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FINITE PROJECTIONS IN MULTIPLIER ALGEBRAS

E. PARDO

A la Fina.

ABSTRACT. We give a characterization of finiteness of projections in the multiplier algebra of a σ-unital C^*-algebra of real rank zero and stable rank one.

1. Introduction. The behavior and properties of projections are some of the most interesting topics in the theory of C^*-algebras and also objects of intensive study (see [**3**] for a complete survey on this topic).

This is particularly important in the case of C^*-algebras with real rank zero, a class introduced by Brown and Pedersen in 1991, [**5**], although this property, under different names, was the object of intensive study some years ago (e.g., [**4**] or [**14**]). This class includes AF C^*-algebras, von Neumann algebras, Rickart C^*-algebras, irrational rotation algebras and purely infinite simple C^*-algebras among others (see, for example, [**5**], [**6**], [**20**]), and because of [**5**, Theorem 2.6] the structure of these algebras is closely related to the structure and properties of their projections.

One of the points of interest on this topic is to know whether projections in a C^*-algebra A are finite or not. Recall that a projection $p \in A$ is *infinite* if there exist p', q' nonzero orthogonal subprojections of p such that $p' + q' = p$ and $p' \sim p$ (where "$\sim$" means Murray-von Neumann equivalent), and otherwise we say that p is *finite*. Also, if there exists a projection $q \in A$ such that $2 \cdot p \oplus q \sim p$ (where "$\oplus$" means orthogonal sum, viewing the projection in $M_\infty(A)$, see [**2**, Chapter 5]), then we say that p is *properly infinite*. The existence

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of infinite projections appears as an obstruction for the existence of order-preserving functionals on C^*-algebras, so that some interesting questions are to know whether the property that all projections in a C^*-algebra A are finite is preserved by matrix rings over A, whether the existence of a (properly) infinite projection implies that every nonzero projection in a simple C^*-algebra A is (properly) infinite, or whether the sum of two nonzero finite projections can be (properly) infinite in a C^*-algebra A. So, to find criteria for deciding when a projection of a C^*-algebra is finite becomes a main point in this context. In this note we adapt some arguments of Kutami [**9**], given in the context of von Neumann regular rings, in order to characterize when a given projection in the multiplier algebra of a σ-unital C^*-algebra of real rank zero and stable rank one is finite.

Now we will fix some notation. For $p, q \in A$ projections, we write $p \leq q$ if there exists a projection $r \in A$ such that p and r are orthogonal and $p + r = q$, and we write $p \lesssim q$ if there exists a projection $r \in A$ such that $p \oplus r \sim q$. We will denote by $\oplus$ the orthogonal sum of two projections, and the context will determine if the orthogonality of the projections is the natural one (in A), or the formal one (in $A \otimes \mathcal{K}$). Also, for a projection p and a positive integer n, we denote by $n \cdot p$ the orthogonal sum of n copies of p. For a C^*-algebra A, we denote by $V(A)$ the abelian monoid of Murray-von Neumann equivalence classes of projections in $M_\infty(A)$ (see [**2**, Chapter 5]), and we consider this monoid endowed with the so-called algebraic preordering that corresponds to the ordering induced by the relation $\lesssim$. Finally, we say that a preordered abelian monoid M satisfies the Riesz decomposition property (it is a Riesz monoid) if, for any $x, y_1, y_2 \in M$ satisfying $x \leq y_1 + y_2$, there exist $x_1, x_2 \in M$ such that $x = x_1 + x_2$ and $x_i \leq y_i$ for $i = 1, 2$. If A is a C^*-algebra with real rank zero, then $V(A)$ is a Riesz monoid, [**19**, Theorem 1.1].

2. Checking finiteness of projections. We start this section with a general result characterizing whether a projection is finite in the multiplier algebra of a stable C^*-algebra. This result is analogous to [**10**, Lemma 1], modified to be applied to the case of C^*-algebras of real rank zero. To do that, we will need a result about cancellation of some kind of projections in multiplier algebras. It derives directly from results of [**8**, Section 1], but we include here a proof of this result for the convenience of the reader.

Lemma 2.1. *Let A be a σ-unital C^*-algebra with real rank zero and stable rank one, and let $p \in A$ and $P, Q \in \mathcal{M}(A)$ be projections. Then*:

1. *If $p \oplus P \sim p \oplus Q$, then $P \sim Q$.*
2. *If $p \oplus P \lesssim p \oplus Q$, then $P \lesssim Q$.*

Proof. (1) According to [**8**, Lemma 1.3], there exist $\{p_n\}_{n\geq 1}$ and $\{q_n\}_{n\geq 1}$ approximate units for $P \cdot A \cdot P$ and $Q \cdot A \cdot Q$, respectively, consisting of increasing sequences of projections. Clearly, $\{p \oplus p_n\}_{n\geq 1}$ and $\{p \oplus q_n\}_{n\geq 1}$ become approximate units for $(p\oplus P)\cdot M_2(A)\cdot(p\oplus P)$ and $(p \oplus Q) \cdot M_2(A) \cdot (p \oplus Q)$, respectively, consisting of increasing sequences of projections. Since $p \oplus P \sim p \oplus Q$, applying recurrently [**8**, Lemma 1.3(b)] and relabeling the indexes if necessary, we can assume that

$$p \oplus p_1 \lesssim p \oplus q_1 \lesssim p \oplus p_2 \lesssim \cdots p \oplus p_n \lesssim p \oplus q_n \lesssim \cdots .$$

Since $\mathrm{sr}\,(A) = 1$, by [**16**] we can cancel the projection p in the above inequalities and hence $p_i \lesssim q_i$ and $q_i \lesssim p_{i+1}$ for every $i \geq 1$. Thus we get

$$p_1 \lesssim q_1 \lesssim p_2 \lesssim \cdots p_n \lesssim q_n \lesssim \cdots .$$

Then by [**8**, Proposition 1.7, Theorem 1.10], we conclude that $P \sim Q$.

(2) It is an analogous argument to that used to proof part (1). □

Proposition 2.2. *Let A be a σ-unital C^*-algebra with real rank zero and stable rank one. For a projection $P \in \mathcal{M}(A)$ the following conditions are equivalent*:

1. *P is infinite.*
2. *There exists a nonzero projection $p \in A$ such that, for any projection $q \in A$ satisfying $q \leq P$, we have $p \lesssim P - q$.*

Proof. (1) $\Rightarrow$ (2). Since P is infinite, a nonzero projection $R \in \mathcal{M}(A)$ exists such that $P \sim P \oplus R$. Also, as A is a σ-unital C^*-algebra with real rank zero, by [**8**, Lemma 1.3], $R \cdot A \cdot R$ is a σ-unital C^*-algebra with an approximate unit consisting of an increasing sequence of projections.

Pick one of them, say p, that clearly is in A, and notice that $p \lesssim R$. Now for any projection $q \in A$ with $q \leq P$, we have

$$q \oplus (P - q) = P \sim P \oplus R = q \oplus (P - q) \oplus R.$$

By Lemma 2.1, q cancels from direct sums, whence $P - q \sim (P - q) \oplus R$. Thus, $p \lesssim R \leq (P - q) \oplus R \sim P - q$.

(2) $\Rightarrow$ (1). Assume that there exists $p \in A$, a projection satisfying (2). By [**8**, Lemma 1.3] there exists $\{p_n\}_{n\geq 0}$, an approximate unit of $P \cdot A \cdot P$, consisting of an increasing sequence of projections so that, if we define $p_{-1} = 0$, then $P = \sum_{i=0}^{\infty}(p_i - p_{i-1})$. Since $p \lesssim P$, again by [**8**, Lemma 1.3], there is a nonnegative integer n_0 such that $p \lesssim p_{n_0}$ (and we can assume that $n_0 = 0$). By hypothesis $p \lesssim P - p_0$. Since $P - p_0 = \sum_{i=1}^{\infty}(p_i - p_{i-1})$, we have that $\{(p_n - p_0)\}_{n\geq 0}$ is an approximate unit consisting of an increasing sequence of projections for $(P - p_0) \cdot A \cdot (P - p_0)$. Thus, again by [**8**, Lemma 1.3], there exists a positive integer n_1 (we can assume $n_1 = 1$) such that $p \lesssim p_1 - p_0$. Thus, projections $p'_0 \leq p_0$ and $p'_1 \leq p_1 - p_0$ exist with $p'_i \sim p$. As p_0 and $p_1 - p_0$ are orthogonal, so are p'_0 and p'_1. Also, $p \lesssim P - p_1$. Hence, applying recurrently this argument and reindexing, we show that for any positive integer n there exists a projection $p'_n \leq p_n - p_{n-1}$ such that $p'_n \sim p$. Thus P dominates an infinite sequence of pairwise orthogonal projections $\{p'_n\}_{n\geq 0}$ such that $p'_n \sim p$, and also $p'_n \in (p_n - p_{n-1}) \cdot A \cdot (p_n - p_{n-1})$, for every $n \geq 0$. Since $\{(p_n - p_{n-1})\}_{n\geq 0}$ are pairwise orthogonal projections whose partial sums converge to P in the strict topology of $\mathcal{M}(A)$, we conclude that $Q = \sum_{i=0}^{\infty} p'_i$ is a subprojection of P because of [**21**, Proposition 1.7]. Notice that the same holds for $\{p'_{2n}\}_{n\geq 0}$ and $\{p'_{2n+1}\}_{n\geq 0}$ so that $Q' = \sum_{i=0}^{\infty} p'_{2i}$ and $Q'' = \sum_{i=0}^{\infty} p'_{2i+1}$ are pairwise orthogonal subprojections of Q with $Q = Q' \oplus Q''$. Moreover, as $p'_i \sim p$ for all $i \geq 1$, again by [**21**, Proposition 1.7] we conclude that $Q' \sim Q'' \sim Q$. Hence, Q is infinite, and therefore P is infinite. □

Notice that the equivalent properties that appear in Proposition 2.2 imply that there exists a projection $p \in A$ such that P dominates an infinite family of pairwise orthogonal projections equivalent to p. Nevertheless, the last property does not suffice, in general, to guarantee that P is an infinite projection. For example, let $\mathcal{H}$ be a countable Hilbert space, let $\mathcal{K}$ be the C^*-algebra of compact operators on $\mathcal{H}$, and let $\mathcal{K}^\sim$ be the unitification of $\mathcal{K}$. Then $\mathrm{RR}(\mathcal{K}^\sim) = 0$ and $\mathrm{sr}(\mathcal{K}^\sim) = 1$ (so that it contains no infinite projections), but the rank one projections of $\mathcal{K}^\sim$ define a countable set of nonzero pairwise equivalent projections. Clearly $1_{\mathcal{K}^\sim}$ dominates this set, but it is a finite projection.

Now we will show that, in some cases, the equivalence of these three properties holds, providing a powerful tool for checking finiteness of projections.

Definition 2.3. Let R be a ring. The index of nilpotence of a nilpotent element $x \in R$ is the least positive integer n such that $x^n = 0$. The index of nilpotence of R is the supremum of the indices of all nilpotent elements of R. If this supremum is finite, then R is said to have bounded index of nilpotence. We denote the index of nilpotence of R by $i(R)$.

Now we will quote a result that we will need in the sequel.

Proposition 2.4. *For a C^*-algebra A, the following are equivalent*:

1. *$i(A) \leq n$.*

2. *Every irreducible representation of A has dimension less than or equal to n.*

Proof. This is a restatement of [**1**, Lemma 6.1.4]. □

As a consequence, we obtain the following results.

Corollary 2.5. *If A is a C^*-algebra with bounded index of nilpotence and k is a positive integer, then $i(M_k(A)) = k \cdot i(A)$.*

Proof. Let $i(A) = n$ for some positive integer n. Then, by Proposition 2.4, we have that A is isomorphic to a C^*-subalgebra of $\prod M_{t_i}(\mathbf{C})$, where the product ranges over the set of irreducible representations of A, $t_i \leq n$ for all i, there exists a subindex i such that $t_i = n$, and for each i there exists an epimorphism $A \to M_{t_i}(\mathbf{C})$. Then $M_k(A)$ is isomorphic to a C^*-subalgebra of $\prod M_{k \cdot t_i}(\mathbf{C})$ and for each i, there is an epimorphism from $M_k(A)$ to $M_{k\cdot t_i}(\mathbf{C})$. Thus $i(M_k(A)) \leq k \cdot i(A)$ by Proposition 2.4, and the equality holds because A has an irreducible representation of dimension $k \cdot n$. □

Corollary 2.6. *If A is a C^*-algebra with real rank zero and bounded index of nilpotence, then A has stable rank one.*

Proof. Since $RR(A) = 0$, A is spectral in the sense of [**12**]. Since A has bounded index of nilpotence, every irreducible representation of A is finite-dimensional by Proposition 2.4, so that every primitive factor of A is finite dimensional (e.g., [**13**, Theorem 5.4.2]). Thus, the result holds because of [**12**, Corollary 4.6]. □

Examples of C^*-algebras of real rank zero with bounded index of nilpotence are, among others:

(i) AF-C^*-algebras with bounded index of nilpotence.

(ii) Given any natural number n, the algebra $M_n(\mathcal{C}_0(X))$, where X is a locally compact, Hausdorff space of (covering) dimension 0.

(iii) More in general, the C^*-algebra of a continuous field of C^*-algebras of real rank zero with bounded index of nilpotence over locally compact, Hausdorff spaces of (covering) dimension 0.

We will need the following result.

Proposition 2.7 [**7**, Theorem 7.2]. *Let R be a unital ring with index of nilpotence $n < +\infty$. Then R_R contains no direct summands consisting of a direct sum of $(n+1)$ nonzero pairwise isomorphic submodules.*

The next result is analogous to [**9**, Lemma 1], but the proof we present here is different from Kutami's one.

Lemma 2.8. *Let A be a unital C^*-algebra with real rank zero and index of nilpotence $n < +\infty$, let k be a positive integer, and let $s_k = nk + 1$. Let $p, p_1, \dots, p_{s_k} \in M_k(A)$ and $q_2, \dots, q_{s_k} \in A \otimes \mathcal{K}$ be projections. If $p_1 \sim p_2 \oplus q_2 \sim \cdots \sim p_{s_k} \oplus q_{s_k}$ and $p_1 \oplus \cdots \oplus p_{s_k} \leq p$,* *then* $p_1 \lesssim q_2 \oplus \cdots \oplus q_{s_k}$.

Proof. Since $RR(A \otimes \mathcal{K}) = 0$, [**5**, Corollary 3.3], there exist subprojections p'_j of p_j and q'_j of q_j for $j = 3, \dots, s_k$ such that $p'_{j-1} \sim p'_j \oplus q'_j$, where $p'_2 = p_2$ because of [**19**, Theorem 1.1]. Notice that $\{p_1, p'_2, \dots, p'_{s_k}\}$ is a family of orthogonal subprojections of p with $p'_{s_k} \lesssim \cdots \lesssim p'_2 \lesssim p_1$. Since p is a projection of $M_k(A)$ and $i(A) = n$, then $i(M_k(A)) = nk$ by Corollary 2.5, and hence Proposition 2.7 implies that $p'_{s_k} = 0$, so that $p_1 \sim p_2 \oplus q_2 \sim \cdots \sim q'_{s_k} \oplus \cdots \oplus q'_3 \oplus q_2 \leq q_2 \oplus \cdots \oplus q_{s_k}$, which ends the proof. □

The next result is analogous to [**9**, Theorem 2], slightly modified to be applied to the case of C^*-algebras with real rank zero.

Theorem 2.9. *Let A be a unital C^*-algebra with real rank zero and bounded index of nilpotence. For a projection $P \in \mathcal{M}(A \otimes \mathcal{K})$ the following conditions are equivalent*:

1. *P is infinite.*

2. *There exists a nonzero projection $p \in A$ such that $n \cdot p \lesssim P$ for all $n \in \mathbf{N}$.*

3. *There exists a nonzero projection $p \in A$ such that, for any projection $q \in A \otimes \mathcal{K}$ satisfying $q \leq P$, we have $p \lesssim P - q$.*

Proof. Notice that (2) is equivalent to the fact that there exists a nonzero projection $p \in A$ such that P dominates an infinite family of pairwise orthogonal projections equivalent to p, because of Corollary 2.6 and Lemma 2.1 (see, for example, [**7**, Proposition 4.8]).

(1) $\Rightarrow$ (2). Since P is infinite, a standard argument (e.g., [**7**, Proposition 5.5]) shows that it dominates a countable infinite sequence of nonzero pairwise orthogonal projections that are pairwise equivalent. Let Q be one of them. Since $A \otimes \mathcal{K}$ is a σ-unital C^*-algebra with real rank zero, by [**8**, Lemma 1.3], $Q \cdot (A \otimes \mathcal{K}) \cdot Q$ is a σ-unital C^*-algebra with an approximate unit consisting of an increasing sequence of projections. Pick one of them, say q, that clearly is in $A \otimes \mathcal{K}$, and notice that by [**19**, Theorem 1.1], $q \sim q_1 \oplus \cdots \oplus q_k$ for $q_1, \dots, q_k \in A$ nonzero projections. Thus the result holds by taking $p = q_1$.

(2) ⇒ (3). Assume that a nonzero projection $p \in A$ exists such that P dominates an infinite sequence of pairwise orthogonal projections equivalent to p, and let $q \in A \otimes \mathcal{K}$ satisfy $q \leq P$. By [**8**, Lemma 1.3] $P \cdot (A \otimes \mathcal{K}) \cdot P$ contains an approximate identity $\{p_i\}_{i \geq 0}$ consisting of an increasing sequence of projections. Moreover, there is a positive integer n such that $p \lesssim p_n$ and $q \lesssim p_n$. Since $\mathrm{sr}\,(A) = 1$ by Corollary 2.6, q cancels from direct sums by Lemma 2.1, whence $P - p_n \lesssim P - q$. Thus, if we change q to p_n, then there is no loss of generality assuming that $p \lesssim q$. Hence there exists a decomposition $q = q^0 \oplus q^{(0)}$ such that $p \sim q^0$. Also we can assume that $q \in M_k(A)$ for some $k \in \mathbf{N}$. Set $Q = P - q$ so that $p \lesssim q \oplus Q$. By hypothesis $2 \cdot p \lesssim P = q \oplus Q = q^0 \oplus q^{(0)} \oplus Q$. Since p cancels from direct sums by Lemma 2.1, we have that $p \lesssim q^{(0)} \oplus Q$. Now, as $A \otimes \mathcal{K}$ is σ-unital with real rank zero, again by [**19**, Theorem 1.1] we have that $V(\mathcal{M}(A \otimes \mathcal{K}))$ is a Riesz monoid. Then decompositions $q^{(0)} = q^1 \oplus q^{(1)}$ and $Q = Q^1 \oplus Q^{(1)}$ exist such that $p \sim q^1 \oplus Q^1$. Again by the hypothesis we have $3 \cdot p \lesssim P = q^0 \oplus q^1 \oplus q^{(1)} \oplus Q^1 \oplus Q^{(1)}$, and since p cancels from direct sums by Lemma 2.1, we conclude that $p \lesssim q^{(1)} \oplus Q^{(1)}$. We continue this procedure and we obtain, for each $m \geq 1$, decompositions $q^{(m)} = q^{m+1} \oplus q^{(m+1)}$ and $Q^{(m)} = Q^{m+1} \oplus Q^{(m+1)}$ such that $p \sim q^{m+1} \oplus Q^{m+1}$. Thus, applying Lemma 2.8, we have a positive integer m_0 such that $p \lesssim Q^1 \oplus \cdots \oplus Q^{m_0} \leq Q$, whence the result holds.

(3) ⇒ (1). By Corollary 2.6 and Proposition 2.2. □

Finally, in the case of the algebra A being simple, we can give a characterization of finiteness of projections in $\mathcal{M}(A)$.

Theorem 2.10. *Let A be a σ-unital, nonunital, simple C^*-algebra of real rank zero and stable rank one. For a projection $P \in \mathcal{M}(A)$ the following conditions are equivalent*:

1. *$P \cdot A \cdot P$ is stable.*
2. *P is properly infinite.*
3. *P is infinite.*
4. *There exists a nonzero projection $p \in A$ such that $n \cdot p \lesssim P$ for all $n \in \mathbf{N}$.*
5. *For any projection $q \in A \otimes \mathcal{K}$ we have $q \lesssim P$.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3). It is direct from [**17**, Theorem 3.5, Proposition 3.6].

(3) $\Rightarrow$ (4). It is the same proof as Theorem 2.9.

(4) $\Rightarrow$ (5). Let $p \in A$ satisfy (4), let $q \in A \otimes \mathcal{K}$ be a nonzero projection, and notice that by [**19**, Theorem 1.1], $q \sim q_1 \oplus \cdots \oplus q_k$ for $q_1, \ldots, q_k \in A$ nonzero projections. Since A is simple, for each $i = 1, \ldots, k$, there exists $n_i \in \mathbf{N}$ such that $q_i \lesssim n_i \cdot p$. Thus, if we take $n = n_1 + \cdots + n_k$, we get

$$q \sim q_1 \oplus \cdots \oplus q_k \lesssim n_1 \cdot p \oplus \cdots \oplus n_k \cdot p = n \cdot p \lesssim P.$$

(5) $\Rightarrow$ (3). Fix a nonzero projection $p \in A$, and let $q \in A \otimes \mathcal{K}$ be any nonzero projection satisfying $q \leq P$. Now, since A is simple, a positive integer n exists such that $p \lesssim n \cdot q$. By hypothesis,

$$n \cdot q \oplus q = (n+1) \cdot q \lesssim P = q \oplus (P - q),$$

whence, by Lemma 2.1, $n \cdot q \lesssim (P - q)$. Thus, $p \lesssim (P - q)$, and then we conclude the desired result because of Proposition 2.2. $\square$

Moreover, if we add to the above result the extra hypothesis of the algebra A having $V(A)$ strictly unperforated, then we can characterize the existence of orthogonal finite projections whose sum is infinite in terms of the structure of the space of states of the algebra. To do that we recall some definitions and results of [**15**]. Given (M, u) an abelian monoid with order-unit, endowed with the algebraic preorder, we denote by S_u the compact convex space of states on M, by $\mathrm{Aff}\,(S_u)^+$ the monoid of positive, affine and continuous functions from S_u to $\mathbf{R}$, and by $\phi_u : M \to \mathrm{Aff}\,(S_u)^+$ the natural evaluation map. Also, let $\mathrm{LAff}_\sigma(S_u)^{++}$ be the monoid of strictly positive, affine, lower semi-continuous functions from M to $\mathbf{R}$ that are pointwise suprema of increasing sequences of functions in $\mathrm{Aff}\,(S_u)^+$. Thus, if A is a σ-unital, nonunital, simple, nonelementary, C^*-algebra of real rank zero, stable rank one and $V(A)$ strictly unperforated, and we fix $\{e_n\}_{n \geq 1}$ any approximate unit of A consisting of an ascending chain of projections, $u = [e_1] \in V(A)$, $d = \sup_{n \in \mathbf{N}} \phi_u([e_n])$ (this function is the *scale of* A) and $W^d_\sigma(S_u) = \{f \in \mathrm{LAff}_\sigma(S_u)^{++} \mid f + g = nd$ for some $g \in \mathrm{LAff}_\sigma(S_u)^{++}$ and $n \in \mathbf{N}\}$, we have a normalized monoid isomorphism

$$\varphi(V(\mathcal{M}(A), [1_{\mathcal{M}(A)}])) \longrightarrow V(A) \sqcup W^d_\sigma(S_u)$$

given by the rule $\varphi(p) = p$ for any $p \in A \otimes \mathcal{K}$, and by $\varphi([P]) = \sup_{n\in\mathbf{N}} \phi_u([p_n])$ for any $P \in \mathcal{M}(A) \otimes \mathcal{K}$, being $\{p_n\}_{n\geq 1}$ any approximate unit of $P \cdot A \cdot P$ consisting of an ascending chain of projections ([**15**, Theorem 3.10]). In this context, we say that the algebra has *identically infinite scale* if the function d defined above takes the value ∞ on every element of S_u. Notice that, in particular, identically infinite scale implies $d = 2d$, and through the above-mentioned isomorphism, that $1_{\mathcal{M}(A)} \sim 2 \cdot 1_{\mathcal{M}(A)}$. Equivalently, this means that A is stable [**17**, Proposition 3.6]. Then we have the following result.

Proposition 2.11. *Let A be a unital, simple, nonelementary C^*-algebra of real rank zero, stable rank one and $V(A)$ strictly unperforated. Then the following are equivalent*:

1. *The space state S_u has more than one element.*

2. *There exists a projection $P \in \mathcal{M}(A \otimes \mathcal{K})$ such that P and $1_{\mathcal{M}(A\otimes\mathcal{K})} - P$ are finite.*

3. *There exist finite projections $P, Q \in \mathcal{M}(A \otimes \mathcal{K})$ such that the sum $P \oplus Q$ is infinite.*

Proof. (1) $\Rightarrow$ (2). Suppose that S_u does not consist of a single element. Then at least two different extremal states $s, t \in \partial_e(S_u)$ exist. Thus, by [**15**, Proposition 4.13], there exist $f, g \in W_\sigma^d(S_u)$ such that $f(s) = 1$, $g(t) = 1$ and $f + g = d$. Since $A \otimes \mathcal{K}$ is stable, the remark before Proposition 2.11 shows that the scale d is identically infinite. Let $Q, Q' \in \mathcal{M}(A \otimes \mathcal{K})$ be projections satisfying $\varphi(Q) = f$ and $\varphi(Q') = g$. Then, using the above-mentioned isomorphism we conclude that both Q and Q' are finite. But $\varphi(Q \oplus Q') = d = \varphi(1_{\mathcal{M}(A\otimes\mathcal{K})})$, whence $Q \oplus Q' \sim 1_{\mathcal{M}(A\otimes\mathcal{K})}$. Thus there exists a projection P such that $P \sim Q$ and $1_{\mathcal{M}(A\otimes\mathcal{K})} - P \sim Q'$, whence the result holds.

(2) $\Rightarrow$ (3). It is immediate, because under our hypothesis, $1_{\mathcal{M}(A\otimes\mathcal{K})}$ is properly infinite by [**17**, Proposition 3.6].

(3) $\Rightarrow$ (1). If S_u consists of a single element, we have that $W_\sigma^d(S_u) \cong \mathbf{R}^{++} \sqcup \{+\infty\}$. Hence, through the above-mentioned isomorphism, finite projections correspond to real numbers and $\varphi(1_{\mathcal{M}(A\otimes\mathcal{K})}) = +\infty$. Thus, for any finite projections $P, Q \in \mathcal{M}(A \otimes \mathcal{K})$, $\varphi(P) = \alpha$ and $\varphi(Q) = \beta$ for some $\alpha, \beta \in \mathbf{R}$, whence $\varphi(P \oplus Q) = \alpha + \beta \neq +\infty$. ◻

We end the note by showing the application of our results to a couple of examples. The first one is an example whose existence was quoted implicitly by Lin (see [**11**]), although we will quote here an analogous example due to Rørdam that appears explicitly in [**17**], of the multiplier algebra of a simple AF C^*-algebra containing a couple of orthogonal finite projections whose sum is properly infinite.

Example 2.12 ([**11**], [**17**]). A unital simple AF C^*-algebra A such that $\mathcal{M}(A \otimes \mathcal{K})$ contains a projection P such that both P and $1_{\mathcal{M}(A\otimes\mathcal{K})} - P$ are finite, but $1_{\mathcal{M}(A\otimes\mathcal{K})} \sim 2 \cdot 1_{\mathcal{M}(A\otimes\mathcal{K})}$.

Proof. Let A be the unique unital AF C^*-algebra associated to the dimension group

$$G = \mathbf{Q} \oplus \mathbf{Q}, \quad G^+ = \{(s,t) \in G \mid s > 0,\ t > 0\} \cup \{0\},$$

with order-unit $u = (1,1)$. Since (G, G^+) is a simple dimension group, it follows that A is a simple C^*-algebra. Let τ_1, τ_2 be the (extremal) tracial states on A given by $K_0(\tau_1)(s,t) = s$ and $K_0(\tau_2)(s,t) = t$. Then S_u contains more than one element, and then the result holds because of Proposition 2.11. □

In [**17**, Example 4.3] an explicit description of a projection can be found, as in the example above. The second example is an adaptation of a construction given by Kutami ([**9**]) for von Neumann regular rings, to the context of C^*-algebras that allows us to obtain an example of the same pathology, but starting with a nonsimple AF C^*-algebra.

Example 2.13. A unital nonsimple AF C^*-algebra A such that the algebra $\mathcal{M}(A \otimes \mathcal{K})$ contains a projection P with both P and $1_{\mathcal{M}(A\otimes\mathcal{K})} - P$ are finite, but $1_{\mathcal{M}(A\otimes\mathcal{K})} \sim 2 \cdot 1_{\mathcal{M}(A\otimes\mathcal{K})}$.

Proof. Consider, for each positive integer $n \geq 0$, the C^*-algebra $A_n = \oplus_{i=1}^{2^n} \mathbf{C}$ and the maps $\varphi_{n,n+1} : A_n \to A_{n+1}$ defined by the rule $\varphi_{n,n+1}(x) = (x,x)$. Let $A = C^* - \varinjlim A_n$ and notice that A is a unital AF C^*-algebra so that $RR(A) = 0$ and $\mathrm{sr}\,(A) = 1$. Moreover, as A is abelian, $i(A) = 1$. Now consider $\bar{p}_1 = (1,0) \in A_1$ and $\bar{q}_1 = (0,1) \in A_1$, and define inductively the set $\{\bar{p}_i, \bar{q}_i\}_{i\geq 2}$

$$\begin{aligned} \bar{p}_{2i} &= (\bar{p}_i, 0) \in A_{n+1}, \qquad & \bar{q}_{2i} &= (0, \bar{p}_i) \in A_{n+1} \text{ for } \bar{p}_i \in A_n, \\ \bar{p}_{2i+1} &= (\bar{q}_i, 0) \in A_{n+1}, \qquad & \bar{q}_{2i+1} &= (0, \bar{q}_i) \in A_{n+1} \text{ for } \bar{q}_i \in A_n. \end{aligned}$$

Let $\varphi_i : A_i \to A$ be the canonical maps, and let $p_i = \varphi_i(\bar{p}_i)$, $q_i = \varphi_i(\bar{q}_i)$ be the projections of A coming from the above family. Notice that for each $k \geq 1$, we have $p_k = p_{2k} + q_{2k}$ and $q_k = p_{2k+1} + q_{2k+1}$. Moreover, for every $k \geq 1$, p_k cannot be obtained by sums of p_j's with $j > k$ or by sums of q_j's with $j > k$, and the same holds for q_k. Now, since A is abelian, if $r \lesssim p_k$, then $r \leq p_k$, whence $r \geq p_j$ or $r \geq q_j$ for some p_j or q_j subprojection of p_k. Thus it is clear from the previous remarks that if $r \lesssim p_k$, then

$$(*) \qquad r \not\lesssim \oplus_{i=j+1}^{n+j+1} p_i \quad \text{for any } n \geq 0.$$

Also observe that, for any $k \geq 1$, the set $\{p_{2^{k-1}}, \dots, p_{2^k-1}, q_{2^{k-1}}, \dots, q^{2^k-1}\}$ consists of pairwise orthogonal projections of A with

$$\sum_{i=2^{k-1}}^{2k-1} p_i + \sum_{i=2^{k-1}}^{2^k-1} q_i = 1.$$

Let $\{e_{ij}\}_{i,j\geq 1}$ be a complete set of matrix units for $\mathcal{K}$, choose $\{\sum_{i=1}^{n} 1 \otimes e_{ii}\}_{n\geq 1}$ be an approximate unit for $A \otimes \mathcal{K}$ consisting of an increasing sequence of projections and let $\{\tilde{p}_i\}_{i\geq 1}$ and $\{\tilde{q}_i\}_{i\geq 1}$ be projections of $A \otimes \mathcal{K}$ defined as follows:

$$\tilde{p}_i = \left(\sum_{j=2^{i-1}}^{2^i-1} p_j \right) \otimes e_{ii},$$

$$\tilde{q}_i = \left(\sum_{j=2^{i-1}}^{2^i-1} q_j \right) \otimes e_{ii}.$$

Notice that for each $i \geq 1$, $\tilde{p}_i$ and $\tilde{q}_i$ are orthogonal and $\tilde{p}_i + \tilde{q}_i = 1 \otimes e_{ii}$, so that $\tilde{p}_i, \tilde{q}_i \in (1 \otimes e_{ii})(A \otimes \mathcal{K})(1 \otimes e_{ii})$. Since the projections $1 \otimes e_{ii}$ are pairwise orthogonal and $\sum_{i=1}^{\infty} 1 \otimes e_{ii} = 1_{\mathcal{M}(A\otimes\mathcal{K})}$, we conclude from [**21**, Proposition 1.7] that $P = \sum_{i=1}^{\infty} \tilde{p}_i$ and $Q = \sum_{i=1}^{\infty} \tilde{q}_i$ are orthogonal projections of $\mathcal{M}(A \otimes \mathcal{K})$ such that $P \oplus Q = 1_{\mathcal{M}(A\otimes\mathcal{K})}$. In particular, the elements $\sum_{i=1}^{n} \tilde{p}_i$ form an approximate unit for $P \cdot (A \otimes \mathcal{K}) \cdot P$ consisting of an increasing sequence of projections, and the same holds for the elements $\sum_{i=1}^{n} \tilde{q}_i$ of $Q \cdot (A \otimes \mathcal{K}) \cdot Q$.

Now let $r \in A$ be a projection such that $r \lesssim P$. Then, by [**8**, Lemma 1.3], $r \lesssim p'_n := \tilde{p}_1 \oplus \cdots \oplus \tilde{p}_n$ for some $n \in \mathbf{N}$. Since $RR(A\otimes\mathcal{K}) = 0$, there exist $r_1, \dots, r_n$ orthogonal subprojections of r such that $r = r_1 \oplus \cdots \oplus r_n$ and $r_i \lesssim \tilde{p}_i$. Thus, in view of $(*)$, for each $i = 1, \dots, n$, there exists $k_i \geq 1$ such that $r_i \not\lesssim P - p'_{k_i}$. If we take $k = \min\{k_1, \dots, k_n\}$, then $r = r_1 \oplus \cdots \oplus r_n \not\lesssim P - p'_k$. The same holds for Q, whence we have that for any nonzero projection $r \in A$, projections $p, q \in A \otimes \mathcal{K}$ exist such that $p \leq P$, $q \leq Q$ and $r \not\lesssim P - p$, $r \not\lesssim Q - q$. Thus, by Theorem 2.9, we conclude that both P and Q are finite projections. On the other hand, it is well known that $1_{\mathcal{M}(A\otimes\mathcal{K})} \sim 2 \cdot 1_{\mathcal{M}(A\otimes\mathcal{K})}$ (see, for example, [**18**, Theorem 15.4.6]).

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Departamento de Matemáticas, Universidad de Cádiz, Apartado 40, 11510 Puerto Real (Cádiz), Spain.
E-mail address: enrique.pardo@uca.es