p-atomic. $\square$

3. Generalized CK rings. Let R be a commutative ring with identity. It is well known that an atomic ring R has every atom prime, i.e., R is p-atomic, if and only if R is a finite direct product of UFD's and special principal ideal rings (SPIR's) [**10**, Theorem 3.6]. In this section we characterize the atomic rings in which almost all atoms are prime.

Definition 3.1. R is a *Cohen-Kaplansky* (*CK*) *ring* if R is an atomic ring with only a finite number of nonassociate atoms. R is a *generalized Cohen-Kaplansky* (*CK*) *ring* if R is an atomic ring with only finitely many nonassociate atoms that are not prime.

In [**1**, Theorem 2] it was shown that R is a CK ring if and only if R is a finite direct product of finite local rings, SPIR's and one-dimensional semi-local domains D with the property that, for each nonprincipal maximal ideal M of D, D/M is finite and D_M is analytically irreducible. Thus a finite direct product of CK rings is a CK ring. For a detailed study of CK domains, see [**8**]. Generalized CK domains were introduced in [**5**]. Examples of generalized CK domains besides UFD's include $\mathbf{Z}[2\sqrt{-1}]$, $k+XK[[X]]$ and $k+XK[X]$ where $k \subseteq K$ are finite fields. Unfortunately, the characterization of generalized CK domains given in [**5**, Theorem 6] is incomplete, as pointed out to us by Picavet-L'Hermitte, see [**6**] and [**16**]. An integral domain R is a generalized CK domain if and only if (1) $\overline{R}$, the integral closure of R, is a UFD, (2) $R \subseteq \overline{R}$ is a root extension, (3) $[R : \overline{R}]$ is a principal ideal of $\overline{R}$, (4) $\overline{R}/[R : \overline{R}]$ is finite and (5) $\operatorname{Pic}(R) = 0$. Condition (5) may be replaced by $\operatorname{Cl}_t(R) = 0$ or each height-one prime ideal P of R with $P \not\supseteq [R : \overline{R}]$ is principal.

Theorem 3.2. *R is a generalized* CK *ring if and only if R is a finite direct product of* CK *rings and generalized* CK *domains.*

Proof. Suppose that $R = R_1 \times \cdots \times R_n$. A nonzero nonunit a of R is irreducible (prime) $\Leftrightarrow a = (u_1, \dots, u_{i-1}, a_i, u_{i+1}, \dots, u_n)$ where all coordinates u_j, $j \neq i$, but one are units and that one nonunit coordinate a_i is irreducible (prime) in R_i [**10**, Theorem 2.15]. It easily follows that R is a generalized CK ring $\Leftrightarrow$ each R_i is. This gives ($\Leftarrow$).

($\Rightarrow$). By [**10**, Theorem 3.3] an atomic ring is a finite direct product of indecomposable atomic rings. Thus it suffices to show that an indecomposable generalized CK ring is either a CK ring or an integral domain. So let R be an indecomposable generalized CK ring and let $\{a_1, \dots, a_n\}$ be the finite set of nonassociate nonprime atoms of R. Let P be a minimal prime ideal of R. Then since each nonzero element of P is a product of atoms, either P is principal or P is generated by a subset of $\{a_1, \dots, a_n\}$. So all the minimal prime ideals of R are finitely generated. By [**2**, Theorem], R has only finitely many minimal prime ideals $P_1, \dots, P_m$. Let (p) be a principal prime of R.

Claim. $\operatorname{ht}(p) \leq 1$.

Proof. Suppose $\operatorname{ht}(p) > 1$. Now $Q = \cap_{n=1}^{\infty}(p^n)$ is the unique prime ideal directly below (p) and $pQ = Q$ [**7**, Corollary 2.3]. Suppose there is a principal prime $(q) \subseteq Q$. Then $(q)_{(p)} \subseteq (p)_{(p)} \Rightarrow (q)_{(p)} = (q)_{(p)}(p)_{(p)} \Rightarrow (q)_{(p)} = 0_{(p)}$ by Nakayama's lemma. So Q contains at most one principal prime ideal. Thus Q is generated by some subset of $\{a_1, \dots, a_n\}$ and possibly a single principal prime; so Q is finitely generated. Thus $pQ_{(p)} = Q_{(p)}$ gives $Q_{(p)} = 0_{(p)}$ by Nakayama's lemma. So $\mathrm{ht}\,(p) = 1$.

Let M be a maximal ideal of R. Suppose that M contains a nonminimal principal prime ideal (p). Then $\mathrm{ht}\,(p) = 1$ so $\cap_{n=1}^{\infty}(p^n) = P_i$ for some i and $pP_i = P_i$. Since P_i is finitely generated, $(P_i)_M = 0_M$ by Nakayama's lemma. Thus, R_M is an integral domain and M contains a unique minimal prime ideal. Suppose that M contains no nonminimal principal prime ideal. Let $P \subseteq M$ be a prime ideal. Then P is a finite union of principal ideals, each of which is either a height-zero principal prime ideal or an (a_i). The same follows for P_M in R_M. Hence R_M is a local CK ring [**1**, Theorem 2] and thus R_M is a zero-dimensional local ring or a one-dimensional local domain. Thus M contains a unique minimal prime and R_M is a domain unless M is also minimal. Thus the minimal prime ideals of R are comaximal and hence the minimal prime ideals of $R/\mathrm{nil}\,(R)$ are comaximal. So $R/\mathrm{nil}\,(R)$ is a finite direct product of integral domains. Since R is indecomposable, so is $R/\mathrm{nil}\,(R)$. So $R/\mathrm{nil}\,(R)$ is an integral domain, i.e., R has a unique minimal prime ideal. If P is also maximal, R is a zero-dimensional local CK ring. If P is not maximal, then R_M is a domain for each maximal ideal M; i.e., $P_M = 0_M$ for each maximal ideal M. Hence $P = 0$, so R is an integral domain. $\square$

Remark 3.3. (a) Note that the proof of Theorem 3.2 gives another proof of the opening statement of this section that a ring R is p-atomic if and only if R is a finite direct product of UFD's and SPIR's.

(b) Since a CK ring R is a finite direct product of local rings and integral domains, each irreducible element of R is actually strongly irreducible. Moreover, using [**10**, Theorem 3.4] it is easy to characterize the generalized CK rings that are m-atomic or very strongly atomic.

4. U-factorizations, BFR's and U-BFR's. Let R be a commutative ring with identity. Let $a \in R$ be a nonunit, possibly 0. By a *factorization of* a we mean $a = a_1 \cdots a_n$ where each $a_i \in R$ is a nonunit. Let $\alpha \in \{$irreducible, strongly irreducible, m-irreducible, very strongly irreducible, prime$\}$. By an *α-factorization of* a we mean a factorization

$a = a_1 \cdots a_n$ where each a_i is α.

Recall [**14**] that for $a \in R$, $U(a) = \{r \in R \mid \exists s \in R$ with $rsa = a\} = \{r \in R \mid r(a) = (a)\}$. A *$U$-factorization of a* is a factorization $a = (a_1 \cdots a_n)(b_1 \cdots b_m)$ where $a_i \in U(b_1 \cdots b_m)$ for $1 \leq i \leq n$, we allow $\{a_1, \dots, a_n\}$ to be empty and then write $a = (\ )(b_1 \cdots b_m)$, and $b_i \notin U(b_1 \cdots \hat{b}_i \cdots b_m)$, for $1 \leq i \leq m$. We call $a_1, \dots, a_n$ the *irrelevant factors* and $b_1, \dots, b_m$ the *relevant factors*. A U-factorization is called an *α-U-factorization* if each a_i, b_j is α. An irreducible-U-factorization of a is called a *U-decomposition of a*. The notion of a U-decomposition was used by Fletcher [**14, 15**] in his study of unique factorization in rings with zero divisors. Our next lemma is a slight generalization of [**14**, Proposition 2].

Proposition 4.1. *Any factorization of a can be rearranged to a U-factorization of a. Hence, any α-factorization of a can be rearranged to an α-U-factorization of a.*

Proof. Let $a = a_1 \cdots a_n$ be a factorization of a. If $a_i \notin U(a_1 \cdots \hat{a}_i \cdots a_n)$ for each i, then $a = (\ )(a_1 \cdots a_n)$ is a U-factorization of a. So assume some $a_i \in U(a_1 \cdots \hat{a}_i \cdots a_n)$. With a change of notation we can take $i = 1$. By induction, after a change of notation, we have $a_2 \cdots a_n = (a_2 \cdots a_s)(a_{s+1} \cdots a_n)$, a U-factorization of $a_2 \cdots a_n$. We claim that $(a_1 a_2 \cdots a_s)(a_{s+1} \cdots a_n)$ is a U-factorization of a. By definition $a_i \notin U(a_{s+1} \cdots \hat{a}_i \cdots a_n)$ for $s+1 \leq i \leq n$ and $a_i \in U(a_{s+1} \cdots a_n)$ for $2 \leq i \leq s$. And $a_1 \in U(a_2 \cdots a_n) \Rightarrow a_1 R a_2 \cdots a_n = R a_2 \cdots a_n = a_2 \cdots a_s R a_{s+1} \cdots a_n = a_2 \cdots a_{s-1} R a_s (a_{s+1} \cdots a_n) = a_2 \cdots a_{s-1} R a_{s+1} \cdots a_n = \cdots = R a_{s+1} \cdots a_n \Rightarrow a_1 \in U(a_{s+1} \cdots a_n)$. $\square$

However, the resulting U-factorization is not necessarily unique. For, let $e = (1, 0)$ in $\mathbf{Z} \times \mathbf{Z}$. Then $-e = -e \cdot e = (-e)(e) = (e)(-e)$ are two different U-factorizations derived from the factorization $-e = -e \cdot e$. If $a = a_1 \cdots a_n$ is a factorization of a and $a = (\ )(a_1 \cdots a_n)$ is a U-factorization, then this is the only way to convert $a = a_1 \cdots a_n$ to a U-factorization. Recall that R is *présimplifiable* if $xy = x \Rightarrow x = 0$ or $y \in U(R)$. R is présimplifiable if and only if $U(x) = U(R)$ for each $0 \neq x \in R$, or in the terminology of [**15**], R is a *pseudo-domain.* Hence,

R is présimplifiable if and only if for each nonzero nonunit $a \in R$ each factorization $a = a_1 \cdots a_n$ has $a = (\ )(a_1 \cdots a_n)$ as its unique conversion to a U-factorization. Note that $0 = (a_1 \cdots a_n)(b_1 \cdots b_m)$ is a U-factorization of $0 \Leftrightarrow b_1 \cdots b_m = 0$, but $b_1 \cdots \hat{b}_i \cdots b_m \neq 0$ for each $i = 1, \dots, m$. If R is an integral domain, any U-factorization of 0 has the form $0 = (a_1 \cdots a_n)(0)$.

For $a \in R$, put $L(a) = \sup\{n \mid a = a_1 \cdots a_n \text{ is a factorization of } a\}$. So (1) $L(a) = 0 \Leftrightarrow a$ is a unit, (2) $L(0) = \infty$ and (3) $L(a) < \infty \Rightarrow a$ is a product of irreducibles and then $L(a) = \sup\{n \mid a = a_1 \cdots a_n, \text{ each } a_i \text{ is irreducible}\}$. Note that R is a BFR if and only if $L(a) < \infty$ for each $0 \neq a \in R$.

For a nonunit $a \in R$ put $L_U(a) = \sup\{m \mid a = (a_1 \cdots a_n)(b_1 \cdots b_m)$ is a U-factorization of $a\}$. Note that for a nonunit $a \in R$, $a = (\ )(a)$, so $L_U(a) \geq 1$. For $a \in U(R)$, we define $L_U(a) = 0$. Since each U-factorization of a is a factorization of a, we have $L_U(a) \leq L(a)$. We say that a is *U-bounded* if $L_U(a) < \infty$ and that R is a U-BFR if each nonzero element of R is U-bounded (note that we are not assuming that a U-BFR is atomic). Clearly a BFR is a U-BFR, in fact, a U-BFR is a BFR if and only if it is présimplifiable (see Theorem 4.2). Note that $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ is a U-BFR (for $L_U((0,1)) = L_U((1,0)) = 1$), but is not a BFR. The following theorem gives several characterizations of BFR's.

Theorem 4.2. *For a commutative ring R the following conditions are equivalent.*

1. *R is a* BFR.

2. *R is présimplifiable and for $0 \neq a \in R$, there is a fixed bound on the lengths of chains of principal ideals starting at Ra.*

3. *R is présimplifiable and a U*-BFR.

4. *For each nonzero nonunit $a \in R$, natural numbers $N_1(a)$ and $N_2(a)$ exist so that if $a = (a_1 \cdots a_n)(b_1 \cdots b_m)$ is a U-factorization of a, then $n \leq N_1(a)$ and $m \leq N_2(a)$.*

5. *There is a function $l : R \to \mathbf{N}_0 \cup \{\infty\}$ that satisfies* (i) *$l(a) = \infty \Leftrightarrow a = 0$,* (ii) *$l(a) = 0 \Leftrightarrow a \in U(R)$ and* (iii) *$l(ab) \geq l(a) + l(b)$ for $a, b \in R$.*

Proof. (1) $\Rightarrow$ (2). Clearly a BFR is présimplifiable. If $Ra = Ra_1 \subsetneq \cdots \subsetneq Ra_n \neq R$ is a proper ascending chain of principal ideals, then each $a_i = r_i a_{i+1}$ where r_i is a nonunit. Hence $a = a_1 = r_1 a_2 = r_1 r_2 a_3 = \cdots = r_1 \cdots r_{n-1} a_n$. So $n \leq L(a) < \infty$.

(2) $\Rightarrow$ (3). Let a be a nonzero nonunit of R. If $a = (a_1 \cdots a_n)(b_1 \cdots b_m)$ is a U-factorization of a, then $Ra = Rb_1 \cdots b_m \subsetneq Rb_1 \cdots b_{m-1} \subsetneq \cdots \subsetneq Rb_1 b_2 \subsetneq Rb_1$. Thus $L_U(a) < \infty$.

(3) $\Rightarrow$ (4). Let a be a nonzero nonunit of R. Since R is présimplifiable, a U-factorization of a has the form $a = (\ )(b_1 \cdots b_m)$. Hence we can take $N_1(a) = 0$ and $N_2(a) = L_U(a)$.

(4) $\Rightarrow$ (1). Let a be a nonzero nonunit of R. Let $a = a_1 \cdots a_n$ be a factorization of a. Then $a = a_1 \cdots a_n$ can be rearranged to a U-factorization, say $a = (a_{s_1} \cdots a_{s_i})(a_{s_{i+1}} \cdots a_{s_n})$ is a U-factorization of a. Then $n = i + (n-i) \leq N_1(a) + N_2(a)$. Hence $L(a) \leq N_1(a) + N_2(a)$.

(1) $\Rightarrow$ (5). Take $l(a) = L(a)$.

(5) $\Rightarrow$ (1). Let a be a nonzero nonunit of R, and let $a = a_1 \cdots a_n$ be a factorization of a. Then $l(a) = l(a_1 \cdots a_n) \geq l(a_1) + \cdots + l(a_n) \geq 1 + \cdots + 1 = n$. Hence $L(a) \leq l(a) < \infty$. □

Concerning (4) of Theorem 4.2, note that R is présimplifiable if and only if for each nonzero nonunit $a \in R$ a natural number $N_1(a)$ exists so that if $a = (a_1 \cdots a_n)(b_1 \cdots b_m)$ is a U-factorization of a, then $n \leq N_1(a)$.

Theorem 4.3. *Let R be a commutative ring and S a regular multiplicative set of R such that $R \subset R_S$ is weakly inert. If R is a* BFR*, then R_S is a* BFR.

Proof. Suppose R_S is not a BFR. Let $a \in R_S$ be a nonzero nonunit with $L(a) = \infty$. After suitable multiplication by an element of S we can assume $a \in R$.

Suppose we have a factorization $a = x_1 \cdots x_n$ of a in R_S. Since $R \subset R_S$ is weakly inert, we can write $a = (x_1 u_1)(x_2 \cdots x_n u_1^{-1})$ where $x_1 u_1, x_2 \cdots x_n u_1^{-1} \in R$ and $u_1 \in U(R_S)$. Now $x_2 \cdots x_n u_1^{-1} = x_2(x_3 \cdots x_n u_1^{-1})$ so there is a $u_2 \in U(R_S)$ with $x_2 u_2 \in R$ and $x_3 \cdots x_n u_1^{-1} u_2^{-1} \in R$. Continuing, there are units $u_1, \ldots, u_{n-1} \in R_S$

with $x_1u_1, \dots, x_{n-1}u_{n-1}, x_nu_1^{-1}\cdots u_{n-1}^{-1} \in R$. Hence $a = (x_1u_1)(x_2u_2)\cdots(x_{n-1}u_{n-1})(x_nu_1^{-1}\cdots u_{n-1}^{-1})$ is a factorization of a in R of length n. Hence, $L(a) = \infty$ in R, a contradiction. □

The following proposition uses the functions L and L_U to characterize two forms of irreducibility.

Proposition 4.4. *Let $a \in R$, R a commutative ring with identity.*

1. *a is irreducible $\Leftrightarrow L_U(a) = 1$.*
2. *For $a \neq 0$, a is very strongly irreducible $\Leftrightarrow L(a) = 1$.*

Proof. (1) ($\Leftarrow$). Suppose $L_U(a) = 1$. Let $a = bc$ where b and c are nonunits. By Proposition 4.1, we get the following possible U-factorizations: $a = (\ )(bc)$, $a = (b)(c)$ or $a = (c)(b)$. Since $L_U(a) = 1$, the first situation cannot occur. So $a = (b)(c)$ which implies $(a) = (c)$ or $a = (c)(b)$ which implies $(a) = (b)$. So a is irreducible.

($\Rightarrow$). Assume that a is irreducible. Let $a = (a_1\cdots a_n)(b_1\cdots b_m)$ be a U-factorization of a. Hence $(a) = (b_1\cdots b_m) = (b_1)\cdots(b_m)$, so say $(a) = (b_1)$. Hence, if $m > 1$, $(a) = (b_1\cdots b_m) \subsetneq (b_1\cdots b_{m-1}) \subseteq (b_1) = (a)$, a contradiction.

(2) For $a \neq 0$, $L(a) = 1 \Leftrightarrow a = bc$ implies b or $c \in U(R) \Leftrightarrow a$ is very strongly irreducible. □

We next wish to show that for R atomic and $a \in R$, $L_U(a) = \sup\{m \mid a = (a_1\cdots a_n)(b_1\cdots b_m)$ is a U-decomposition of $a\}$. We need the following lemma.

Lemma 4.5. *Let $a \in R$ be a nonunit, and let $a = (a_1\cdots a_n)(b_1\cdots b_m)$ be a U-factorization of a.*

1. *If $a_i = a_i'a_i''$ with a_i', a_i'' nonunits, then $a = (a_1\cdots a_{i-1}a_i'a_i''a_{i+1}\cdots a_n)(b_1\cdots b_m)$ is a U-factorization of a.*
2. *If $b_i = b_i'b_i''$ with b_i', b_i'' nonunits, then at least one of*

(a) $a = (a_1\cdots a_n)(b_1\cdots b_{i-1}b_i'b_i''b_{i+1}\cdots b_m)$

(b) $a = (a_1\cdots a_nb_i')(b_1\cdots b_{i-1}b_i''b_{i+1}\cdots b_m)$

(c) $a = (a_1 \cdots a_n b_i'')(b_1 \cdots b_{i-1} b_i' b_{i+1} \cdots b_m)$

is a U-factorization of a.

3. *For $i < j$, $a = (a_1 \cdots \hat{a}_i \cdots \hat{a}_j \cdots a_n(a_i a_j))(b_1 \cdots b_m)$ and*

$a = (a_1 \cdots a_n)(b_1 \cdots \hat{b}_i \cdots \hat{b}_j \cdots b_m(b_i b_j))$ are U-factorizations of a. Hence $a = ((a_1 \cdots a_n))(b_1 \cdots b_m)$ is a U-factorization of a.

Proof. (1) $a_i(b_1 \cdots b_m) = (b_1 \cdots b_m)$ and $c | a_i \Rightarrow c(b_1 \cdots b_m) = (b_1 \cdots b_m)$.

(2) Suppose that the decomposition in 4.5 (2a) is *not* a U-factorization.

Hence $b_i' \in U(b_1 \cdots b_{i-1} b_i'' b_{i+1} \cdots b_m)$ or $b_i'' \in U(b_1 \cdots b_{i-1} b_i' b_{i+1} \cdots b_m)$. Assume the former; so $b_i'(b_1 \cdots b_{i-1} b_i'' b_{i+1} \cdots b_m) = (b_1 \cdots b_{i-1} b_i'' b_{i+1} \cdots b_m)$. Hence $(b_1 \cdots b_{i-1} b_i b_{i+1} \cdots b_m) = (b_1 \cdots b_{i-1} b_i'' b_{i+1} \cdots b_m)$. So $a_i(b_1 \cdots b_{i-1} b_i'' b_{i+1} \cdots b_m) = a_i(b_1 \cdots b_{i-1} b_i b_{i+1} \cdots b_m) = (b_1 \cdots b_{i-1} b_i \cdot b_{i+1} \cdots b_m) = (b_1 \cdots b_{i-1} b_i'' b_{i+1} \cdots b_m)$. Therefore $a = (a_1 \cdots a_n b_i') \cdot (b_1 \cdots b_{i-1} b_i'' b_{i+1} \cdots b_m)$ is a U-factorization of a unless $b_i''(b_1 \cdots b_{i-1} \cdot b_{i+1} \cdots b_m) = (b_1 \cdots b_{i-1} b_{i+1} \cdots b_m)$. But then $(b_1 \cdots b_{i-1} b_{i+1} \cdots b_m) = (b_1 \cdots b_{i-1} b_i'' b_{i+1} \cdots b_m) = (b_1 \cdots b_{i-1} b_i b_{i+1} \cdots b_m)$, a contradiction.

(3) Clear. □

Theorem 4.6. *Let R be a commutative ring with identity. Suppose that R is atomic, respectively strongly atomic, very strongly atomic, m-atomic, p-atomic. Let a be a nonunit of R. Then $L_U(a) = \sup\{t \mid a = (a_1 \cdots a_s)(b_1 \cdots b_t)$ is an irreducible, respectively strongly irreducible, very strongly irreducible, m-irreducible, prime-U-factorization of $a\}$.*

Proof. We do the atomic case; the other cases are similar. Since each U-decomposition of a is a U-factorization of a, we have $L_U(a) \geq \sup\{t \mid a = (a_1 \cdots a_n)(b_1 \cdots b_t)$ is a U-decomposition of $a\}$. Let $a = (a_1 \cdots a_n)(b_1 \cdots b_m)$ be a U-factorization of a. After factoring each a_i, b_j into irreducibles, repeated applications of Lemma 4.5 give a U-decomposition $a = (c_1 \cdots c_s)(d_1 \cdots d_t)$ where $t \geq m$. Hence $\sup\{t \mid a = (a_1 \cdots a_n)(b_1 \cdots b_t)$ is a U-decomposition of $a\} \geq L_U(a)$, and so we have equality. □

Corollary 4.7. *Suppose that R is atomic. Then R is a U-*BFR *if* *and only if for each nonzero nonunit $a \in R$ a natural number $N(a)$ exists such that, for each U-decomposition $a = (a_1 \cdots a_n)(b_1 \cdots b_m)$ of a, $m \leq N(a)$.*

We next consider direct products of rings.

Lemma 4.8. *Let $R_1, \dots, R_n$ be commutative rings, and let $R = R_1 \times \cdots \times R_n$.*

1. *Let $a \in R$ be a nonunit, and let $a = (a_1 \cdots a_s)(b_1 \cdots b_m)$ be a U-factorization of a. Then a has a U-factorization $a = (a'_1 \cdots a'_{s'})(b'_1 \cdots b'_{m'})$ where $mn \geq m' \geq m$ and each $a'_i, b'_j \in R_1 \times \cdots \times R_n$ has all coordinates except one equal to 1_{R_k}, for the appropriate R_k.*

2. *Let $a = (a_1, \dots, a_n) \in R_1 \times \cdots \times R_n$. Then $L_U(a) \leq L_U(a_1) + \cdots + L_U(a_n)$.*

Proof. For $c = (c_1, \dots, c_n) \in R_1 \times \cdots \times R_n$, put $c^{(i)} = (1, \dots, 1, c_i, 1, \dots, 1)$. So $c = c^{(1)} \cdots c^{(n)}$, each $c^{(i)}$ has at most one coefficient a nonunit, and if c is a nonunit at least one $c^{(i)}$ is a nonunit. In the U-factorization $a = (a_1 \cdots a_s)(b_1 \cdots b_m)$, factor each a_i, b_j into the $a_i^{(k)}, b_j^{(k)}$'s. By Lemma 4.5, a has a U-factorization $a = (a'_1 \cdots a'_{s'})(b'_1 \cdots b'_{m'})$ where each factor has all coordinates but one equal to 1 and that coordinate is a nonunit and $m \leq m' \leq mn$.

(2) If a is a unit, the result is obvious. So assume that a is a nonunit. Let $a = (c_1 \cdots c_s)(d_1 \cdots d_t)$ be a U-factorization of a. By (1), there is a U-factorization $a = (c'_1 \cdots c'_{s'})(d'_1 \cdots d'_{t'})$ where $t' \geq t$ and each c'_i, d'_j has exactly one coordinate not equal to 1. Let N_i be the number of d'_j's that have a nonunit in the ith coordinate. So $N_1 + \cdots + N_n = t'$. If a_i is a unit, then $N_i = 0 = L_U(a_i)$. So suppose a_i is a nonunit. Now $a_i = c'_{1i} \cdots c'_{s'i} d'_{1i} \cdots d'_{t'i}$ where c'_{ji} and d'_{ji} are the ith coordinates of c'_j and d'_j, respectively. After removing the c'_{ji}'s and d'_{ji}'s that are units, we get a U-factorization $a_i = (c'_{j_1 i} \cdots c'_{j_{s''} i})(d'_{k_1 i} \cdots d'_{k_{N_i} i})$. Hence $N_i \leq L_U(a_i)$. So $t \leq t' = N_1 + \cdots + N_n \leq L_U(a_1) + \cdots + L_U(a_n)$. Hence $L_U(a) \leq L_U(a_1) + \cdots + L_U(a_n)$. $\square$

Theorem 4.9. *Let $R_1, \dots, R_n$ be commutative rings, $n > 1$, and let $R = R_1 \times \cdots \times R_n$. Then R is a U-*BFR *$\Leftrightarrow$ each R_i is a U-*BFR *and*

0_{R_i} *is* U*-bounded. Hence,* 0_R *is* U*-bounded.*

Proof. ($\Rightarrow$). Let $a \in R_i$ be a nonunit, possibly 0, and let $a = (a_1 \cdots a_n)(b_1 \cdots b_m)$ be a U-factorization of a. For $c \in R_i$, put $\tilde{c} = (1, \ldots, 1, c, 1, \ldots, 1) \in R_1 \times \cdots \times R_n$ where each coordinate is 1 except for the ith coordinate which is c. Now $\tilde{a} = (\tilde{a}_1 \cdots \tilde{a}_n)(\tilde{b}_1 \cdots \tilde{b}_m)$ is easily seen to be a U-factorization of $\tilde{a}$. Hence, $m \leq L_U(\tilde{a})$. So R_i is a U-BFR and 0_{R_i} is U-bounded.

($\Leftarrow$). This follows from Lemma 4.8(2).

The last statement also follows from Lemma 4.8(2). □

Corollary 4.10. *Let R be a U-*BFR*. If R is not indecomposable, then 0 is U-bounded.*

Proof. If R is not indecomposable, we can decompose R into a direct product $R = R_1 \times R_2$. By Theorem 4.9, 0_R is U-bounded. □

Corollary 4.11. *A finite direct product of* BFD*'s is a U-*BFR*.*

Proof. Let $D_1, \ldots, D_n$ be BFD's. Since each D_i is an integral domain, $L_U(0_{D_i}) = 1$ (Proposition 4.4 or note that in an integral domain a U-factorization of 0 has the form $0 = (a_1 \cdots a_n)(0)$). By Theorem 4.9, $D_1 \times \cdots \times D_n$ is a U-BFR. □

However, as the next example shows, in an indecomposable U-BFR R, 0 need not be U-bounded. This example also shows that a finite direct product of BFR's need not be a U-BFR.

Example 4.12 (A quasilocal BFR in which 0 is *not* U-bounded). Take $R = k[[X,Y]] \oplus N$ (idealization) where k is a field and $N = \oplus\{k[[X,Y]]/(p) \mid (p)$ is a height-one prime ideal of $k[[X,Y]]\}$. Now $\cap_{n=1}^{\infty}((X,Y) \oplus N)^n = 0$, so R is a quasilocal BFR. Let $\{p_i\}$ be a countable set of nonassociate nonzero primes of $k[[X,Y]]$, and let $e_{p_i} = 1_{k[[X,Y]]/(p_i)}$ in N. Then $(0,0) = (\ )((p_1,0) \cdots (p_n,0)(0, e_{p_1} + \cdots + e_{p_n}))$ is a U-factorization of (0,0). Also, by Theorem 4.9, $R \times R$ is not a U-BFR.

Theorem 4.13. *Let R be a U-BFR. Then R is a finite direct product of indecomposable U-BFR's.*

Proof. We may assume that R is not indecomposable, so let $R = R_1 \times R_2$. Let $a = (1, 0) \in R_1 \times R_2$. Suppose $R_2 = S_1 \times \cdots \times S_t$, and let $f_i = (1, \dots, 1, 0, 1, \dots, 1) \in S_1 \times \cdots \times S_t$ where each coordinate is 1 except for the ith coordinate which is 0_{s_i}. Then $a = (\ )((1, f_1) \cdots (1, f_t))$ is a U-factorization of a. Hence $t \le L_U(a)$. Thus R_2 is a finite direct product of indecomposable rings. Likewise, R_1 is a finite direct product of indecomposable rings. By Theorem 4.9, each of the indecomposable factors is a U-BFR. $\square$

The proof of Theorem 4.13 yields the following result.

Corollary 4.14. *Let R be a commutative ring with $L_U(0) < \infty$. Then R is a finite direct product of indecomposable rings R_i, $R = R_1 \times \cdots \times R_n$, with each $L_U(0_{R_i}) < \infty$.*

Proof. Let $R = S_1 \times \cdots \times S_t$. In the notation of the proof of Theorem 4.13, note that $0 = (\ )(f_1 \cdots f_t)$ is a U-factorization of 0 in R. Hence $t \le L_U(0)$. So R is a finite direct product of indecomposable rings, say $R = R_1 \times \cdots \times R_n$ where each R_i is indecomposable. Suppose in R_i, $0_{R_i} = (a_1 \cdots a_n)(b_1 \cdots b_m)$ is a U-factorization. Then $0_R = ((a_1, 0, \dots, 0) \cdots (a_n, 0, \dots, 0))((b_1, 0, \dots, 0) \cdots (b_m, 0, \dots, 0))$ is a U-factorization of 0_R in R. Hence, $m \le L_U(0_R)$. Thus $L_U(0_{R_i}) \le L_U(0_R)$. $\square$

We next show that any Noetherian ring is a U-BFR. We need two lemmas.

Lemma 4.15. *Let R be a commutative ring, $a \in R$, and B an ideal of R with $B \subseteq (a)$. Then for $a + B \in R/B$, $L_U(a + B) \ge L_U(a)$.*

Proof. If a is a unit, then $L_U(a + B) = L_U(a) = 0$. So assume a is a nonunit. Let $a = (a_1 \cdots a_n)(b_1 \cdots b_m)$ be a U-factorization of a. Then in R/B, $\bar{a} = \bar{a}_1 \cdots \bar{a}_n \bar{b}_1 \cdots \bar{b}_m$ and each factor is a nonunit. Now $a_i(b_1 \cdots b_m) = (b_1 \cdots b_m) \Rightarrow \bar{a}_i(\bar{b}_1 \cdots \bar{b}_m) = (\bar{b}_1 \cdots \bar{b}_m)$ and $\bar{b}_i(\bar{b}_1 \cdots \hat{\bar{b}}_i \cdots \bar{b}_m) = (\bar{b}_1 \cdots \hat{\bar{b}}_i \cdots \bar{b}_m) \Rightarrow (b_1 \cdots b_m) = (b_1 \cdots b_m) + B = (b_1 \cdots \hat{b}_i \cdots b_m) + B = (b_1 \cdots \hat{b}_i \cdots b_m)$, a contradiction. So $\bar{a} = (\bar{a}_1 \cdots \bar{a}_n)(\bar{b}_1 \cdots \bar{b}_m)$ is a U-factorization of $\bar{a} = a + B$. Hence, $L_U(a+B) \geq L_U(a)$. ◻

Lemma 4.16. *Let R be a commutative ring and A an ideal of R with $A = Q_1 \cap \cdots \cap Q_n$ where Q_i is P_i-primary. Suppose that $P_i^{s_i} \subseteq Q_i$. Let $t_1, \ldots, t_m \in R$ with $t_1 \cdots t_m \in A$. Then some subproduct of $t_1 \cdots t_m$ of length at most $s_1 + \cdots + s_n$ lies in A.*

Proof. Let $t_{i_1}, \ldots, t_{i_s}$ be the t_j's that lie in P_i. If $s \geq s_i$, then $t_{i_1} \cdots t_{i_{s_i}} \in P_i^{s_i} \subseteq Q_i$. So suppose $s < s_i$. Let $\bar{t}_i = \prod\{t_j \mid j \notin \{i_1, \ldots, i_s\}\}$, so $\bar{t}_i \notin P_i$ and $t_{i_1} \cdots t_{i_s}\bar{t}_i = t_1 \cdots t_m \in Q_i$. Since Q_i is P_i-primary, $t_{i_1} \cdots t_{i_s} \in Q_i$. So in either case we have a subproduct $t_{i_1} \cdots t_{i_{k_i}}$ of $t_1 \cdots t_m$ of length $k_i \leq s_i$ that lies in Q_i. Let $t = \prod\{t_j \mid j \in \cup_{i=1}^n \{i_1, \ldots, i_{k_i}\}\}$. Then t is a subproduct of $t_1 \cdots t_m$ of length at most $k_1 + \cdots + k_m \leq s_1 + \cdots + s_n$ which lies in $Q_1 \cap \cdots \cap Q_n = A$. ◻

Theorem 4.17. *Let R be a Noetherian ring. Then R is a U-*BFR* with 0_R U-bounded.*

Proof. It suffices to prove that $L_U(0) < \infty$. For then if $a \in R$ is a nonunit, Lemma 4.15 gives that $L_U(a) \leq L_U(a + (a)) = L_U(0_{R/(a)}) < \infty$ (since $R/(a)$ is Noetherian). Let $0 = Q_1 \cap \cdots \cap Q_n$ be a primary decomposition of 0 where Q_i is P_i-primary and $P_i^{s_i} \subseteq Q_i$. Let $0 = (a_1 \cdots a_n)(b_1 \cdots b_m)$ be a U-factorization of 0. Then $b_1 \cdots b_m = 0$ but $b_1 \cdots \hat{b}_i \cdots b_m \neq 0$ for $i = 1, \ldots, m$. Suppose that $m > s_1 + \cdots + s_n$. Then by Lemma 4.16 some proper subproduct of $b_1 \cdots b_m$ is 0, a contradiction. Hence $L_U(0) \leq s_1 + \cdots + s_n$. ◻

The proof of Theorem 4.17 shows that if R is a commutative ring in which 0 has a strong primary decomposition (i.e., $0 = Q_1 \cap \cdots \cap Q_n$ where Q_i is P_i-primary and $P_i^{s_i} \subseteq Q_i$ for some $s_i \geq 1$), then 0_R is U-bounded and that a strongly Laskerian ring (or more generally, a ring in which every principal ideal has a strong primary decomposition) is a U-BFR. Thus, Example 4.12 was chosen as a BFR in which 0 does not have a primary decomposition. Hence, while every Noetherian ring R is a U-BFR, R is a BFR $\Leftrightarrow$ R is présimplifiable [**10**, Theorem 3.9]. We next give an example of a Noetherian U-BFR which is not a finite direct product of BFR's.

Example 4.18 (A Noetherian U-BFR that is not a finite direct product of BFR's.) Let $R = k[x, y] = k[X, Y]/(XY)$ where k is a field. Now R is not présimplifiable since $x(1 + y) = x \neq 0$, but $1 + y$ is not a unit. Hence R is not a BFR. Since R is indecomposable, R is not a finite direct product of BFR's.

5. Finite factorization rings. Let R be a commutative ring with identity. Recall from [**10**] that R is a *finite factorization ring* (*FFR*) if every nonzero nonunit of R has only a finite number of factorizations up to order and associates; R is a *weak finite factorization ring* (*WFFR*) if every nonzero nonunit of R has only a finite number of nonassociate divisors, and R is an *atomic idf-ring* if R is atomic and each nonzero element of R has at most a finite number of nonassociate irreducible divisors. Clearly if R is an FFR, then R is a WFFR, and if R is a WFFR then R is an atomic idf-ring. But $\mathbf{Z}_2 \times \mathbf{Z}_2$ is a WFFR that is not an FFR, consider $(0, 1) = (0, 1)^n$, and $\mathbf{Z}_{(2)} \times \mathbf{Z}_{(2)}$ is an atomic idf-ring that is not a WFFR, consider $(0, 1) = (0, 1)(2, 1)^n$. In [**10**, Proposition 6.6], it was shown that the following conditions on a commutative ring R are equivalent: (1) R is an FFR, (2) R is a BFR and WFFR, (3) R is présimplifiable and a WFFR, (4) R is a BFR and an atomic idf-ring, and (5) R is présimplifiable and an atomic idf-ring. In [**11**, Theorem 1.7] we proved that if $R[X]$ is an atomic idf-ring or $R[[X]]$ is a WFFR, then either R is an integral domain or R is a finite local ring. For a finite local ring R, $R[X]$ and $R[[X]]$ are both BFR's. So $R[X]$ (respectively, $R[[X]]$) is an FFR if and only if $R[X]$ (respectively, $R[[X]]$) is an atomic idf-ring. We give a partial answer to the question of when $R[X]$ or $R[[X]]$ is an FFR for R a finite local ring.

Theorem 5.1. *Let (R, M) be a finite local ring with $a, b \in M$ such that Ra and* $\operatorname{ann}(b)$ *are not comparable. Then $R[X]$ and $R[[X]]$ are not* FFR*'s.*

Proof. Since $Ra \not\subset \operatorname{ann}(b)$, $ab \neq 0$. Choose $c \in \operatorname{ann}(b) - Ra$. Then for $1 \leq n < m$, $0 \neq ab = (a + cX^n)b = (a + cX^m)b$. Suppose $a + cX^n \sim a + cX^m$. So since $R[X]$ and $R[[X]]$ are both présimplifiable for R a finite local ring, $(a + cX^n)l(X) = a + cX^m$ where $l(X)$ is a unit of $R[X]$ or $R[[X]]$. In either case, $l(0)$ is a unit of R. Then $a(l(X) - 1) = cX^n(-l(X) + X^{m-n})$. Hence, $c \in Ra$, a contradiction. □

Thus for $R = k[X_1, X_2, \ldots, X_n]/(X_1, \ldots, X_n)^m$ where k is a finite field and $n \geq 2$ and $m \geq 3$, $R[X]$ and $R[[X]]$ are not FFR's since $R\overline{X}_1$ and $\operatorname{ann}(\overline{X}_2)$ are not comparable. On the positive side, we show that it is possible for $R[[X]]$ to be an FFR for some finite local rings R. We need the following lemma. Recall that for a polynomial $f \in R[X]$ or power series $f \in R[[X]]$, A_f is the ideal of R generated by the coefficients of f.

Lemma 5.2 (Weierstrauss preparation theorem) [**12**]. *Let (R, M) be a complete local ring. Let $f \in R[[X]]$ with $A_f = R$. Suppose n is the degree of the first power of X whose coefficient is a unit. Then $f = pu$ where $p = a_0 + a_1X + \cdots + a_{n-1}X^{n-1} + X^n$ with each $a_i \in M$ and $u \in R[[X]]$ is a unit.*

Theorem 5.3. *Let (R, M) be a finite local ring. If either R is an* SPIR *or $M^2 = 0$, then $R[[X]]$ is an* FFR.

Proof. Let $f \in R[[X]]$ be irreducible. Suppose that $A_f = R$. So, by Lemma 5.2, $f = pu$ where $p = a_0 + a_1X + \cdots + a_{n-1}X^{n-1} + X^n$ where $a_0, \ldots, a_{n-1} \in M$ and n is the degree of the first power of X whose coefficient is a unit in R and $u \in U(R[[X]])$. So if $a_0 = 0$, $f \sim X$. Suppose $A_f \neq R$. If $M^2 = 0$, f is irreducible $\Leftrightarrow f(0) = a_0 \neq 0$ and if R is an SPIR, f is irreducible $\Leftrightarrow f \sim q$ where $M = (q)$.

(1) Case $M^2 = 0$. Let $f \in R[[X]]$ be a nonzero nonunit. First suppose $A_f = R$. Since $g|f \Rightarrow A_g = R$, a factorization of f into irreducibles has the form $X^s p_1 \cdots p_t u$ where $0 \leq s$, $0 \leq t$, each p_i is a polynomial (irreducible as a power series) with leading coefficient 1, and $u \in U(R[[X]])$. Note that $n = s + \deg p_1 + \cdots + \deg p_t$ is the order of the first term of f having a unit coefficient. Hence, for each i, $\deg p_i \leq n$.

However, the number of polynomials in $R[X]$ of degree $\leq n$ is finite, so the number of nonassociate irreducible factors of f is also finite. Next suppose that $A_f \subseteq M$. Then $f = X^s p_1 \cdots p_t g$, where $0 \leq s$, $p_1, \dots, p_t$ are as above and $g \in R[[X]]$ is irreducible with $0 \neq g(0) \in M$ and $A_g \subseteq M$. Now $p_1 \cdots p_t = b_0 + b_1 X + \cdots + b_{n-1}X^{n-1} + X^n$ where each $b_i \in M$ and $n = \deg p_1 + \cdots + \deg p_t$. So $f = X^s p_1 \cdots p_t g = X^{s+n} g$. Hence $g = X^{-s-n} f$ is uniquely determined. Hence R is an FFR.

(2) Case $(R, (q))$ is an SPIR. Let $f \in R[[X]]$ be a nonzero nonunit. So $f = q^i f'$ where $A_{f'} = R$. Let i_0 be the order of the first unit coefficient of f'. So if $f'' \in R[[X]]$ is irreducible with $A_{f''} = R$ and $f'' \mid f$, then the first unit coefficient of f'' has order $n \leq i_0$. So $f'' \sim a_0 + a_1 X + \cdots + a_{n-1} X^{n-1} + X^n$ where $a_0, \dots, a_{n-1} \in M$ and $n \leq i_0$. But there are only finitely many such polynomials. $\square$

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