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NONCLASSICAL GORENSTEIN CURVES

E. BALLICO AND A. COSSIDENTE

1. Introduction. Let $\mathcal{C}$ be a complete irreducible algebraic curve of arithmetic genus g, defined over an algebraically closed field $\mathbf{K}$ of characteristic $p > 0$.

Let $\hat{\mathcal{C}}$ be its nonsingular model. Let D be a canonical positive divisor of the curve $\mathcal{C}$, and let $d_0, d_1, \dots, d_{g-1}$ be a basis of the space $H^0(\mathcal{C}, D)$. The canonical morphism

$$(d_0, d_1, \dots, d_{g-1}) : \hat{\mathcal{C}} \longrightarrow \mathbf{P}^{g-1},$$

is uniquely determined by $\mathcal{C}$ up to projectives.

We will always assume that $\mathcal{C}$ is a *Gorenstein curve*, i.e., the canonical morphism induces a morphism $\mathcal{C} \to \mathbf{P}^{g-1}$, and that $\mathcal{C}$ is not *hyperelliptic*, i.e., the canonical morphism induces an isomorphism of $\mathcal{C}$ onto a curve in $\mathbf{P}^{g-1}$. Let $\{b_i\}$, $0 \le i \le g-1$, be the generic Hasse sequence of invariants of $\mathcal{C}$. Hence $b_0 = 0$ and b_i, $1 \le i \le g-1$, is the intersection multiplicity at a general point $Q \in \mathcal{C}$ of the i-dimensional linear subspace of $\mathcal{C}$ at Q. It follows that $b_i \ge i+1$ and $b_{i-1} < b_i$ for every i. If $p = 0$, then $b_i = i$ and $\mathcal{C}$ is said to be *classical*; otherwise $\mathcal{C}$ is said to be *nonclassical*. For further information, see [**23**].

The curves of genus smaller than three or, more generally, hyperelliptic curves are always classical Gorenstein curves. In the nonsingular case, the nonclassical curves of genus three and four have been classified by Komiya [**14**]. In [**9**], the authors extended the classification list by Komiya to nonclassical Gorenstein curves of arithmetic genus three and four. In [**18**], among other results, Rosa classified the nonclassical trigonal Gorenstein curves of genus g when $\operatorname{char} \mathbf{K} = g-1$, $g-2$ or $g-3$. If $\operatorname{char} \mathbf{K} = 2$, she completely solved the case $g = 2^n + 1$, too. See also [**16**], [**17**].

Our aim here is to extend (under suitable assumptions) the classification by Freitas and Stöhr to nonclassical Gorenstein curves of arithmetic genus g by at least 5. We do not give equations of the curves,

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explicitly, but we have a good description of such curves. However, we consider the problem as solved if we can say, "the curve $\mathcal{C}$ is strange, the characteristic is a certain prime p and $\mathcal{C}$ lies on a surface S of a certain type (a ruled surface or a cone)." This is because, in this case, we may describe all such curves in S, [**1**], [**2**].

We will prove the following theorem.

Theorem 1.1. *Let $\mathcal{C} \subset \mathbf{P}^{g-1}$ be a canonically embedded Gorenstein irreducible curve with arithmetic genus $p_a(\mathcal{C}) = g \geq 5$. Let q be the genus of the normalization $\mathcal{C}''$ of $\mathcal{C}$. Let $\{b_i\}$, $0 \leq i \leq g-1$, be the generic Hasse invariants of $\mathcal{C}$. Then we have*

(i) *if $b_1 \geq 3$, then $b_1 = p-3$, $\mathcal{C}$ is strange and contained in a smooth rational normal scroll $S \subset \mathbf{P}^{g-1}$, see* **a)**;

(ii) *Assume $g = 6$, $b_1 = 2$ and $b_2 \geq 4$ and that $\mathcal{C}$ is contained in the Veronese surface of $\mathbf{P}^5$. Thus $\mathcal{C}$ is isomorphic to a plane quintic curve. If $p \neq 2$, then $p = b_2 = 5$, $\mathcal{C}$ is strange and the equations of all such strange curves are given in* [**4**, Section 3] *and in* [**13**, Theorem 3.4]. *If $p = 2$, we have $b_2 = 4$ and there are examples which are strange and, hence, singular, see* **b)**. *If $\mathcal{C}$ is smooth, then it is isomorphic to a Fermat quintic curve with equation $x^5 + y^5 + z^5 = 0$.*

(iii) *The only cases with $b_1 = 2$ and $b_2 \geq 5$ are the ones considered in* **b)** *for $g = 6$.*

(iv) *Assume $b_1 = 2$, $b_2 = 4$ and $p = 2$. Then either $\mathcal{C}$ is strange, see* **e)**, *or there is a double covering $f : \mathcal{C} \to E$ with E an integral curve with $p_a(E) = 1$, see* **g)**. *If E is smooth, then $\mathcal{C}$ is strange. If E is singular, then $q = 0$, $\mathcal{C}$ is contained in a cone T with vertex P and as a base rational normal curve of $\mathbf{P}^{g-2}$. Furthermore, P is a double point of $\mathcal{C}$ and the rational map, $\pi : \mathcal{C} \to E$ induced by the projection from P has degree 2. Any canonical curve $\mathcal{C}$ contained in T, with multiplicity two at P and such that π is inseparable, has $b_2 \geq 4$, see* **f)**.

(v) *If $b_1 = 2$, $b_2 = 4$, $p = 2$, $q = 0$ and $\mathcal{C}$ is not a double covering $f : \mathcal{C} \to E$, with $p_a(E) = 1$, then there is a purely inseparable morphism of degree four $m : \mathcal{C} \to \mathbf{P}^1$.*

(vi) *Assume $q > 0$ and $b_2 \geq 4$. Then $g \leq 9$. If $q = 3, 4$, then $\mathcal{C}$ is classified in* [**14**]. *If $q \geq 5$, then there is a degree two morphism $\mathcal{C} \to E$ with E smooth elliptic curve. If $q > 0$, $g \geq 5$, $b_1 = 2$ and $b_2 = 4$, then*

$\mathcal{C}$ is bielliptic, see **g)**.

Notice that the letters **a)**, **b)**, **e)**, **g)** that appear in the statement of Theorem 1.1, refer to the corresponding steps in the proof given in the next section.

Remark 1.2. We stress that nonclassical smooth canonical curves cannot have $b_2 \geq 4$ if the genus is large.

2. The proof of Theorem 1.1. We divide the proof in several steps according to the type of the generic Hasse sequence of invariants.

We begin with a remark.

Remark 2.1. By [**22**], if $C \subset \mathbf{P}^{g-1}$ is not scheme-theoretically cut out by quadrics, then $\mathcal{C}$ is contained in a minimal degree irreducible surface $S \subset \mathbf{P}^{g-1}$. Such surfaces have been classified [**7**]: S is either a minimal degree rational ruled surface or a cone over the normal rational curve of $\mathbf{P}^{g-2}$ or $g = 6$ and S is the Veronese surface. Furthermore, S is the intersection of the quadrics containing $\mathcal{C}$. In the latter case, $\mathcal{C}$ is isomorphic to a plane quintic and conversely.

a) Assume $b_1 \geq 3$**.** In this case there is an integer $e \geq 1$ such that $b_1 = p^e$. This is a consequence of the p-adic criterion (see [**9**], [**13**, Proposition 3.2a]). Since $b_1 \geq 3$, every quadric hypersurface containing $\mathcal{C}$ contains the ruled variety union of the general tangent lines of $\mathcal{C}$. Hence $\mathcal{C} \subset S$, and S is not the Veronese surface, even if $g = 6$. Firstly assume that S is smooth. Then $S \simeq F_e$ (the Segre-Hirzebruch surface) for an integer e with $0 \leq e \leq g-3$ and $e \equiv g \bmod 2$. Furthermore, the restriction to $\mathcal{C}$ of the ruling $S \to \mathbf{P}^1$ is purely inseparable of degree 3. It follows that $p = b_1 = 3$. In this case all curves $\mathcal{C}$ and their equations have been determined in [**1**], [**2**]. Now assume that S is singular, i.e., let S be the cone over the rational normal curve of $\mathbf{P}^{g-2}$. Let $u : Y \to S$ be the blowing-up of the vertex of the cone and $\mathcal{C}'$ the strict transform of $\mathcal{C}$ in u. We have $Y \simeq F_{g-2}$. Take as a basis of $\mathrm{Pic}\,(S) \simeq \mathbf{Z}^{\oplus 2}$, a minimal degree section h of the ruling and a fiber f of the ruling. Hence, $h^2 = -e$, $hf = 1$ and $f^2 = 0$. The morphism $Y \to S \subset \mathbf{P}^{g-2}$

is induced by the complete linear system $|h + (g - 2)f|$. We have $h = u^{-1}(v)$ where v is the vertex of S. Let μ be the multiplicity of $\mathcal{C}$ at v. It follows that $\mu \geq 0$, $\mu = 0$ if and only if $v \notin \mathcal{C}$ and $\mu = 1$ if and only if $v \in \mathcal{C}_{\text{reg}}$. We have $\mathcal{C}' \in |ah + (\mu + a(g-2)f|$ and $a = sb_1$ with $s \geq 1$ (s is the number of points outside the vertex such that a general line of S intersects $\mathcal{C}$). Since $\deg(\mathcal{C}) = 2g - 2$, we have $2g - 2 = (ah + (\mu + a(g-2))f)(h + (g-2)f) = \mu + a(g-2)$. Hence this case cannot occur.

From now on, we assume $b_1 = 2$ and $b_2 \geq 4$. It follows from [**9**], [**13**], that $p \mid b_2$.

b) Assume $g = 6$ and let $\mathcal{C}$ be a plane quintic curve. We have $b_1 = 2$ because the Veronese surface S has no trisecant lines. We have $b_2 \geq 4$ if and only if in the embedding of degree 5, $j : \mathcal{C} \to \mathbf{P}^2$, $j(\mathcal{C})$ has $b_1(j(\mathcal{C})) \geq 4$ and indeed, in that case, the image of the general tangent line of $j(\mathcal{C})$ is a conic osculating $\mathcal{C}$. From [**11**, Section 3], $\mathcal{C}$ is a strange curve with degree of inseparability equal to $\deg(j(\mathcal{C})) = 5 = p$.

For $p \geq 3$, the equations of all such curves are described in [**4**]. It remains that the case $p = 2$ and $b_2 = 4$ and curves $j(\mathcal{C})$ with $b_1(j(\mathcal{C})) = 4$. Let $\mathcal{D} \subset \mathbf{P}^2$ be an irreducible curve of degree 5 with $b_1(\mathcal{D}) = 4$. Hence $p = 2$. Assume $\mathcal{D}$ strange with P as a strange point. Due to the degree we have that P is a regular point of $\mathcal{D}$. The equations of all such curves are described in [**4**, Section 3]. By Bezout's theorem, the tangent line $T_P(\mathcal{D})$ to $\mathcal{D}$ at P has intersection multiplicity $b_1(\mathcal{D}, P)$ at most 5 with $\mathcal{D}$ at P. Since P is a regular point, $b_1(\mathcal{D}, P)$ is the Hermite invariant of $\mathcal{D}$ at P, and we have $b_1(\mathcal{D}, P) \geq b_1(\mathcal{D}) = 4$ [**13**, Theorem 2.4], [**15**, Proposition 4].

Hence, by Bezout's theorem $T_P(\mathcal{D})$ cannot intersect $\mathcal{D}$ at a singular point P': if $\mathcal{D}$ has multiplicity $\mu \geq 2$, we would have $4 + \mu \leq \deg(\mathcal{D}) = 5$. Since $8 > \deg(\mathcal{D})$, $T_P(\mathcal{D})$ cannot intersect $\mathcal{D}$ at another smooth point. Hence $b_1(\mathcal{D}, P) = 5$.

Let $\mathcal{D} \subset \mathbf{P}^2$ be an irreducible degree 5 curve with $b_1(\mathcal{D}) = 4$. If $\mathcal{D}$ is smooth [**13**, Theorem 6.1], or has a very small number of singularities [**11**, Corollary 5.10], we know all the possible equations of $\mathcal{D}$. In particular, if $\mathcal{D}$ is smooth, then it is projectively equivalent to the Fermat curve $x^5 + y^5 + z^5 = 0$.

c) Assume $b_2 \geq 5$. Since a zero-dimensional subscheme of length $b_2 \geq 5$ of a plane is not cut out by quadrics, $\mathcal{C}$ is contained in a minimal degree surface S. By step **b)**, we may assume that S is not the Veronese surface.

Since $b_1 = 2$, no irreducible line (even a double line) may have order of contact ≥ 4 with $\mathcal{C}$ at a general point of $\mathcal{C}$. Hence, a general osculating conic D to $\mathcal{C}$ is irreducible. Since S is the intersection of the quadrics containing $\mathcal{C}$, every osculating conic is contained in S. We distinguish two cases.

(i) The surface S is a smooth ruled surface. Hence $S \simeq F_e$ for an integer e with $0 \leq e \leq g-3$ and $e \equiv g \bmod 2$. Take as a basis of $\mathrm{Pic}\,(S) \simeq \mathbf{Z}^{\oplus 2}$, a minimal degree section h of the ruling and a fiber f of the ruling. Hence, $h^2 = -e$, $hf = 1$ and $f^2 = 0$. Since S is embedded as a degree $g-2$ ruled surface, we have $\mathcal{O}_S(1) \simeq \mathcal{O}_C(h+((g+e-2)/2)f)$. Assume that $\mathcal{C} \in |ah+bf|$. Since $\mathcal{C}$ is not hyperelliptic, we have $a \geq 3$. Notice that a general fiber F of the ruling of S intersects $\mathcal{C}$ only at regular points. Hence, any such F induces a length a divisor $F \cap \mathcal{C}$ of $\mathcal{C}$.

Since F is a line, by the geometric version of the Riemann-Roch theorem, we have $h^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(F \cap \mathcal{C})) = a-1$. Since $\mathcal{C}$ is not hyperelliptic, by Clifford's theorem for singular curves, we have $a = 3$. Thus $\mathcal{C}$ is trigonal. Since $\mathcal{C}$ is integral, if $e > 0$ then $b \geq ae$. Since $\mathcal{C}$ is not hyperelliptic, if $e = 0$ we have $b \geq 3$. Since $\deg(\mathcal{C}) = 2g-2$, we have $2g-2 = (ah+bf)(h+((g+e-2)/2)f)$, and so $b = 2g-2-a(g-e-2)/2$.

The general osculating conic, D, is a member of $|yh+xf|$ with $y \geq 0$ and $y = 0$ if $e = 0$ and $x = 1$ (since D is irreducible). Since $\deg(D) = 2$, we have $2 = (yh+xf)(h+((g+e-2)/2)f) = -ey+x+y(g+e-2)/2$. If $y = 0$, then $x = 2$, contradicting the fact that D is irreducible. The case $g = 5$, $e = 1$, $a+b = 8$, $b \geq a \geq 3$ and $2y+x = 2$ remains. Since the case $y = 0$ and $x = 2$ is excluded by the irreducibility of D, we have $y = 1$ and $x = 0$. However, such a curve is unique in S, and the plane Π spanned by it is also unique. Since Π must be the osculating plane to $\mathcal{C}$ at a general point of $\mathcal{C}$, we have $\mathcal{C} \subset \Pi$, a contradiction.

(ii) The surface S is a cone over the normal rational curve of $\mathbf{P}^{g-2}$. Let $u : Y \to S$ be the blowing-up of the vertex of the cone, and let $\mathcal{C}'$ be the strict transform of $\mathcal{C}$ in u.

We have $Y \simeq F_{g-2}$ and, with the convention used in the previous steps for $\mathrm{Pic}\,(Y) \simeq \mathbf{Z}^{\oplus 2}$, the morphism $Y \to S$ is induced by the complete linear system $|h+(g-2)f|$. We have $h = u^{-1}(v)$ where v is the vertex of S. Let μ be the multiplicity of $\mathcal{C}$ at v. Hence $\mu \geq 0$, $\mu = 0$, if and only if $v \notin \mathcal{C}$ and $\mu = 1$ if and only if v is a regular point of $\mathcal{C}$.

We have $\mathcal{C}' \in |ah + (\mu + a(g-2))f|$. Since $\mathcal{C}'$ is assumed not to be hyperelliptic, if $\mu = 0$ we have $a \geq 3$. Assume $\mu = 1$. Since $\mathcal{C}$ is smooth at v and not hyperelliptic, we obtain $a \geq 2$. Since $\deg(\mathcal{C}) = 2g-2$, we have

$$2g-2 = (ah + (\mu + a(g-2))f)(h + (g-2)f) = \mu + a(g-2).$$

Hence, either $\mu = 2$ and $a = 2$ or $a = 1$ and $\mu = g$. The case $a = 2$ and $\mu = 2$ has been studied (with no assumptions on b_2) in [**19**, Theorem 3.6].

If $a = 1$, the normalization of $\mathcal{C}$ is rational. Let D' be the strict transform in Y of a general osculating conic D of $\mathcal{C}$. Then D' is irreducible. Let $|xh+yf|$ be the linear system containing D'. Since D is irreducible and it is not a line, we have $y \geq x(g-2)$. Since $\deg(D) = 2$, we have $2 = (xh+yf)(h+(g-2)f) = -xe + x(g-2) + y = y$.

Hence, for $g \geq 5$, we obtain a contradiction and so this case cannot occur.

d) Assume $b_2 = 4$ (and hence $p = 2$). By the geometric version of the Riemann-Roch theorem, for a general regular point Q of $\mathcal{C}$, we have $h^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(4Q)) \geq 2$. Since $b_1 = 2$, again by the geometric version of the Riemann-Roch theorem, we have $h^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(3Q)) = 1$. Hence $h^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(4Q)) = 2$ and the line bundle $\mathcal{O}_{\mathcal{C}}(4Q)$ is spanned.

Here we assume that the normalization $\mathcal{C}''$ of $\mathcal{C}$ is not rational. For the case in which $\mathcal{C}''$ is rational, see the next step. Since C'' is not isomorphic to $\mathbf{P}^1$, for a general regular point Q' of $\mathcal{C}$, $\mathcal{O}_{\mathcal{C}''}(4Q)$ and $\mathcal{O}_{\mathcal{C}''}(4Q')$ are not isomorphic. We stress this point because it is a characteristic p phenomenon. For any integral singular curve $\mathcal{C}$ with normalization $\mathcal{C}''$, the kernel K of the surjective homomorphism $\mathrm{Pic}^0(\mathcal{C}) \to \mathrm{Pic}^0(\mathcal{C}'')$ is a unipotent group and, if $\mathrm{Pic}^0(\mathcal{C}'') = \{d\}$ and K

is not an extension of copies of the multiplicative group of the ground field, then K contains a one-dimensional family, say $\{L_\alpha\}$ of nontrivial line bundles, with $L_\alpha^{\otimes p} \simeq \mathcal{O}_{\mathcal{C}}$ for every α, just because for every $\alpha \in \mathbf{K}$, we have $p\alpha = 0$.

Hence, $\mathcal{C}$ has infinitely many complete base point free pencils of degree 4. If $\mathcal{C}$ is smooth, this implies an arbitrary characteristic that $\mathcal{C}$ is bielliptic.

We will check the existence of a double covering $v : \mathcal{C} \to \mathcal{E}$ with $\mathcal{E}$ an irreducible curve with $p_a(\mathcal{E}) = 1$. Thus $\mathcal{E}$ is either a smooth elliptic curve or a rational curve with an ordinary node as unique singularity, or a rational curve with an ordinary cusp as unique singularity.

The case $g \geq 10$ is easy to check because of the following reason. Take two different degree 4 spanned line bundles L and L' on $\mathcal{C}$. The two associated pencils induce a morphism $b : \mathcal{C} \to \mathbf{P}^1 \times \mathbf{P}^1$. If b is rational, then $b(\mathcal{C})$ is an irreducible curve of degree $(4, 4)$ on the quadric $\mathbf{P}^1 \times \mathbf{P}^1$ and hence $9 = p_a(b(\mathcal{C})) \geq p_a(\mathcal{C}) = g$. So we may assume $g \leq 9$.

Now assume that the normalization $\mathcal{C}''$ of $\mathcal{C}$ is rational and that the previous discussion cannot start, i.e., we assume that all line bundles $\mathcal{O}_{\mathcal{C}}(4Q)$, Q general on $\mathcal{C}$, are isomorphic.

We stress that they are isomorphic as line bundles on $\mathcal{C}$ because their pull-backs to $\mathcal{C}''$ are isomorphic since they have the same degree.

Fix a general point $Q \in \mathcal{C}$ and call $m : \mathcal{C} \to \mathbf{P}^1$ the induced morphism. We have $h^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(4Q)) = 2$ because $\mathcal{C}$ is assumed to be nonhyperelliptic.

By assumption, for a general point $Q' \in \mathcal{C}$, the divisors $4Q$ and $4Q'$ are linearly equivalent, and hence $4Q$ is a scheme theoretic fiber $m^{-1}(m(Q'))$ of Q'. It follows that m is purely inseparable of degree 4.

Vice versa, take any Gorenstein nonhyperelliptic integral curve $\mathcal{C}$ with a degree 4 purely inseparable morphism $m : \mathcal{C} \to \mathbf{P}^1$. Set $g := p_a(\mathcal{C}')$ and embed $\mathcal{C}$ in P^{g-1}. We have that for a general regular point Q of $\mathcal{C}$, the scheme theoretic fiber $m^{-1}(m(Q'))$ is the divisor $4Q$. By the geometric version of the Riemann-Roch theorem for a general $Q \in \mathcal{C}$, the subscheme $4Q$ of $\mathcal{C}$ spans at most a plane of $\mathbf{P}^{g-1}$. By definition, this means $b_2(\mathcal{C}) \geq 4$.

e) The canonically embedded curve $\mathcal{C}$ is strange. Call P the strange point of $\mathcal{C}$ and $\mathcal{C}' \subset \mathbf{P}^{g-2}$ the image of $\mathcal{C}$ by the projection from P.

Since $\mathcal{C}'$ spans $\mathbf{P}^{g-2}$, we have $\deg(\mathcal{C}') \geq g-2$. Since $\deg(\mathcal{C}) = 2g-2$ and the projection π of $\mathcal{C}$ from P has at least degree p, we see that $p = 2$, the projection is purely inseparable, generically injective and that only two cases are a priori possible:

(i) $P \notin \mathcal{C}$, $\deg(\pi) = 2$, $\deg(\mathcal{C}') = g-1$;

(ii) $P \in \mathcal{C}$, $\mathcal{C}$ has multiplicity 2 at P, $\deg(\pi) = 2$ and $\deg(\mathcal{C}') = g-2$, i.e., $\mathcal{C}'$ is a normal rational curve.

Furthermore, in case (i) we have to distinguish the following subcases.

(i1) $\mathcal{C}'$ is smooth and rational (and hence not linearly normal);

(i2) $\mathcal{C}'$ is smooth and elliptic and hence linearly normal;

(i3) $\mathcal{C}'$ has a node (hence $\mathcal{C}'$ is linearly normal and $p_a(\mathcal{C}') = 1$);

(i4) $\mathcal{C}'$ has a cusp (hence $\mathcal{C}'$ is linearly normal and $p_a(\mathcal{C}') = 1$).

Call $T \subset \mathbf{P}^{g-1}$ the cone with vertex P and base $\mathcal{C}'$. Thus $\deg(T) = g-1$ in case (i) and $\deg(T) = g-2$ in case (ii). Let W be the strict transform of T in the blowing-up of $\mathbf{P}^{g-1}$ at P.

In case (i2) W is the ruled surface $\mathbf{P}(\mathcal{O}_{\mathcal{C}'} \oplus \mathcal{O}_{\mathcal{C}'}(1))$ over $\mathcal{C}'$; call $v : W \to \mathcal{C}'$ the ruling, h the counterimage of the vertex P and $R := v^*(\mathcal{O}_{\mathcal{C}'}(-1))$.

We have $h^2 = -g+1$. The map $W \to \mathbf{P}^{g-1}$ is induced by the linear system $|h+R|$.

In case (ii) we have a similar description of T and W, but here the embedding is not linearly normal. In cases (i3) and (i4), the cone T arises from a nongeneral projection of a linearly normal cone $T' \subset \mathbf{P}^g$ of degree $g-1$ over a normal rational curve of $\mathbf{P}^{g-1}$ and the blowing-up, W', of T' at the vertex is isomorphic to F_{g-1}.

Now we exclude case (ii).

Let $A \subset \mathbf{P}^{g-1}$ be a quadric hypersurface containing $\mathcal{C}$. Since a general line of T contains at least a length three subscheme of $\mathcal{C}$, we have $T \subset A$. Since every quadric hypersurface containing T is the cone with vertex P of a quadric hypersurface of $\mathbf{P}^{g-2}$ containing $\mathcal{C}'$, we have $h^0(\mathbf{P}^{g-1}, I_T(2)) = h^0(\mathbf{P}^{g-2}, I_{\mathcal{C}'}(2)) = g(g-1)/2 - 2(g-$

$2) - 1$. Since $h^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(2)) = g - 3$, by the Riemann-Roch theorem we have $h^0(\mathbf{P}^{g-1}, I_{\mathcal{C}}(2)) \geq g(g+1)/2 - 3g + 3 > h^0(\mathbf{P}^{g-2}, I_T(2))$, a contradiction.

The case (i1) is not possible because $\mathcal{C}$ would be hyperelliptic, contradicting our assumptions. Here we use the fact that $P \notin \mathcal{C}'$, i.e., we use the fact that the rational map from $\mathcal{C}$ onto $\mathcal{C}'$ induced by the projection from P is regular. The same would be true if P is a regular point of $\mathcal{C}$ but would be false if P is a singular point.

Now we classify the possible curves $\mathcal{C}$ in case (i2).

We have $h^0(\mathbf{P}^{g-1}, I_T(2)) = h^0(\mathbf{P}^{g-2}, I_{\mathcal{C}'}(2)) = g(g-1)/2 - 2(g-1) < g(g+1)/2 - 3g - 3 \leq h^0(\mathbf{P}^{g-1}, I_{\mathcal{C}}(2))$.

Hence there is a quadric hypersurface A of $\mathbf{P}^{g-1}$ containing $\mathcal{C}$ but not T. Since $\deg(\mathcal{C}) = 2\deg(T)$, $\mathcal{C} = A \cap T$, as schemes. Since $P \notin T$, every singular point of $\mathcal{C}$ is a planar singularity. Vice versa, for every quadric hypersurface A of $\mathbf{P}^{g-1}$ not containing P, with $\mathcal{C} \cap T$ with no multiple components and irreducible, the curve $A \cap T$ is a Gorenstein bielliptic curve.

The curve $A \cap T$ is strange (and hence $p = 2$) if and only if every line of T is tangent to the curve $A \cap T$. In this case P is the strange point of $A \cap T$. We described all the equations of such curves inside the linear space $\mathbf{P}(H^0(T, \mathcal{O}_T(2))$ in [**1**] and [**2**].

Proposition 2.2. *Let T be as in subcases* (i2), (i3) *and* (i4)*, and let A be a quadric hypersurface of $\mathbf{P}^{g-1}$ not containing T with $P \notin A$ and with $A \cap T$ irreducible and with no multiple component. Set $\mathcal{C} := A \cap T$. Assume that $\mathcal{C}$ is strange, i.e., assume that $p = 2$ and that the projection $f : \mathcal{C} \to \mathcal{C}'$ from P is purely inseparable of degree* 2*. Let $\{b_i\}$, $0 \leq i \leq g-1$ be the Hasse sequence of invariants of $\mathcal{C}$. Then $b_i = 2i$ for every $i \leq g-1$.*

Proof. Take a general $Q \in \mathcal{C}'$ and let $b_i(\mathcal{C}')$, $0 \leq i \leq g-2$ be the Hasse sequence of invariants of $\mathcal{C}'$. Let $B := (f^{-1}(Q))_{\mathrm{red}}$. B may be considered a general point of $\mathcal{C}$. Since the line, W_1, of T through Q has multiplicity two with $\mathcal{C}$ at B, we have $b_1 = 2$. Since $\mathcal{C}'$ is linearly normal and $h^0(\mathcal{C}', \mathcal{O}_{\mathcal{C}'}(tQ)) = t$ for every $t > 0$, we have $b_i(\mathcal{C}') = i + 1$ for every i.

Let V_i, $1 \le i \le g-2$, be the osculating linear subspace of $\mathcal{C}'$ at Q with $\dim(V_i) = i$. Let $W_{i+1} := \langle\{P, V_i\}\rangle$ be the linear span of V_i and P. We have $\dim(W_{i+1}) = i+1$. Since f is purely inseparable of degree two, the connected component of the scheme $W_{i+1} \cap \mathcal{C}$ supported by P has length twice the length $i+1$ of the subscheme $V_i \cap \mathcal{C}$ with Q as support. Since B is general, we obtain $b_i \ge 2i$ for $2 \le i \le g-2$. Inductively, we obtain that $\{W_j\}$, $1 \le j \le g-2$, is the osculating flag to $\mathcal{C}$ at B. Thus $b_i = 2i$ for every $i \le g-1$. □

f) The case $\mathcal{C}$ bielliptic. Here we study the case in which $\mathcal{C}$ is bielliptic, i.e., we assume the existence of a degree two morphism $f : \mathcal{C} \to E$ with E an integral curve and $p_a(E) = 1$. Thus either E is a smooth elliptic curve or E is a rational curve with an ordinary node or an ordinary cusp as unique singularity.

Also, in this step and the next steps, we extend to the case of Gorenstein bielliptic curves the well-known characterization of smooth bielliptic curves as the curves whose canonical model in $\mathbf{P}^{g-1}$ lies on the cone over a degree $g-1$ linearly normal elliptic curve E of a hyperplane of $\mathbf{P}^{g-1}$ (see, e.g., [**6**, Theorem 2.1]).

Here we assume that Y is a Gorenstein nonhyperelliptic curve with a double covering $f : \mathcal{C} \to E$ with $p_a(\mathcal{C}) = 1$. By [**20**, Theorem 1.5], the canonical morphism is a degree $2g-2$ embedding of Y into $\mathbf{P}^{g-1}$.

See $\mathcal{C}$ as a canonically embedded curve. For a general point $Q \in E$, either $\mathrm{card}\,(f^{-1}(Q)) = 2$ (and hence f is separable) or $\mathrm{card}\,(f^{-1}(Q)) = 1$ (and hence f is purely inseparable). In the latter case, the scheme $f^{-1}(Q)$ is a length two subscheme of $\mathcal{C}$.

Thus, in both cases, for a general $Q \in E$, there is a unique line $D(Q) \subset \mathbf{P}^{g-1}$ spanned by the scheme $f^{-1}(Q)$.

Fix general $Q, Q' \in E$. Since $h^0(E, \mathcal{O}_E(Q + Q')) = 2$, we have $h^0(\mathcal{C}, \mathcal{O}_C(f^{-1}(Q) + f^{-1}(Q'))) \ge 2$. By the geometric version of the Riemann-Roch theorem, we obtain that for general $Q, Q' \in E$, we have $D(Q) \cap D(Q') \ne \varnothing$. Since E has infinitely many points, this implies that either all lines $D(Q)$ are contained in the same plane or $P \in \mathbf{P}^{g-1}$ exists with $P \in D(Q)$ for all Q.

Since $\mathbf{P}^{g-1}$ is not a plane and $\mathcal{C}$ spans $\mathbf{P}^{g-1}$, we obtain that $\mathcal{C}$ is contained in a cone T which is the closure of the union of the lines $D(Q)$

with $Q \in E$, and Q general. Call P the vertex of T. Let $\mathcal{C}' \subset \mathbf{P}^{g-2}$ be the image of the projection, say π, of $\mathcal{C}$ from P.

Since f is a morphism, $f^{-1}(Q) \cap f^{-1}(Q') \neq \varnothing$, for general points $Q, Q' \in E$, $Q \neq Q'$. This implies that the rational map from $\mathcal{C}$ onto $\mathcal{C}'$ induced by the projection from P has at least degree two. Since C' spans $\mathbf{P}^{g-2}$, $\deg(\mathcal{C}') \geq g-2$. Hence, for $g \geq 5$, only two cases are a priori possible.

(i) $P \notin \mathcal{C}$, $\deg(\pi) = 2$, $\deg(\mathcal{C}') = g-1$;

(ii) $P \in \mathcal{C}$, $\mathcal{C}$ has multiplicity two at P, $\deg(\pi) = 2$ and $\deg(\mathcal{C}') = g-2$, i.e., $\mathcal{C}'$ is a normal rational curve.

Remark 2.3. Every integral canonical curve $\mathcal{C}$ with $\mathcal{C} \subset T$, $\mathcal{C}$ with multiplicity two at P and with π inseparable has $b_2 \geq 4$. Indeed, take a general point $Q \in \mathcal{C}$. The plane $\Pi(Q)$ spanned by P and the tangent line to $\mathcal{C}'$ at $\pi(P)$ contains at least the double of the scheme cut out on $\mathcal{C}$ at Q by the line $\langle\{P, Q\}\rangle$. Since this scheme has length two, by the inseparability of the degree two map π, we obtain $b_2 \geq 4$. Since a rational normal curve is ordinary and π has degree two, the subscheme, Z, of $\Pi \cap \mathcal{C}$ supported by P and has length < 6. Since $_2$ is divisible by $p = 2$, we obtain $b_2 = 4$.

Remark 2.4. In both cases $\mathcal{C}'$ is birational to E. Hence, if E is smooth, then we are in case i) and C' is an elliptic linearly normal curve.

Remark 2.5. The case E smooth is completely classified in (i1).

Here we assume the case (i). Since the case $\mathcal{C}$ strange has been analyzed in e), we may assume that $\mathcal{C}$ is not strange. If $\mathcal{C}'$ is linearly normal (i.e., not a smooth rational curve), the proof given to exclude the case (i2) gives $\mathcal{C} = A \cap T$ (as schemes) with A a quadric hypersurface with $P \notin T$. The proof of Proposition 2.2 would give that $\mathcal{C}$ is ordinary unless $\mathcal{C}$ is strange with vertex P, i.e., unless we are in a case completely classified in e). Hence, we may assume that $\mathcal{C}' \subset \mathbf{P}^{g-2}$ is a nonlinearly normal smooth rational curve of degree $g-1$. Since $P \notin \mathcal{C}$, π is a morphism and hence $\mathcal{C}$ is hyperelliptic, a contradiction.

Examples with $g = 9$, $b_1 = 2$ and $b_2 = 4$. Here we construct a family of examples with $g = 9$, $b_1 = 2$ and $b_2 = 4$ (and hence $p = 2$). Let $S \simeq \mathbf{P}^1 \times \mathbf{P}^1 \subset \mathbf{P}^3$ be a smooth quadric surface. Every irreducible curve D of type $(4, 4)$ on S has $\omega_S \equiv \mathcal{O}_S(2)$, and hence $p_a(D) = 9$. Consider the projection $f : S \to \mathbf{P}^1$ on the first factor. In the linear system $\mathcal{O}_S(2)$ we have described in [**2**] the "equations" of all such curves, D, such that $f|D$ is purely inseparable of degree 4.

Consider the canonical embedding, C, of D. Since $h^1(S, \mathcal{O}_S) = 0$ and $h^1(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}) = 0$, the restriction maps $\alpha : H^0(S, \mathcal{O}_S(2)) \to H^0(D, \mathcal{O}_D(2))$ and $\beta : H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(2)) \to H^0(S, \mathcal{O}_S(2))$ are surjective.

Hence the restriction map $H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(2)) \to H^0(D, \mathcal{O}_D(2))$ is surjective and its kernel is one-dimensional.

This means that the canonical embedding of D embeds $\mathcal{C}$ as a curve in a hyperplane $H \simeq \mathbf{P}^8$ of $\mathbf{P}^9$ where $\mathbf{P}^9$ is the ambient space for the double Veronese embedding j of $\mathbf{P}^3$. This implies $b_1(\mathcal{C}) = 2$ (use reducible quadrics). Fix a general $Q \in D$ and call Q' the corresponding point of $\mathcal{C}$. Let $D(Q)$ be the line of $\mathbf{P}^3$ tangent to D at Q. The line $D(Q)$ has intersection multiplicity four with D at Q.

The line $D(Q)$ is sent by j into a conic $j(D(Q))$. Since $j(D(Q))$ has intersection multiplicity four with $\mathcal{C}$ at Q', $j(D(Q))$ is the osculating conic of $\mathcal{C}$ at Q'.

Then $b_1(\mathcal{C}(Q')) = 4$ as wanted. Notice that the normalization of $\mathcal{C}$ is $\mathbf{P}^1$ and that, for all $A, B \in \mathcal{C}_{\mathrm{reg}}$, we have $\mathcal{O}_{\mathcal{C}}(4A) \simeq \mathcal{O}_C(4B)$.

Remark 2.6. Take any example D with $g = 9$, $b_1 = 2$ and $b_2 = 4$ given before, and any partial normalization Y of D, i.e., any integral curve with a birational morphism $Y \to D$. Assume that Y is Gorenstein and not hyperelliptic. Since the canonical series of Y is a subseries of ω_D (subadjunction theory), the canonical embedding Y' of Y has $b_2(Y') \geq b_2(\mathcal{C}) = 4$, and hence every such Y gives an example with lower genus.

Remark 2.7. The examples of curves with $g = 9$, $b_1 = 2$ and $b_2 = 4$ and parts e) and f) give a complete classification of all curves $\mathcal{C}$ of genus nine with $b_2 \geq 4$.

Indeed, if $\mathcal{C}$ is a double covering of a curve with arithmetic genus one, then we may apply the classification made in f) and e). If this is not the case, then by the case $b_2 = 4$, we know that two general degree four pencils send $\mathcal{C}$ onto a curve D of type $(4,4)$ on $S \simeq \mathbf{P}^1 \times \mathbf{P}^1$. Since $p_a(D) = 9$, the morphism $\mathcal{C} \to D$ is an isomorphism.

Furthermore, we know that for a general point $Q \in D$, we have $h^0(D, \mathcal{O}_D(4Q)) \geq 2$, i.e., by Serre duality, $h^0(D, \omega_D(-4Q)) > 0$. By the adjunction formula, we have $\mathcal{O}_D(2) \simeq \mathcal{O}_D(2,2)$. Since $H^1(S, \mathcal{O}_S(2,2)) = 0$ (e.g., by Künneth's formula), the restriction map $H^0(S, \mathcal{O}_S(2,2) \to H^0(D, \mathcal{O}_D(2,2))$ is surjective. We obtain $H^1(S, \mathcal{O}_S(2,2) \otimes I_{\{4Q\}}) \neq 0$, where $\{4Q\}$ denotes the zero-dimensional length four subscheme of S associated to the effective Cartier divisor $4Q$ on D.

Using reducible curves of type $(2,2)$ on S, we see that this is equivalent to the fact that $\{4Q\}$ is contained either in a line of type $(0,1)$ or on a line of type $(1,0)$. Since this is true for a general $Q \in D$, we see that the restriction to Q of the projection from S onto one of its factors $\mathbf{P}^1$ is purely inseparable of degree four. Thus, $\mathcal{C}$ is given by the examples with $g = 9$, $b_1 = 2$ and $b_2 = 4$.

g) Again on the case $b_2 \geq 4$. Here we will conclude the case $b_2 \geq 4$ under the assumption that the normalization $\mathcal{C}''$ of $\mathcal{C}$ has genus $q \geq 4$.

In particular, since $q > 0$, we may apply results in part a) and hence $\mathcal{C}$ has many base point free degree four spanned line bundles. By parts e) and f), we may assume that there is no degree two morphism from $\mathcal{C}$ onto a curve E with $p_a(E) = 1$ and such that all these spanned degree four line bundles come from E.

In particular, by part d) we may assume $g \leq 9$. However, we will only assume $q > 0$ unless otherwise specified.

Remark 2.8. Notice that the canonical series of $\mathcal{C}''$ is in a natural way a subseries of the canonical series of $\mathcal{C}$. Hence, if $q = 3, 4$ and $\mathcal{C}''$ is not hyperelliptic, then $\mathcal{C}''$ is classified in [**14**].

Remark 2.9. Assume $q \geq 5$. Since $q > 0$, the pull-back to $\mathcal{C}''$ of the line bundles $\mathcal{O}_{\mathcal{C}}(4Q)$, Q a general regular point of $\mathcal{C}$, are spanned and hence, either $\mathcal{C}''$ is hyperelliptic or $\mathcal{C}''$ is bielliptic and these line bundles come from a degree two morphism $f : \mathcal{C}'' \to E$. This implies that, for general points $Q, Q' \in \mathcal{C}_{\mathrm{reg}}$, the corresponding pencils $\mathcal{O}_{\mathcal{C}''}(4Q)$ and $\mathcal{O}_{\mathcal{C}''}(4Q')$ induce a degree two morphism $u : \mathcal{C}'' \to \mathbf{P}^1 \times \mathbf{P}^1$ with $u(\mathcal{C}'') = E$. Since not only the line bundles but also the spanning sections come from $\mathcal{C}$, the morphism u factors through the normalization map $\mathcal{C}'' \to \mathcal{C}$.

Thus we have found a degree two morphism $\mathcal{C} \to E$ with E smooth elliptic curve. We have solved the case $\mathcal{C}''$ not hyperelliptic.

Now we will exclude the case $q \geq 2$ and $\mathcal{C}''$ hyperelliptic. By assumption, for a general $Q \in \mathcal{C}$, the line bundle $\mathcal{O}_{\mathcal{C}}(4Q)$ is spanned. Hence, for a general $B \in \mathcal{C}''$, the line bundles $\mathcal{O}_{\mathcal{C}''}(4B)$ is spanned.

This is false for every smooth hyperelliptic curve $\mathcal{C}''$ of genus ≥ 2.

Now we assume that $q > 0$, $g \geq 5$, $b_1 = 2$ and $b_2 = 4$, hence $p = 2$, and prove that $\mathcal{C}$ is bielliptic.

Let $M \subset \mathrm{Pic}^4(\mathcal{C})$ be the one-dimensional family of spanned line bundles $\{\mathcal{O}_{\mathcal{C}}(4Q)\}$, $Q \in \mathcal{C}_{\mathrm{reg}}$ and Q general. Note that $\dim(M) = 1$ because $q > 0$ (see part d)). Since the line bundles $\mathcal{O}_{\mathcal{C}}(4Q)$ and $\mathcal{O}_{\mathcal{C}}(4Q')$ are not isomorphic for general Q, Q', by the base point free pencil trick, we obtain $h^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(4Q + 4Q')) \geq 4$. By the geometric version of the Riemann-Roch theorem, we obtain that two planes, M and M', spanned by a general divisor of $\mathcal{O}_{\mathcal{C}}(4Q)$ and a general divisor of $\mathcal{O}_{\mathcal{C}}(4Q')$, respectively, $M \cup M'$ spans at most a 4-dimensional linear space. Thus we have that $M \cap M' \neq \varnothing$. If the curve $\mathcal{C}'$ is not hyperelliptic or trigonal, it is easy to obtain that $M \cap M'$ is a point, say $P(M, M')$. Without assuming that $\mathcal{C}$ is not trigonal, one has to consider separately (but in a similar way) the far easier case in which for general M, M', the set $M \cap M'$ is a line.

We have a continuous family of mutually intersecting planes which, in general, intersect mutually in exactly one point. Since M and M' are planes in $\mathbf{P}^{g-1}$, $g \geq 7$, and their union spans $\mathbf{P}^{g-1}$, we obtain that all the planes M pass through a common point [**5**].

Call P the common point of all planes M, M'. First assume that $P \notin \mathcal{C}$. We assume that $\mathcal{C}$ is not strange (this case has been classified in part e)).

We assume that $\mathcal{C}$ is not contained in a minimal degree surface because this case was studied in part a).

Hence $h^0(\mathbf{P}^{g-1}, I_{\mathcal{C}\cup\{P\}}(2)) = h^0(\mathbf{P}^{g-1}, I_{\mathcal{C}}(2)) - 1$.

Since $P \notin \mathcal{C}$, every quadric hypersurface containing $\mathcal{C} \cup \{P\}$ contains at least five points of such general plane M. Hence, the base locus, W, of the linear system U of all quadric hypersurfaces containing $\mathcal{C} \cup \{P\}$ contains either a conic of each general plane M or a general such M. Since $\mathcal{C}$ is not strange and not contained in a minimal degree surface, $\mathcal{C}$ is scheme-theoretically cut out by quadrics and hence $W \cap B = \mathcal{C}$ as schemes for any quadric hypersurface B containing $\mathcal{C}$, but not P.

This implies that W is, except for finitely many points, an irreducible surface, W', with $2\deg(W') = \deg(\mathcal{C})$, i.e., with $\deg(W') = g-1$. Since $\mathcal{C} = W' \cap B$ as schemes, we have $\mathcal{C}_{\text{reg}} \subset W'_{\text{reg}}$. Two different planes through P intersect only in P or in a line. Since planes associated to different degree four divisors of the linear system $\mathbf{P}(H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(4Q)))$ are different and planes associated to divisors of different line bundles $\mathcal{O}_{\mathcal{C}}(4Q)$ and $\mathcal{O}_{\mathcal{C}}(4Q')$ are different, we have a two-dimensional family of such planes such that, the union of the conics containing the divisors and P gives W. Since B is a Cartier divisor of $\mathbf{P}^{g-1}$ and $W' \cap B = \mathcal{C}$, we have $\dim(W') = 2$, as remarked before.

The only way W' may be the union of such a family of conics is if the conics are reducible so that each of the associated lines is contained in a one-dimensional family of our two-dimensional family of planes. Since at least one of the two lines of each reducible conic of $M \cap W'$, M one of our planes, must contain P, the projection of W' from P into $\mathbf{P}^{g-1}$ is a curve. Hence, W' is a cone with vertex P and basis an irreducible curve, E, with $\deg(E) = \deg(W') = g-1$ and E spanning $\mathbf{P}^{g-2}$. Hence $p_a(E) \leq 1$.

Since E is irreducible and the family of lines contained in W' is parametrized by E, there is a Zariski dense open set Ω of such lines such that every $D \in \Omega$ intersects $\mathcal{C}$ in a scheme with the same length, say a. Since $P \notin \mathcal{C}$ and a general plane M in our family is 4-secant to $\mathcal{C}$, we have $a = 2$. Since $p \notin \mathcal{C}$ this means that the projection from P induces a degree two morphism $f : \mathcal{C} \to E$. It is sufficient to exclude the case $p_a(E) = 0$.

Assume $p_a(E) = 0$, i.e., assume that E is a smooth degree $g-1$ curve spanning a hyperplane, H, of $\mathbf{P}^{g-1}$. Thus the embedding of E in H is not linearly normal and E is an isomorphic projection of a rational normal cone E'' of $\mathbf{P}^{g-1}$. The cone W' is the projection of a cone

$W'' \subset \mathbf{P}^g$ with base E''. Call P'' the vertex of W''. W' and W'' are not isomorphic, but $W' \setminus \{P\}$ and $W'' \setminus \{P''\}$ are isomorphic as abstract varieties. Since $P \notin \mathcal{C}$, $\mathcal{C}$ is the projection of a curve $D \subset W'' \setminus \{P''\}$ with $D \simeq \mathcal{C}$, $\deg(D) = 2g-2$. Since $\mathcal{O}_{\mathcal{C}}(1) \simeq \omega_{\mathcal{C}}$, we have $\mathcal{O}_D(1) \simeq \omega_D$.

Since $\deg(D) > \deg(W'')$, the Bezout theorem implies that D is not contained in a hyperplane of $\mathbf{P}^{g-2}$. Thus $g = p_a(\mathcal{C}) = p_a(D) = h^0(D, \omega_D) = h^0(D, \mathcal{O}_D(1)) \geq g+1$, a contradiction.

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Dipartimento di Matematica, Università di Trento, 38050 Povo (TN), Italy
E-mail address: ballico@science.unitn.it

Dipartimento di Matematica, Università della Basilicata, Via Nazario Sauro, 85, 85100 Potenza, Italy
E-mail address: cossidente@unibas.it