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REAL GENUS ACTIONS OF FINITE SIMPLE GROUPS

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ABSTRACT. A finite group G can be represented as a group of dianalytic automorphisms of a compact bordered Klein surface, that is, G acts effectively on a bordered surface. The *real genus* $\rho(G)$ is the minimum algebraic genus of any bordered surface on which G acts. A *real genus action* of G is an action of G on a bordered surface of (algebraic) genus $\rho(G)$. In this paper we consider real genus actions of finite simple groups. Let G be a finite simple group, and let X be a bordered surface of least genus on which G acts. We show that if G is $(2,s,t)$-generated, then G is normal in $\mathrm{Aut}(X)$, $[\mathrm{Aut}\,X : G]$ divides 4, and $\mathrm{Aut}\,X$ embeds faithfully in $\mathrm{Aut}\,G$. We also consider the real genus actions of each projective special linear group $\mathrm{PSL}(2,q)$.

1. Introduction. In connection with group actions on bordered surfaces, there is a natural parameter associated with each finite group. A finite group G can be represented as a group of dianalytic automorphisms of a compact bordered Klein surface, that is, G acts effectively on a bordered surface. The *real genus* $\rho(G)$ [**12**] is the minimum algebraic genus of any bordered surface on which G acts. A *real genus action* of G is an action of G on a bordered surface of (algebraic) genus $\rho(G)$.

There is now a considerable body of work on the real genus parameter, and genus formulas have been obtained for several classes of groups [**12**], [**14**], [**15**], [**16**], [**17**], [**18**]. Almost all of this work has concentrated on solvable groups. Most notably, McCullough calculated the real genus of each finite abelian group [**18**].

Here we consider actions of finite simple groups on bordered surfaces; in this paper simple always means nonabelian simple. We are particularly interested in the real genus actions of these groups. Our main result is the following.

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Theorem 1. *Let G be a finite simple group, and let X be a bordered surface of least genus on which G acts. If G is $(2, s, t)$-generated, then*

1) *G is normal in* $\mathrm{Aut}\, X$*,*
2) $[\mathrm{Aut}\, X : G]$ *divides* 4*, and*
3) $\mathrm{Aut}\, X$ *embeds faithfully in* $\mathrm{Aut}\, G$*.*

The requirement that G be $(2, s, t)$-generated is not at all restrictive. There is a long-standing conjecture that every finite simple group can be $(2, s, t)$-generated.

We also consider the real genus actions of each projective special linear group $\mathrm{PSL}\,(2, q)$. Singerman [**21**] has shown that most of these groups are M^*-groups. An M^*-group is a group G that acts on a bordered surface of genus $g \geq 2$ such that the order of G is $12(g-1)$, the largest possible. The real genus is then determined for an M^*-group by its order. The projective special linear groups that are not M^*-groups are $\mathrm{PSL}\,(2, 7)$, $\mathrm{PSL}\,(2, 9) \cong A_6$, $\mathrm{PSL}\,(2, 11)$, and the infinite family $\mathrm{PSL}\,(2, 3^n)$ with n odd, $n \geq 3$. We find the real genus of each of these exceptional groups.

Symmetric genus actions of finite simple groups on Riemann surfaces were considered by Woldar in [**22**]. Woldar's results provided the motivation for our work here.

2. Preliminaries. We shall assume that all surfaces are compact. A bordered surface X can carry a dianalytic structure [**1**, p. 46] and be considered a Klein surface or a nonsingular real algebraic curve. Thus the surface X has an algebraic genus g. The algebraic genus appears naturally in bounds for the order of the automorphism group of a Klein surface, and the real genus of a group is defined in terms of the algebraic genus.

We use the standard representation of a group G as a quotient of a non-Euclidean crystallographic (NEC) group Γ by a bordered surface group K; then G acts on the Klein surface U/K, where U is the open upper half-plane. A summary of this approach is in [**12**, Section 2]. Also see the monograph [**2**], which is an excellent general reference for the work on Klein surfaces.

There is an upper bound for the real genus of a finite group in terms of the orders of the elements in a generating set [**12**, p. 712]. This bound will be helpful here.

Theorem A [**12**]. *Let G be a finite group with generators $z_1, \dots, z_c$, where $o(z_i) = m_i$. Then*

$$\rho(G) \leq 1 + o(G)\left[c - 1 - \sum_{i=1}^{c} \frac{1}{m_i}\right].$$

A group G is called an (r, s, t)-group in case G is generated by three distinct elements A, B, C with partial presentation

$$A^r = B^s = C^t = ABC = 1,$$

where $t \geq s \geq r \geq 2$. An (r, s, t)-group is clearly a two-generator group. We shall be particularly interested in the case $r = 2$.

Corollary 1. *Let G be a finite $(2, s, t)$-group. Then*

$$\rho(G) \leq 1 + o(G)\left[\frac{1}{2} - \frac{1}{s}\right].$$

Lots of interesting groups are $(2, s, t)$-groups. Among the simple groups, the alternating groups [**3**], the sporadic groups, and certain two-dimensional projective linear groups [**8**] are generated by an involution and one additional element. Of course, many other groups have this property as well.

3. Large groups. Especially important in the study of automorphisms of bordered Klein surfaces are the quadrilateral groups. An *extended quadrilateral group* is an NEC group with signature

$$(0; \quad +; \quad [\,]; \quad \{(l, m, n, t)\}),$$

where

$$1/l + 1/m + 1/n + 1/t < 2.$$

We denote a group with this signature $\Gamma[l, m, n, t]$.

Let the finite group G act on the bordered Klein surface X of algebraic genus $g \geq 2$. Then the order of G is at most $12(g-1)$ [**9**]. If the order of G is the largest possible, then G is called an M^*-*group* [**10**]; in this case, $G = \operatorname{Aut} X$ of course. This general upper bound was established by considering all possible ramification indices of the covering $\pi : X \to X/G$ and applying the Riemann-Hurwitz formula. An examination of [**9**, Section 3] shows that $o(G) \leq 6(g-1)$ in all but three cases. For more details, see [**13**, Section 3].

Proposition 1. *Let the finite group G act on the bordered Klein surface X of algebraic genus $g \geq 2$. If $o(G) > 6(g-1)$, then $o(G)$ is one of the following; in each case G is a quotient of the extended quadrilateral group by a bordered surface group.*

1) $o(G) = 12(g-1)$ $\quad\Gamma[2,2,2,3]$
2) $o(G) = 8(g-1)$ $\quad\Gamma[2,2,2,4]$
3) $o(G) = 20(g-1)/3$ $\quad\Gamma[2,2,2,5]$

In particular, an M^*-group is a quotient of $\Gamma[2,2,2,3]$, and a large group of automorphisms of a bordered Klein surface must be a quotient of one of these three quadrilateral groups.

Corollary. *Let G be a finite group with $\rho(G) \geq 2$. If G is not a quotient of one of the three quadrilateral groups $\Gamma[2,2,2,3]$, $\Gamma[2,2,2,4]$, and $\Gamma[2,2,2,5]$ by a bordered surface group, then*

$$\rho(G) \geq 1 + o(G)/6.$$

Proof. Let G act on a bordered surface of genus $\rho = \rho(G)$. Then by Proposition 1, $o(G) \leq 6(\rho - 1)$.

If $\rho(G) \geq 2$, then we always have $\rho(G) \geq 1 + o(G)/12$ [**12**, Section 4].

Since an extended quadrilateral group is generated by reflections, Proposition 1 also guarantees that in a large automorphism group, a normal subgroup has even index.

Proposition 2. *Let the finite group G be a quotient of an extended quadrilateral group. If N is a proper normal subgroup of G, then the index of N in G is even.*

Proof. The group G and its quotient G/N are generated by involutions.

Large groups of automorphisms also must have a certain type of partial presentation [**13**, Section 3]; also see [**2**, Section 4.1].

Proposition 3 [**13**]. *Let G be a finite group and $\Gamma = \Gamma[2,2,2,n]$ an extended quadrilateral group. If G is a quotient of Γ by a bordered surface group, then G is generated by three distinct nontrivial elements T, U, V satisfying the relations*

$$T^2 = U^2 = V^2 = (TU)^2 = (TV)^n = 1. \tag{3.1}$$

One consequence of Proposition 3 is that large automorphism groups must be quotients of groups in an infinite family that has been studied quite extensively. Let $G^{n,q,r}$ [**4**] be the group with generators A, B and C and defining relations

$$A^n = B^q = C^r = (AB)^2 = (BC)^2 = (CA)^2 = (ABC)^2 = 1.$$

If we set $T = BC$, $U = CA$, and $V = BCA$, then we obtain the presentation

$$T^2 = U^2 = V^2 = (TU)^2 = (TV)^n = (UV)^q = (TUV)^r = 1.$$

Thus G is a quotient of $\Gamma[2,2,2,n]$ by a bordered surface group if and only if G is a quotient of the group $G^{n,q,r}$ for some q and r. This can be useful if there are enough limitations on n, q and r. The complete table of the known finite groups $G^{n,q,r}$ is in [**5**, pp. 139, 140].

4. Simple groups. Let G be a finite simple (nonabelian) group. Then the real genus of G is not too small. The groups with real genus $\rho \leq 5$ have been classified [**12**], [**13**], [**16**], and no simple group appears in the lists. Also, the smallest simple groups have been considered. The smallest, A_5, is an M^*-group with real genus 6 [**10**, p. 9]. The next two simple groups, PSL $(2, 7)$ and A_6, are not M^*-groups [**11**, p. 384]. Other small simple groups are considered in [**7**, p. 277]; also see [**21**].

Now we consider the real genus action of a simple group G. Let X be a surface of least real genus $\rho = \rho(G)$ on which G acts. We know $\rho \geq 6$, and the standard representation $X = U/K$ applies. First we show that if the simple group is a relatively large group of automorphisms of X, then it must be normal in $\operatorname{Aut} X$.

Proposition 4. *Let G be a finite simple group with real genus action on the bordered Klein surface X of genus $g = \rho(G) \geq 6$. If $o(G) > 2(g-1)$, then G is normal in* $\operatorname{Aut} X$ *and, further, the index* $[\operatorname{Aut} X : G]$ *divides* 4.

Proof. Write $H = \operatorname{Aut} X$. Then $G \subset H$, and we always have $\rho(G) \leq \rho(H)$. But $\rho(H) \leq g$, the genus of X. Thus $\rho(G) = \rho(H)$.

The basic upper bound for the size of an automorphism group gives $o(H) \leq 12(g-1)$ or $g \geq 1 + o(H)/12$. But by hypothesis $g < 1 + o(G)/2$. It follows that $6 > o(H)/o(G) = [H : G]$. Therefore $5 \geq [H : G]$.

There is a representation $\theta : H \to S_5$, where $N =$ kernel $\theta \subset G$ [**19**, p. 48]. Since G is simple, we must have either $N = 1$ or $N = G$. If $N = G$, then G is normal in H, of course. Suppose $N = 1$. Then H is isomorphic to a subgroup of S_5 and H contains the simple group G. The only possibility is $G \cong A_5$, an M^*-group with $\rho = 6$; then further $H = G$. Thus the simple group G is normal in $H = \operatorname{Aut} X$ in any case.

We still need to show that the index $[H : G]$ cannot be 3 or 5. Assume $[H : G] \geq 3$. By hypothesis $o(G) > 2(g-1)$. Then $o(H) > 6(g-1)$ and H is a quotient of an extended quadrilateral group by Proposition 1. Hence, by Proposition 2 the index $[H : G] = 4$ in this case. Thus the only possibilities for $[H : G]$ are 1, 2 and 4.

Next we consider extensions of real genus actions. Let the finite group G act on the bordered surface X. We say that the finite group H *extends the action* of G on X if G is a subgroup of H and H acts on X such that the action G inherits as a subgroup of H is consistent with the original action of G on X.

Under certain conditions a real genus action cannot be properly extended.

Proposition 5. *Let G be a finite group with real genus action on the bordered Klein surface X of genus $g = \rho(G) \geq 2$. Suppose there is a group H that extends the action of G on X. If G is a quotient group of H, then $G = H$.*

Proof. Since H extends the action of G on X, we have $G \subset H \subset \operatorname{Aut} X$, and it follows that $\rho(H) = \rho(G)$. Also, by hypothesis, there is a normal subgroup N of H such that $H/N \cong G$.

Let N act on X and set $X' = X/N$. Let g' be the genus of X' and let $\phi : X \to X'$ be the quotient map. Since X' is a quotient space of X, we have $g' \leq g$. But $H/N \cong G$ acts on X', so that $g' \geq \rho(G) = g$. Thus $g' = g$. Applying the Riemann-Hurwitz formula for coverings of bordered surfaces [**9**, p. 201] to ϕ yields

$$(2g-2)/o(N) = 2g - 2 + \sum \left(1 - \frac{1}{e_i}\right) n_i,$$

where the e_i's are the ramification indices and each n_i is 1 or 2 depending upon whether the ramification is above a boundary or an interior point of X'. Since $g \geq 2$, $2g-2$ is positive and it is easy to see that the only possibility is $o(N) = 1$. Hence $H = G$.

Proposition 6. *Let G be a finite simple group with real genus action on the bordered Klein surface X of genus $g \geq 2$. Suppose G is normal in a group H that extends the action of G on X. Then $C_H(G) = 1$. In particular, if G is normal in* $\operatorname{Aut} X$*, then* $\operatorname{Aut} X$ *embeds faithfully in* $\operatorname{Aut} G$.

Proof. Assume there were a non-identity element x in $C_H(G)$. Since G is simple, $Z(G) = 1$ and $x \notin G$. But now $G \times \langle x \rangle$ is a subgroup of H that extends the action of G on X. Since G is obviously a quotient of $G \times \langle x \rangle$, this contradicts Proposition 5. Hence, $C_H(G) = 1$.

Now let $H = \operatorname{Aut} X$, and assume that G is normal in H so that $N_H(G) = H$. But $N_H(G)/C_H(G)$ is isomorphic to a subgroup of $\operatorname{Aut} G$ [**20**, p. 50]. Thus H embeds faithfully in $\operatorname{Aut} G$.

5. The main result. It is now easy to establish Theorem 1. Let G be a finite simple group with real genus action on the bordered Klein surface X of genus $g = \rho(G)$. We know $g \geq 6$. Assume G is $(2, s, t)$-generated. Then by the corollary to Theorem A, $2(g-1) < o(G)$. Now by Proposition 4, G is normal in $\operatorname{Aut} X$ and the index $[\operatorname{Aut} X : G]$ divides 4. Finally $\operatorname{Aut} X$ embeds faithfully in $\operatorname{Aut} G$ by Proposition 6. This completes the proof of Theorem 1.

The simple group A_5 is an M^*-group that acts on a real projective plane W with 6 holes, a nonorientable Klein surface with genus 6 [**10**, p. 9]. Here A_5 is the full group, that is, $A_5 = \operatorname{Aut} W$.

The group $\operatorname{PGL}(2, 7)$ is an M^*-group with $\rho = 29$ that acts on two different topological types of surfaces, one orientable and one nonorientable [**11**, p. 385]. Let $\operatorname{PGL}(2, 7)$ act on the bordered surface X of genus 29. Then the simple group $G = \operatorname{PSL}(2, 7)$ also acts on X and, as we shall see, $\rho(G) = 29$. Here $[\operatorname{Aut} X : G] = 2$. Note that $\operatorname{Aut}(G) = \operatorname{PGL}(2, 7)$.

We do not know an example of a real genus action of a simple (nonabelian) group G on a surface X for which $[\operatorname{Aut} X : G] = 4$. If this does occur, then we must have $o(G) = 3[\rho(G) - 1]$ and $\operatorname{Aut} X$ is an M^*-group. Further G must be a simple group with $[\operatorname{Aut} G : G] \geq 4$; this of course restricts the possibilities for G.

A real genus action for which $[\operatorname{Aut} X : G] = 4$ can occur. For example, $D_3 \times D_3$ and its normal subgroup $Z_3 \times Z_3$ both have real genus four [**13**, Section 5].

6. PSL Groups. Among the best known simple groups are the projective special linear groups. The group $\operatorname{PSL}(2, q)$ is simple in case $q > 3$ [**19**, p. 163]. Singerman [**21**] has shown that most of these groups are M^*-groups.

Theorem C [**21**]. *Let q be a prime power other than 2, 7, 11 or 3^n, where $n = 2$ or n is odd. Then $\operatorname{PSL}(2, q)$ is an M^*-group and*

$$\rho(\operatorname{PSL}(2, q)) = 1 + (q + 1)(q^2 - q)/12d,$$

where $d = (2, q - 1)$.

The simple groups in this family that are not M^*-groups are

PSL $(2,7)$, PSL $(2,9) \cong A_6$, PSL $(2,11)$ and PSL $(2,3^n)$ with n odd, $n \geq 3$. We find the real genus of each of these groups. The basic properties of the projective special linear groups are in [**6**, Section 15.1].

The group $G = \mathrm{PSL}\,(2,7)$ is the simple group of order 168; we know G is not an M^*-group. The group G has elements of order 2, 3, 4, and 7. Since G has no element of order 5, G cannot be a quotient of the extended quadrilateral group $\Gamma\,[2,2,2,5]$ with kernel a bordered surface group. Suppose G were a quotient of $\Gamma\,[2,2,2,4]$ by a bordered surface group K. Then G would be a quotient of $G^{4,q,r}$ for some q and r, where we may take $q \leq r$ and q and r must be orders of elements of G. All these groups are finite and too small except $G^{4,7,7}$; see [**5**, p. 139] and [**4**, p. 121]. The group $G^{4,7,7}$ is infinite, but if G had partial presentation

$$T^2 = U^2 = V^2 = (TU)^2 = (TV)^4 = (UV)^7 = (TUV)^7 = 1,$$

then $\langle U, V \rangle$ would be a dihedral group of order 14, since the surface group K contains no analytic elements of finite order. But G has no subgroup of order 14 at all, and hence G is not a quotient of $G^{4,7,7}$. Now G is not a quotient of $\Gamma\,[2,2,2,4]$ by a surface group. By the corollary to Proposition 1, $\rho\,(G) \geq 1 + o(G)/6 = 29$. Since there is an action of G on a surface of genus 29 [**11**, p. 385],

$$\rho\,(\mathrm{PSL}\,(2,7)) = 29.$$

The group $G = \mathrm{PSL}\,(2,11)$ is the simple group of order 660; G is not an M^*-group [**21**, p. 149]. Because G has no element of order 4, G is not a quotient of $\Gamma\,[2,2,2,4]$ by a bordered surface group. However, $G \cong G^{5,5,5}$ [**5**, p. 139]. Thus G is a quotient of $\Gamma\,[2,2,2,5]$, and $660 = 20[\rho\,(G) - 1]/3$. Hence

$$\rho\,(\mathrm{PSL}\,(2,11)) = 100.$$

The group $\mathrm{PSL}\,(2,9) \cong A_6$ is the simple group of order 360; A_6 is not an M^*-group either. The group A_6 has no element with order larger than 5, and it is easy to see that A_6 is not a quotient of $G^{n,q,r}$, where n is 4 or 5 and q and r are orders of elements in A_6. Consequently, A_6 is not a quotient of $\Gamma\,[2,2,2,n]$ by a surface group. By the Corollary to Proposition 1, $\rho\,(G) \geq 1 + o(G)/6 = 61$.

One presentation of A_6 is the following [**5**, p. 137]

$$L^2 = M^2 = N^2 = (LM)^3 = (MN)^3 = (LN)^4 = (LMN)^5 = 1.$$

Let $\Gamma = \Gamma\,[2,2,3,3]$. The quadrilateral group Γ is generated by four reflections t, u, v, w with defining relations

$$t^2 = u^2 = v^2 = w^2 = (tu)^2 = (uv)^2 = (vw)^3 = (tw)^3 = 1.$$

There is a homomorphism $\phi : \Gamma \to A_6$ onto A_6 defined by

$$\phi(t) = N, \qquad \phi(u) = 1, \qquad \phi(v) = L, \qquad \phi(w) = M,$$

and kernel ϕ is a bordered surface group. Then $\mu(\Gamma)/2\pi = 1/6$, and A_6 acts on a surface genus $g = 1 + 360/6 = 61$ [**12**, Section 2]. Thus

$$\rho\,(\mathrm{PSL}\,(2,9)) = 61.$$

Finally we consider the infinite family $\mathrm{PSL}\,(2,3^n)$ with n odd, $n \geq 3$. These groups are not M^*-groups [**21**, p. 149].

Theorem 2. *Let $q = 3^n$, where n is odd and $n \geq 3$, $G = \mathrm{PSL}\,(2,q)$ and $\rho = \rho\,(G)$. Then $o(G) = 6(\rho - 1)$ and*

$$\rho\,(\mathrm{PSL}\,(2,q)) = 1 + (q+1)(q^2 - q)/12.$$

Proof. The group G is generated by two elements, one of order 2 and one of order 3 [**8**, p. 29]. Thus, by Theorem A, $\rho\,(G) \leq 1 + o(G)/6$.

The order $o(G) = 3^n(3^{2n} - 1)/2$ [**19**, p. 163]. Then, since n is odd, $3^{2n} \equiv 4 \bmod 5$ and $3^{2n} - 1 \equiv 3 \bmod 5$. Hence 5 does not divide $o(G)$. Since G has no element of order 5, G cannot be a quotient of the extended quadrilateral group $\Gamma = \Gamma\,[2,2,2,5]$ with kernel a bordered surface group.

The Sylow 2-subgroup of G is dihedral and has order 4 [**6**, p. 418]. Thus G has no element of order 4, and G cannot be a quotient of $\Gamma\,[2,2,2,4]$ by a surface group either. Now by the corollary to Proposition 1, $\rho\,(G) \geq 1 + o(G)/6$.

7. Open problems. There are many unsolved problems about real genus actions of simple groups. Perhaps the most interesting question in connection with Theorem 1 is whether or not $[\mathrm{Aut}\, X : G] = 4$ can occur. It would also be interesting to see a proof of the theorem that omits the $(2, s, t)$-generation requirement from the hypothesis.

Conder [**3**] obtained a partial presentation for the alternating group A_n that shows that this simple group is an M^*-group for all $n > 167$. Consequently, for each $n > 167$, $\rho(A_n) = 1 + n!/24$.

Problem 1. Determine $\rho(A_n)$ for $n \leq 167$.

Other families of simple groups could be considered, as well. A family with some of the smaller groups is $\mathrm{PSL}(3, q)$.

Problem 2. Determine $\rho(\mathrm{PSL}(3, q))$ for each prime power q.

Of course, there is a problem of this type for each family of finite simple groups.

Problem 3. Let G be a sporadic simple group. Find $\rho(G)$.

In addition, there is the following general problem.

Problem 4. Determine which simple groups are M^*-groups.

This is equivalent to determining which simple groups are quotients of the extended modular group; see [**7**, p. 277] and [**21**, p. 150].

Finally, another related, more general problem is to find the simple groups that act as full automorphism groups.

Problem 5. Classify all simple groups G for which $\mathrm{Aut}\, X = G$, where X is a bordered surface of genus $\rho(G)$ on which G acts.

If G is an M^*-group, then G is such a group, of course, but there are others.

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