ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 31, Number 3, Fall 2001

ON THE SECOND FUNDAMENTAL TENSOR OF REAL HYPERSURFACES IN QUATERNIONIC HYPERBOLIC SPACE

MIGUEL ORTEGA AND JUAN DE DIOS PÉREZ

ABSTRACT. We study real hypersurfaces with constant quaternionic sectional curvature in the quaternionic hyperbolic space and the action of the curvature operator on the Weingarten endomorphism. We also introduce examples of ruled real hypersurfaces.

1. Introduction. A quaternionic Kaehlerian manifold is called a quaternionic space form if it is connected, simply connected and it is endowed with a complete metric g of constant quaternionic sectional curvature c. The study of real hypersurfaces when $c > 0$ is rather developed (cf. [**1**], [**7**], [**9**]). Besides, these authors deal with both $c < 0$ and $c > 0$. Our purpose is to study real hypersurfaces in quaternionic hyperbolic space ($c < 0$) of constant quaternionic sectional curvature $c = -4$, $\mathbf{Q}H^m$, $m \geq 2$, by paying attention to the second fundamental tensor.

We describe in Section 2 a semi-Riemannian manifold of index 3 that is a semi-Riemannian submersion (cf. [**5**]) over $\mathbf{Q}H^m$ with time-like totally geodesic fibers as well as a principal fiber bundle over $\mathbf{Q}H^m$ with structural group $\mathbf{S}^3$.

A real hypersurface M of a quaternionic space form is said to be ruled if its maximal quaternionic distribution $\mathbf{D}$ of the tangent bundle of M, TM, is integrable. This condition is equivalent to $g(AX, Y) = 0$ for any X, Y in $\mathbf{D}$, where A is the Weingarten endomorphism of M (cf. [**8**]). In Section 4, we construct a family of ruled real hypersurfaces in $\mathbf{Q}H^m$, $m \geq 2$, which proves that the class of such real hypersurfaces is not empty, so Theorem 2 is meaningful.

$\mathbf{D}$ is a linear subbundle of TM which inherits two different metric tensors. The first one is the simple restriction of g, so it seems natural to keep the same name. The second one comes from the

Received by the editors on October 26, 1998, and in revised form on April 28, 2000.

second fundamental form by $g^0(X,Y) = g(AX,Y)$ for any $X, Y \in \mathbf{D}$. Theorem 2 studies the case in which both metrics are proportional by a smooth function defined on M. Anyway, the main result of Section 5 is the classification of real hypersurfaces in $\mathbf{Q}H^m$, $m \geq 3$, which have constant quaternionic sectional curvature.

As it is shown in [**10**], there are no real hypersurfaces in $\mathbf{Q}H^m$ whose second fundamental tensor is parallel. So, in Section 6, we consider the curvature operator R acting as a derivation over the Weingarten endomorphism. That is, for any X, Y tangent to the real hypersurface M, we define $R(X,Y)\cdot A = \nabla_X\nabla_Y A - \nabla_Y\nabla_X A - \nabla_{[X,Y]}A$, where ∇ is the Riemannian covariant derivation on M. Thus, we study condition (33) that allows us to obtain as a corollary the nonexistence of real hypersurfaces in $\mathbf{Q}H^m$ which satisfy $R \cdot A = 0$.

2. The quaternionic hyperbolic space. Let $\mathbf{Q}$ be the algebra of quaternions with quaternionic units $\{j_1, j_2, j_3\}$. On $\mathbf{Q}^{m+1}$, $m \geq 2$, let us consider the Hermitian form $b(z,w) = -\bar{z}_0 w_0 + \sum_{k=1}^m \bar{z}_k w_k$ where $z = (z_0, \dots, z_m)$, $w = (w_0, \dots, w_m) \in \mathbf{Q}^{m+1}$ and $\bar{z}$ is the quaternionic conjugate of $z \in \mathbf{Q}$. The symplectic scalar product $\bar{g} = \operatorname{Re} b$ is an indefinite metric tensor of index 4 on $\mathbf{Q}^{m+1}$. Let us consider the real hypersurface in $\mathbf{Q}^{m+1}$

$$H_3^{4m+3} = \{z \in \mathbf{Q}^{m+1} : b(z,z) = -1\}.$$

The tangent space of H_3^{4m+3} at a point z is given by

$$T_z H_3^{4m+3} = \{a \in \mathbf{Q}^{m+1} : \bar{g}(a,z) = 0\}. \tag{1}$$

This shows that the position vector $\chi : H_3^{4m+3} \to \mathbf{Q}^{m+1}$ is a globally defined normal vector field whose length is $\|\chi\|^2 = -1$. Therefore, H_3^{4m+3} is a semi-Riemannian submanifold in $\mathbf{Q}^{m+1}$ of index 3. Let $D, \tilde{D}$ be the Levi-Civita connections of $\mathbf{Q}^{m+1}$ and H_3^{4m+3}, respectively. The Gauss formula of χ is

$$D_X Y = \tilde{D}_X Y + \bar{g}(X,Y)\chi \tag{2}$$

for any X, Y tangent to H_3^{4m+3}. As the curvature tensor of $\mathbf{Q}^{m+1}$ vanishes, it is very easy to check that H_3^{4m+3} is a space of constant sectional curvature -1. Next we consider $\mathbf{S}^3 = \{\lambda \in \mathbf{Q} : \bar{\lambda}\lambda = 1\}$ and the action $\mathbf{S}^3 \times H_3^{4m+3} \to H_3^{4m+3}$ given by $\lambda \in \mathbf{S}^3$, $z = (z_0, \dots, z_m) \in H_3^{4m+3}$, $\lambda z = (\lambda z_0, \dots, \lambda z_m)$. This action is free and the quotient will be denoted by $\mathbf{Q}H^m$. Moreover, H_3^{4m+3} is a principal fiber bundle over $\mathbf{Q}H^m$ with structural group $\mathbf{S}^3$. Given a point $z \in H_3^{4m+3}$, the horizontal subspace is

$$T'_z = \{X \in T_z H_3^{4m+3} : \bar{g}(X, j_k z) = 0,\ k = 1, 2, 3\}. \tag{3}$$

Given X, Y tangent to $\mathbf{Q}H^m$, we will denote their horizontal lifts by X', Y', respectively, and we will define a metric tensor on $\mathbf{Q}H^m$ by $g(X, Y) = \bar{g}(X', Y')$. This metric tensor makes the fibration π be a semi-Riemannian submersion, (cf. [**5**]). As $\bar{g}(j_k\chi, j_k\chi) = -1$, $k = 1, 2, 3$, the fibers are time-like. Note that they are also totally geodesic. Therefore, $(\mathbf{Q}H^m, g)$ is a Riemannian manifold. Any geodesic on $\mathbf{Q}H^m$ is the projection of a horizontal geodesic on H_3^{4m+3}, so that $\mathbf{Q}H^m$ is complete. Let $\tilde{\nabla}$ be the Levi-Civita connection of $\mathbf{Q}H^m$. The set $\{j_1\chi, j_2\chi, j_3\chi\}$ defines a 3-Sasakian structure on H_3^{4m+3}. As it is shown in [**3**], the projection $\pi : H_3^{4m+3} \to \mathbf{Q}H^m$ induces a structure of quaternion Kaehlerian manifold on $\mathbf{Q}H^m$ by $J_k = \pi_*(j_k\chi)$. This means that the three-dimensional vector bundle $\hat{\mathcal{V}} = \operatorname{Span}\{J_1, J_2, J_3\}$ of tensors of type $(1,1)$ of almost Hermitian structures satisfy

$$\begin{gathered} J_1^2 = J_2^2 = J_3^2 = -\operatorname{Id}, \qquad J_1 J_2 = -J_2 J_1 = J_3 \\ \tilde{\nabla}_X J_i = q_k(X) J_j - q_j(X) J_k, \quad i = 1, 2, 3 \\ (dq_i + q_j \wedge q_k)(X, Y) = 4g(X, J_i Y), \quad i = 1, 2, 3 \end{gathered} \tag{4}$$

for any X, Y tangent to $\mathbf{Q}H^m$, where (i, j, k) is a cyclic permutation of $(1, 2, 3)$ and q_k, $k = 1, 2, 3$, are local 1-forms on $\mathbf{Q}H^m$. One can easily compute the following formula

$$\tilde{D}_{X'} Y' = (\tilde{\nabla}_X Y)' - \sum_{k=1}^{3} g(J_k Y, X) j_k \chi \tag{5}$$

for any X, Y tangent to $\mathbf{Q}H^m$. On the other hand, from the fact that H_3^{4m+3} has constant sectional curvature -1, it is not hard to check that the curvature tensor of $\mathbf{Q}H^m$ is

$$\begin{aligned} \tilde{R}(X, Y)Z = &- g(Y, Z)X + g(X, Z)Y \\ &+ \sum_{k=1}^{3} \{g(J_k Y, Z) J_k X - g(J_k X, Z) J_k Y \\ &\qquad + 2g(X, J_k Y) J_k Z\} \end{aligned} \tag{6}$$

for any X, Y, Z tangent to $\mathbf{Q}H^m$. This implies that $\mathbf{Q}H^m$ has constant quaternionic sectional curvature -4. Finally, if $\mathbf{B}$ is the standard unit ball of $\mathbf{Q}^m$, the identification $H_3^{4m+3} \to \mathbf{B}$ given by $(z_0, \dots, z_m) \mapsto (z_1/z_0, \dots, z_m/z_0)$ lets us construct a diffeomorphism from $\mathbf{B}$ to $\mathbf{Q}H^m$. Therefore, $\mathbf{Q}H^m$ is the quaternionic hyperbolic space.

3. Real hypersurfaces in the quaternionic hyperbolic space. In this section we summarize known facts and notations needed in the sequel. In this paper M will always denote a smooth connected real hypersurface in $\mathbf{Q}H^m$, $m \geq 2$, without boundary. For the sake of simplicity, if we write $X \in TM$, we denote a smooth section X of the tangent bundle TM, or a tangent vector field defined on a suitable open subset of M. We will use the same notation when we consider some other linear bundles on M, such as $\mathbf{D}$ or its orthogonal bundle $\mathbf{D}^\perp$ on TM.

Let N be a local normal unit vector field on M. We will denote $U_k = -J_k N$, $k = 1, 2, 3$. If X is a (local) tangent vector field to M, we will write $J_k X = \phi_k X + f_k(X)N$, $k = 1, 2, 3$, where $\phi_k X$ is the tangential component of $J_k X$ and $f_k(X) = g(X, U_k)$, $k = 1, 2, 3$. By (4)

$$\phi_k^2 X = -X + f_k(X)U_k, \qquad f_k(\phi_k X) = 0, \qquad \phi_k U_k = 0, \qquad k = 1, 2, 3 \tag{7}$$

for any X tangent to M.

(8)
$$\begin{aligned} \phi_i X &= \phi_{i+1}\phi_{i+2}X - f_{i+2}(X)U_{i+1} = -\phi_{i+2}\phi_{i+1}X + f_{i+1}(X)U_{i+2} \\ f_i(X) &= f_{i+1}(\phi_{i+2}X) = -f_{i+2}(\phi_{i+1}X), \\ \phi_i U_{i+1} &= U_{i+2} = -\phi_{i+1}U_i \end{aligned}$$

$i = 1, 2, 3$, for any X tangent to M, where the subindices have to be taken module 3. It is also easy to check

$$\begin{gathered} g(\phi_i X, Y) + g(X, \phi_i Y) = 0, \\ g(\phi_i X, \phi_i Y) = g(X, Y) - f_i(X)f_i(Y) \end{gathered} \tag{9}$$

for any X, Y tangent to M, $i = 1, 2, 3$. If we denote by ∇ the induced connection of $\mathbf{Q}H^m$ on M, the Gauss and Weingarten formulae are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N \tag{10}$$

$$\tilde{\nabla}_X N = -AX \tag{11}$$

for any X, Y tangent to M, where A is the Weingarten endomorphism associated to N. A vector field X tangent to M will be called principal if there is a function λ on M such that $AX = \lambda X$. The function λ is called a principal curvature of M. Given a point $x \in M$, we will denote $T_\lambda(x) = \{X \in T_x M : A_x X = \lambda(x)X\}$. From the expression of the curvature tensor of $\mathbf{Q}H^m$ we can compute the Codazzi equation

$$(\nabla_X A)Y - (\nabla_Y A)X = \sum_{k=1}^{3}\{-f_k(X)\phi_k Y + f_k(Y)\phi_k X + 2g(\phi_k X, Y)U_k\} \tag{12}$$

for any X, Y tangent to M. From (4), (10) and (11), we obtain

$$\begin{aligned} \nabla_X U_i &= -p_j(X)U_k + p_k(X)U_j + \phi_i AX \\ (\nabla_X \phi_i)Y &= p_j(X)\phi_k Y - p_k(X)\phi_j Y + f_i(Y)AX - g(AX, Y)U_i \end{aligned} \tag{13}$$

for any X, Y tangent to M and (i, j, k) being a cyclic permutation of $(1, 2, 3)$. If R is the curvature tensor of M, the Gauss equation takes the form

$$\begin{aligned} R(X,Y)Z = &- g(Y,Z)X + g(X,Z)Y \\ &+ \sum_{k=1}^{3}\{-g(\phi_k Y, Z)\phi_k X + g(\phi_k X, Z)\phi_k Y + 2g(\phi_k X, Y)\phi_k Z\} \\ &+ g(AY, Z)AX - g(AX, Z)AY \end{aligned} \tag{14}$$

for any X, Y, Z tangent to M. Let $\tilde{R}$ be the curvature operator of $\mathbf{Q}H^m$. We recall that a real hypersurface in $\mathbf{Q}H^m$ is curvature-adapted if its normal Jacobi operator $K_N = \tilde{R}(\cdot, N)N$ commutes with the Weingarten endomorphism. We will denote the maximal quaternionic distribution of M by $\mathbf{D}$ and its orthogonal complement in TM by $\mathbf{D}^\perp$. Bendt proved in [**1**] that the following three statements are pairwise equivalent:

a) M is curvature-adapted.

b) $\mathbf{D}$ (or equivalently $\mathbf{D}^{\perp}$) is invariant by the Weingarten endomorphism.

c) $\mathbf{D}^{\perp}$ is an autoparallel subbundle of TM.

We note that condition b) is pointwise. That leads to the following useful notation. Given a subset K of M, we will say that M is curvature adapted on K if for each $p \in K$, $A_p\mathbf{D}_p \subseteq \mathbf{D}_p$ or equivalently, $A_p\mathbf{D}_p^{\perp} \subseteq \mathbf{D}_p^{\perp}$. Obviously, if we simply say M is curvature adapted, we are assuming $K = M$. The following results can be found in [**1**].

Lemma A. *Let M be a real hypersurface in $\mathbf{Q}H^m$, $m \geq 2$. Let us suppose that each U_k is principal with principal curvature μ_k, $k = 1, 2, 3$.*

a) *μ_k is locally constant, $k = 1, 2, 3$.*

b) *If $X \in \mathbf{D}$ and X is principal with principal curvature λ, then $(2\lambda - \mu_k)A\phi_k X = (\lambda\mu_k - 2)\phi_k X$, $k = 1, 2, 3$.*

Lemma B. *Let M be a curvature-adapted real hypersurface in $\mathbf{Q}H^m$, $m \geq 2$. If there exists a nonconstant principal curvature in an open subset G of M, then $A_{|\mathbf{D}^{\perp}} = 2I_{\mathbf{D}^{\perp}}$ on G.*

Theorem A. *Let M be a connected curvature-adapted real hypersurface in $\mathbf{Q}H^m$, $m \geq 2$, with constant principal curvatures. Then M is orientable and locally congruent to one of the following*:

a) *a tube of radius $r > 0$ over a totally geodesic $\mathbf{Q}H^k$, $k \in \{0, \dots, m-1\}$,*

b) *a tube of radius $r > 0$ over a totally geodesic $\mathbf{C}H^m$,*

c) *a horosphere.*

Table 1 displays the principal curvatures of each model space in the list of Theorem A.

The distributions T_{μ_i}, $i = 1, 2$, are included in $\mathbf{D}^{\perp}$ and T_{λ_i}, $i = 1, 2$, are included in $\mathbf{D}$.

TABLE 1.

Model space	Principal curvatures	Multiplicities
a)	$\mu_1 = 2\coth(2r)$	3
	$\lambda_1 = \coth(r)$	$4k$
	$\lambda_2 = \tanh(r)$	$4(m-k-1)$
b)	$\mu_1 = 2\coth(2r)$	1
	$\mu_2 = 2\tanh(2r)$	2
	$\lambda_1 = \coth(r)$	$2m-2$
	$\lambda_2 = \tanh(r)$	$2m-2$
c)	$\mu = 2$	3
	$\lambda = 1$	$4m-4$

4. Examples of minimal ruled real hypersurfaces. We consider a metric tensor on $\mathbf{Q}$ given by $g_0(a,b) = \operatorname{Re}(\bar{a}b)$, where $a,b \in \mathbf{Q}$. Now we can rewrite the metric tensor $\bar{g}$ as $\bar{g}(z,w) = -g_0(z_0,w_0) + \sum_{k=1}^{m} g_0(z_k,w_k)$, where $z=(z_0,\dots,z_m)$, $w=(w_0,\dots,w_m)\in\mathbf{Q}^{m+1}$. It is easy to see that $\mu\in\mathbf{Q}$ satisfies $\mu^2=-1$ if and only if $\operatorname{Re}\mu=0$ and $\mu\in\mathbf{S}^3$. Now, given $\mu\in\mathbf{S}^3$ such that $\mu^2=-1$, let us define the hypersurface in H_3^{4m+3}

$$\begin{aligned}\bar{M} = \Big\{ z = \big(r\cosh(t)q_0, r\sinh(t)q_1, \sqrt{r^2-1}\,q_2,\dots,\sqrt{r^2-1}\,q_m\big) \\ \in H_3^{4m+3} : t\in\mathbf{R},\ r>1,\ |q_0|=|q_1|=1,\ \sum_{k=2}^{m}|q_k|^2=1, \\ g_0(\cosh(t)q_0, \sinh(t)q_1\mu)=0\Big\}.\end{aligned}$$

It is clear that $\bar{M}$ is invariant under the action of $\mathbf{S}^3$ and therefore $M=\pi(\bar{M})$ is a real hypersurface in $\mathbf{Q}H^m$. If $z=(z_0,\dots,z_m)\in\bar{M}$, the tangent space of $\bar{M}$ at z is given by

$$\begin{aligned}T_z\bar{M} = \{X=(X_0,\dots,X_m)\in\mathbf{Q}^{m+1} : \bar{g}(X,z)=0, \\ g_0(X_0,z_1\mu)+g_0(z_0,X_1\mu)=0\}.\end{aligned} \tag{15}$$

Let us consider the following tangent vector fields to $\bar{M}$:

$$\begin{aligned}
\bar{E}_k(z) &= (\cosh(t) j_k q_0, \sinh(t) j_k q_1, 0, \dots, 0), && k = 1,2,3,\\
\bar{E}_{k+3}(z) &= (\sinh(t) j_k q_1 \mu, \cosh(t) j_k q_0 \mu, 0, \dots, 0), && k = 1,2,3,\\
\bar{E}_7(z) &= (\cosh(t) q_0, \sinh(t) q_1, r q_2/\sqrt{r^2-1}, \dots, r q_m/\sqrt{r^2-1})\\
\bar{E}_{k+7}(z) &= (0, 0, j_k \sqrt{r^2-1}\, q_2, \dots, j_k \sqrt{r^2-1}\, q_m), && k = 1,2,3
\end{aligned}$$

for any $z \in \bar{M}$. It is not hard to check the following properties: $g_0(\lambda a, b) = -g_0(a, \lambda b)$, $g_0(a\lambda, b) = -g_0(a, b\lambda)$, for any $a, b \in \mathbf{Q}$ and any $\lambda \in \mathbf{S}^3$. Bearing them in mind, the vectors $\{\bar{E}_1, \dots, \bar{E}_{10}\}$ are indeed tangent to $\bar{M}$ and an orthogonal system. Besides, the linear subspace $\bar{W}_z = \{X = (0, 0, X_2, \dots, X_m) \in \mathbf{Q}^{m+1} : \sum_{k=2}^m X_k \bar{z}_k = 0\}$ satisfies $T_z\bar{M} = \bar{W}_z \oplus \mathrm{Span}\,\{\bar{E}_1(z), \dots, \bar{E}_{10}(z)\}$ for any $z \in \bar{M}$. A global unit vector field $\bar{N}$ to $\bar{M}$ in H_3^{4m+3} is

$$\bar{N}_z = (\sinh(t) q_1 \mu, \cosh(t) q_0 \mu, 0, \dots, 0), \quad z \in \bar{M} \tag{16}$$

which has been computed by using (15). Let $\bar{A}$ be the Weingarten endomorphism of $\bar{M}$ associated to $\bar{N}$. By (2) and (11), we get $-\bar{A}\bar{X} = D_{\bar{X}}\bar{N}$ for any $\bar{X} \in T\bar{M}$. A straightforward computation which uses this last formula and (16) shows

$$\begin{gathered}
\bar{A}\bar{E}_k = -\left(\frac{1}{r}\right)\bar{E}_{k+3}, \qquad \bar{A}\bar{E}_{k+3} = \left(\frac{1}{r}\right)\bar{E}_k, \quad k = 1,2,3\\
\bar{A}\bar{X} = 0 \quad \text{if } \bar{X} \in \bar{W}_z \oplus \mathrm{Span}\,\{\bar{E}_7, \dots, \bar{E}_{10}\}(z), \quad z \in \bar{M}.
\end{gathered} \tag{17}$$

Moreover, $N = \pi_* \bar{N}$ is a globally defined unit normal vector field on M. It is easy to see $j_k \bar{N} = \bar{E}_{k+3}$ for any $k = 1,2,3$. Besides, from (17), given $\bar{X} \in T\bar{M}$, $\bar{A}\bar{X} \in \mathrm{Span}\,\{\bar{E}_k, \bar{E}_{k+3} : k = 1,2,3\}$. Moreover, from (5) then $\bar{A}X' = (AX)' - \sum_{k=1}^3 f_k(X) j_k \chi$ for any $X \in TM$,

$$\begin{gathered}
g(AX, Y) = \bar{g}(\bar{A}X', Y'), \quad \text{for any } X, Y \text{ tangent to } M\\
(AX)' = \bar{A}X', \quad \text{for any } X \in \mathbf{D}.
\end{gathered} \tag{18}$$

From (17) and (18), we see that M is minimal and, given $X, Y \in \mathbf{D}$,

$$\begin{aligned}
g(AX, Y) &= \bar{g}(\bar{A}X', Y')\\
&= \sum_{k=1}^{3} \{\bar{g}(\bar{A}X', \bar{E}_{k+3})\bar{g}(\bar{E}_{k+3}, Y') + \bar{g}(\bar{A}X', \bar{E}_k)\bar{g}(\bar{E}_k, Y')\}\\
&= \sum_{k=1}^{3} \bar{g}(X', -(1/r)\bar{E}_{k+3})\bar{g}(\bar{E}_k, Y') = 0
\end{aligned}$$

and, therefore, M is ruled.

5. On the second fundamental form. We begin by proving a generalization of Lemma 3.6 of [**1**] (see also Lemma 5.1 of [**7**]). We will denote by $(*)_{\mathbf{D}^\perp}$ and $(*)_{\mathbf{D}}$ the $\mathbf{D}^\perp$-component and the $\mathbf{D}$-component of $(*)$, respectively.

Lemma 1. *Let M be a real hypersurface in $\mathbf{Q}H^m$, $m \geq 2$. There is a dense open subset $\tilde{M}$ of M with the following property. For every $p \in \tilde{M}$, there are an open neighborhood $\hat{G}$ of p on $\mathbf{Q}H^m$ and a basis $\{J_1, J_2, J_3\}$ of $\hat{\mathcal{V}}$ defined on $\hat{G}$ such that $p \in G := \hat{G} \cap M \subseteq \tilde{M}$, and there are three smooth vectors $\{E_1, E_2, E_3\}$ in $\mathbf{D}$ and three smooth functions μ_k, $k = 1, 2, 3$, defined on G such that the corresponding $U_k = -J_k N$, $k = 1, 2, 3$, satisfy $AU_k = \mu_k U_k + E_k$, for any $k = 1, 2, 3$.*

Remark 1. If M is curvature-adapted, each vector E_k, $k = 1, 2, 3$, vanishes on G and the lemma takes the form of Lemma 3.6 of [**1**]. Also the vectors $\{E_1, E_2, E_3\}$ can be linearly dependent.

Proof. We define the element $A^0 \in \operatorname{Hom}(\mathbf{D}^\perp, \mathbf{D}^\perp)$ by the following. Given $X \in \mathbf{D}^\perp$, $A^0 X = (AX)_{\mathbf{D}^\perp}$. Now we can copy Berndt's proof of Lemma 3.6 in [**1**], but we use A^0 instead of A. At the end of the proof, we know $A^0 U_k = \mu_k U_k$, $k = 1, 2, 3$, on G. Now as the projection map $p : TM \to \mathbf{D}$ is smooth, given $k \in \{1, 2, 3\}$, the orthogonal decomposition $AU_k = A^0 U_k + p(AU_k)$ yields that the smooth vectors $E_k = p(AU_k)$, $k = 1, 2, 3$, are defined on G and lie in $\mathbf{D}$. This concludes the proof. $\square$

Theorem 1. *Let M be a connected real hypersurface in $\mathbf{Q}H^m$, $m \geq 2$, such that*

$$A\phi_k X = \phi_k AX \quad \textit{for any } X \in \mathbf{D}, \quad k = 1, 2, 3. \tag{19}$$

Then M is an open subset of one of the following.

a) *A tube of radius $r > 0$ over a totally geodesic $\mathbf{Q}H^k$, $k = 0, \ldots, m-1$,*

b) *a horosphere.*

Proof. Let $\{U_1, U_2, U_3\}$ be a local orthonormal basis of $\mathbf{D}^\perp$. Fix $k \in \{1,2,3\}$. Given $X \in \mathbf{D}$, there is a $Y \in \mathbf{D}$ such that $X = \phi_k Y$. Bearing this in mind, by (7) and (19), $g(AX, U_k) = g(A\phi_k Y, U_k) = g(\phi_k AY, U_k) = 0$. Therefore, M is curvature-adapted. Let us suppose that M admits a nonconstant principal curvature λ on an open subset G of M. By Lemma B, $AZ = 2Z$ for any $Z \in \mathbf{D}^\perp$ on G. Take a unit $X \in T_\lambda$. By Lemma A and (19), given $k = 1,2,3$, $(2\lambda - 2)\phi_k X = (2\lambda - 2)A\phi_k X = (2\lambda - 2)\phi_k AX = (2\lambda - 2)\lambda\phi_k X$, from which $\lambda = 1$ at any point of G. Therefore, M has constant principal curvatures and we only have to check which of the real hypersurfaces in Theorem A satisfies (19).

a) A straightforward computation shows that the horosphere and the tube of radius $r > 0$ over a totally geodesic $\mathbf{Q}H^k$, $k = 0, \dots, m-1$, satisfy (19).

b) Tube of radius $r > 0$ over a totally geodesic $\mathbf{C}H^m$. In this case, $\mathbf{D}^\perp = V_{\mu_1} \oplus V_{\mu_2}$, $\mathbf{D} = V_{\lambda_1} \oplus V_{\lambda_2}$, where $\mu_1 = 2\coth(2r)$, $\mu_2 = 2\tanh(2r)$, $\lambda_1 = \coth(r)$ and $\lambda_2 = \tanh(r)$. Given $X \in V_{\lambda_1}$ by Lemma A and (19), $\lambda_1\phi_k X = \phi_k AX = A\phi_k X = (\lambda_1\mu_k - 2)/(2\lambda_1 - \mu_k)\phi_k X$, $k = 1, 2$, from which $\lambda_1 = (\lambda_1\mu_k - 2)/(2\lambda_1 - \mu_k)$, and now $\lambda_1^2 - \mu_1\lambda_1 + 1 = \lambda_1^2 - \mu_2\lambda_1 + 1$. As $\lambda_1 \neq 0$, then $\mu_1 = \mu_2$, that is, $\coth(r) = \tanh(r)$. But this equation has no real solutions, which is a contradiction. This concludes the proof. □

Now we suggest reading the introduction to remember the definition of the metric tensors g and g^0 on the bundle $\mathbf{D}$.

Theorem 2. *Let M be a connected real hypersurface in $\mathbf{Q}H^m$, $m \geq 3$, which satisfies $g^0 = ag$ where a is a smooth function on M. Then a is constant and M is one of the following*:

a) *ruled, $a = 0$,*

b) *an open subset of a tube of radius $r > 0$ over a totally geodesic $\mathbf{Q}H^{m-1}$, $0 < a = \tanh(r) < 1$,*

c) *an open subset of a horosphere, $a = 1$,*

d) *an open subset of a tube of radius $r > 0$ over a point, $1 < a = \coth(r)$.*

Proof. Our hypothesis is equivalent to

$$AX = aX + \sum_{l=1}^{3} f_l(AX)U_l \quad \text{for any } X \in \mathbf{D}. \tag{20}$$

Firstly, if $a \equiv 0$ on M, then $g(AX, Y) = 0$ for any $X, Y \in \mathbf{D}$, and therefore M is ruled. Secondly, let us suppose the open set $\{p \in M : a(p) \neq 0\}$ is not empty. We recall the dense open subset $\tilde{M}$ of M of Lemma 1. Clearly the set $\Delta = \{p \in \tilde{M} : a(p) \neq 0\}$ is not empty. Choose a point $x \in \Delta$. Let G be a connected open neighborhood of x in $\tilde{M}$ as in Lemma 1. We will follow the notations of Lemma 1. Let us suppose that M is not curvature adapted at a certain point $p \in G$. Then at least one of the vectors $E_k(p) \neq 0$, so we can choose an open neighborhood Ω of p included in G on which M is not curvature adapted. In the sequel, all computations will be made in Ω unless otherwise stated. Let us define $V = \operatorname{Span}\{E_1, E_2, E_3\}$ and W the orthogonal complement of V in $\mathbf{D}$. By (20),

$$AX = ax \quad \text{for any } X \in W. \tag{21}$$

Given $X, Y \in W$ and $k \in \{1, 2, 3\}$, we develop $g((\nabla_X A)Y - (\nabla_X A)Y, E_k)$, bearing in mind (12), (13), (20) and (21),

$$0 = \sum_{l=1}^{3} g(E_k, E_l) g(Y, \phi_l X) \tag{22}$$

for any X, Y in W, $k = 1, 2, 3$, on Ω. We can regard (22) as a homogeneous linear system whose coefficients are $g(E_k, E_l)$, so that we have to distinguish three cases. We define the matrix $G = (g(E_k, E_l))_{k,l=1,2,3}$.

Case 1. Let us define $\Omega_1 = \{q \in \Omega : \dim V(q) = 3\} = \{q \in \Omega : \det G(q) \neq 0\}$, which is open. The rank of the matrix G is 3, so that the linear system (22) has the unique solution $0 = g(Y, \phi_l X)$ for any $l = 1, 2, 3$ and any $X, Y \in W$. Therefore, $\phi_1 W \subseteq V$ and then $3 = \dim V \geq \dim W = 4m - 7$, that is to say, $4m \leq 10$, which contradicts $m \geq 3$. Therefore, Ω_1 is empty.

Case 2. Let us define $\Omega_2 = \{q \in \Omega : \dim V(q) = 2\}$. We can suppose without losing any generality that $V = \operatorname{Span}\{E_1, E_2\}$ on an open subset Ω_2^0 included in Ω_2. We can forget the third equation of (22) and rewrite the others as

(23)
$$\begin{aligned} g(Y,\phi_1X)g(E_1,E_1)+g(Y,\phi_2X)g(E_1,E_2) &= -\,g(Y,\phi_3X)g(E_1,E_3)\\ g(Y,\phi_1X)g(E_1,E_2)+g(Y,\phi_2X)g(E_2,E_2) &= -\,g(Y,\phi_3X)g(E_2,E_3), \end{aligned}$$

for any $X,Y\in W$ on Ω_2^0. Now there are two subcases.

Case 2a. There is a point $q\in\Omega_2^0$ and a unit $Z\in(W\cap\phi_3W)(q)$. By computing at q, we put $X=Z$, $Y=\phi_3Z$ in (23), obtaining $0=g(E_1,E_3)$, $0=g(E_2,E_3)$. Introducing them in (23), we get $0=g(Y,\phi_1X)$, $0=g(Y,\phi_2X)$ for any $X,Y\in W(q)$. Now this subcase is finished by a similar reasoning as in Case 1.

Case 2b. $W\cap\phi_3W=\{0\}$ at some point $q\in\Omega_2$. As $(\phi_3W\oplus W)(q)\subset \mathbf{D}(q)=(V\oplus W)(q)$, then $4m-6=\dim W=\dim\phi_3W\le\dim V=2$, and therefore $m\le 2$. This is a contradiction. We conclude that the set of interior points of Ω_2 is empty.

Case 3. Let us define $\Omega_3=\{q\in\Omega:\ \dim V(q)=1\}$. Given a point $q\in\Omega_3$, we can suppose without losing any generality that $E_1(q)\neq 0$. Then we only write the first equation of (22),

$$\begin{aligned} g(Y,\phi_1X)g(E_1,E_1)&+g(Y,\phi_2X)g(E_1,E_2)\\ &+g(Y,\phi_3X)g(E_1,E_3)=0 \end{aligned}\tag{24}$$

for any $X,Y\in W$ at q. As $m\ge 3$, $\dim W_q\cap(V\oplus\phi_1V\oplus\phi_2V\oplus \phi_3V)^{\perp}(q)\ge 4$. Given a nonzero vector X in this subspace, take $Y=\phi_1X\in W_q$ so, by (24), $0=g(E_1,E_1)(q)$, which is a contradiction. Therefore, Ω_3 is empty.

Summing up these three cases, Ω is an open subset included in G, $\Omega=\Omega_2$, and Ω_2 has no interior points. Therefore, Ω is empty, which yields that G is a connected curvature adapted real hypersurface in $\mathbf{Q}H^m$, and the equation (20) becomes $AX=aX$ for any $X\in\mathbf{D}$ on G. This means G is an open subset of one of the real hypersurfaces in the list of Theorem 1. Table 1 shows that only the horosphere, the tube of radius $r>0$ over a totally geodesic $\mathbf{Q}H^{m-1}$ or over a point satisfies the fact that there is a constant λ such that $AX = \lambda X$ for any $X \in \mathbf{D}$. In particular, a is constant on G, that is to say, a is locally constant on Δ. Given a connected component C of Δ, as Δ is an open subset of M, so is C. Besides, a is constant on it and M is curvature adapted on C. In fact, C is an open subset of one of the three above-mentioned model spaces. Let us define $H_0 = \{q \in \tilde{M} : a(q) = 0\}$, and let us suppose it is not empty. As $\tilde{M}$ is a dense subset of M and Δ is not empty, there is a sequence $\{p_n\}_{n\in\mathbf{N}}$ in a connected component of Δ whose limit q lies in H_0. But as a is continuous on M, given $n \in \mathbf{N}$, $0 \neq a(p_n) = a(q)$, which is a contradiction. Therefore, Δ must be $\tilde{M}$. Now, as $\tilde{M}$ is a dense subset of M, a is continuous on M, a is locally constant on $\tilde{M}$, and M is connected, then a is constant on M and M is an open subset of one of the three model spaces that we have mentioned before. This concludes the proof. □

Given a vector X tangent to M, we will denote $Q(X) = \mathrm{Span}\,\{X, \phi_1 X, \phi_2 X, \phi_3 X\}$. If Π is a 2-plane tangent to M, we will say that Π is quaternionic if it admits a basis $\{X, Y\}$ such that $Q(X) = Q(Y)$. The quaternionic sectional curvature of M is the sectional curvature of quaternionic 2-planes tangent to M which are included in $\mathbf{D}$.

Theorem 3. *Let M be a connected real hypersurface in $\mathbf{Q}H^m$, $m \geq 3$, which has constant quaternionic sectional curvature q. Then M is one of the following*:

a) *an open subset of a tube of radius $r > 0$ over a point, $-3 < q = -4 + \coth^2(r)$,*

b) *an open subset of a horosphere, $q = -3$,*

c) *an open subset of a tube of radius $r > 0$ over a totally geodesic $\mathbf{Q}M^{m-1}(c)$, $-4 < q = -4 + \tanh^2(r) < -3$,*

d) *ruled, $q = -4$.*

Proof. Take $p \in M$ a point, and let us denote $\mathbf{UD}_p = \{X \in \mathbf{D}_p : \|X\| = 1\|$. If $X \in \mathbf{UD}_p$, then $q = R(X, \phi_k X, \phi_k X, X)$, $k = 1, 2, 3$. From (14) we obtain

$$q = -4 + g(AX, X)g(A\phi_k X, \phi_k X) - g(AX, \phi_k X^2) \quad \text{for any } X \in \mathbf{UD}_p. \tag{25}$$

Let $X(t)$, $t \in (-\varepsilon, \varepsilon)$, be a curve contained in a great circle of $\mathbf{UD}_p$ and such that $X(0) = X$, $X'(0) = Y$. By (25),

$$0 = \frac{d}{dt}\{g(AX(t), X(t))g(A\phi_k X(t), \phi_k X(t)) - g(AX(t), \phi_k X(t))^2\}_{t=0}.$$

A straightforward computation shows

$$\begin{aligned} g(AX,Y)g(A\phi_k X, \phi_k X) + g(AX, X)g(A\phi_k Y, \phi_k X) \\ - g(AX, \phi_k X)g((A\phi_k - \phi_k A)X, Y) = 0, \end{aligned} \tag{26}$$

for any $X, Y \in \mathbf{UD}$, $k = 1, 2, 3$. Similarly,

$$0 = \frac{d^2}{dt^2}\{g(AX(t), X(t))g(A\phi_k X(t), \phi_k X(t)) - g(AX(t), \phi_k X(t))^2\}_{t=0},$$

and, therefore,

$$\begin{aligned} 2(q+4) = {} & g(AY, Y)g(A\phi_k X, \phi_k X) + 4g(AY, X)g(A\phi_k Y, \phi_k X) \\ & + g(AX, X)g(A\phi_k Y, \phi_k Y) - 2g(AX, \phi_k X)g(AY, \phi_k Y) \\ & - g(Y, A\phi_k X - \phi_k AX)^2, \end{aligned} \tag{27}$$

$k = 1, 2, 3$, for any $X, Y \in \mathbf{UD}$. Now let $\{E_1, \dots, E_{4m-4}, U_1, U_2, U_3\}$ be an orthonormal basis of TM defined on an open subset G of M such that

$$(AE_j)_{\mathbf{D}} = a_j E_j \tag{28}$$

where a_j, $j = 1, \dots, 4m-4$, are continuous functions on G. In fact, the functions a_j and the vector fields E_j can be chosen continuous on G, but smooth on an open and dense subset of G. If we substitute $X = E_j$, $Y = E_l$, $j \neq l$ in (26), then $0 = a_j g(A\phi_k E_j, \phi_k E_l)$. This yields $a_j = 0$ or $g(A\phi_k E_j, \phi_k E_l) = 0$, $j \neq l$, which implies $A\phi_k E_j \in \operatorname{Span}\{\phi_k E_j\} \oplus \mathbf{D}^\perp$. Therefore, we can find an orthonormal basis of TM defined on G of the form $\{E_1, \phi_1 E_1, \phi_2 E_1, \phi_3 E_1, \dots, E_{m-1}, \phi_1 E_{m-1}, \phi_2 E_{m-1}, \phi_3 E_{m-1}\}$ such that

$$(AE_j)_{\mathbf{D}} = a_j E_j, \qquad (A\phi_k E_j)_{\mathbf{D}} = a_{jk}\phi_k E_j, \tag{29}$$

where a_k, a_{jk} are continuous functions on G, $j = 1, \dots, m-1$, $k = 1, 2, 3$. As before, all these functions are also smooth on an open and dense subset of G. If we take $X = E_j$ in (25), by (29),

$$q + 4 = a_j a_{jk}, \quad k = 1, 2, 3,\ j = 1, \dots, m-1. \tag{30}$$

If we put $X = E_j$, $Y = \phi_k E_j$ in (27), by (29),

$$2(q+4) = a_{jk}^2 + a_j^2, \quad k = 1, 2, 3,\ j = 1, \dots, m-1. \tag{31}$$

By (30) and (31), $0 = a_{jk}^2 - 2a_j a_{jk} + a_j^2 = (a_{jk} - a_j)^2$, and then

$$a_j = a_{j1} = a_{j2} = a_{j3}, \quad j = 1, \dots, m-1. \tag{32}$$

Choose $X = E_j$, $Y = E_l$, $j \neq l$, and introduce them in (27), bearing in mind (29), $q + 4 = a_j a_l$. From this, by (30) and (32), $a_l a_j = q + 4 = a_l^2 = a_j^2$ for any $j \neq l$. If, for some $j \neq l$, $a_l \neq a_j$ at some point of G, then $a_l = -a_j$. Thus, $a_l^2 = a_l a_j = -a_l^2$ and then $a_l = a_j = 0$. Consequently, $a_1 = \cdots = a_{m-1} = a$. Moreover, from (31), a is constant on G. From (29) and (32), then $(AX)_{\mathbf{D}} = aX$ for any $X \in \mathbf{D}$ on G, that is to say, $g(AX, Y) = ag(X, Y)$ for any $X, Y \in \mathbf{D}$ on G. So we only have to check which of the real hypersurfaces of Theorem 2 have constant quaternionic sectional curvature. Table 1 and (14) show that all of them satisfy this property. This concludes the proof. $\square$

6. The curvature operator.

Theorem 4. *Let M be a real hypersurface in $\mathbf{Q}H^m$, $m \geq 2$, which satisfies*

$$\begin{gathered}(R(X,Y)A)Z + (R(Y,Z)A)X + (R(Z,X)A)Y = 0 \\ \textit{for any } X, Y, Z \in \mathbf{D}.\end{gathered} \tag{33}$$

Then M is an open subset of either:

a) *a tube of radius $r > 0$ over a totally geodesic $\mathbf{Q}H^k$, $k = 0, \dots, m-1$,*

b) *a horosphere.*

Proof. By the first Bianchi identity, (33) is equivalent to

$$R(X,Y)AZ + R(Y,Z)AX + R(Z,X)AY = 0, \quad \text{for any } X,Y,Z \in \mathbf{D}. \tag{34}$$

By virtue of (14), equation (34) takes the form

$$\begin{aligned} 0 = \sum_{k=1}^{3} \{ & 2g(\phi_k X, Y)\phi_k AZ + 2g(\phi_k Z, X)\phi_k AY + 2g(\phi_k Y, Z)\phi_k AX \\ & + g((A\phi_k + \phi_k A)Z, Y)\phi_k X + g((A\phi_k + \phi_k A)X, Z)\phi_k Y \\ & + g((A\phi_k + \phi_k A)Y, X)\phi_k Z\} \end{aligned} \tag{35}$$

for any $X,Y,Z \in \mathbf{D}$. Let $\{E_1, \dots, E_{4m-4}\}$ be an orthonormal basis of $\mathbf{D}$ defined on an open subset G of M. For each $k,l \in \{1,2,3\}$ we define the function $a_{kl} = \sum_{i=1}^{4m-4} g((A\phi_k + \phi_k A)\phi_l E_i, E_i)$ on G. In the sequel, all the computations will be made on G unless otherwise stated. If we take $X = E_i$, $Y = \phi_l E_i$, $l = 1,2,3$, in (35) and adding in $i = 1, \dots, 4m-4$, we get

$$\begin{aligned} 0 = \sum_{k=1}^{3} \{ & -2\phi_k\phi_l(A\phi_k + \phi_k A)Z + a_{kl}\phi_k Z + \phi_k A\phi_l\phi_k Z \\ & + 2\sum_{j=1}^{3} f_j(\phi_k\phi_l(A\phi_k + \phi_k A)Z)U_j\} + 8(m-1)\phi_l AZ, \end{aligned} \tag{36}$$

for any $Z \in \mathbf{D}$, $l = 1,2,3$. If we multiply (36) scalarly by U_l, by virtue of (7), (8), (9), (35) and the fact that $\phi_k\phi_l(A\phi_k + \phi_k A)Z + \sum_{j=1}^{3} f_j(\phi_k\phi_l(A\phi_k + \phi_k A)Z))U_j \in \mathbf{D}$, we see $0 = -\sum_{k=1}^{3} g(\phi_k A\phi_l\phi_k Z, U_l) = g(Z, \phi_{l+2}AU_{l+2}) + g(Z, \phi_{l+1}AU_{l+1})$ for any $Z \in \mathbf{D}$, $l = 1,2,3$, which yields $\phi_{l+2}AU_{l+2} + \phi_{l+1}AU_{l+1} \in \mathbf{D}^\perp$, that is to say, $\phi_1 AU_1 + \phi_2 AU_2 = Z_1$, $\phi_2 AU_2 + \phi_3 AU_3 = Z_2$, $\phi_1 AU_1 + \phi_3 AU_3 = Z_3$, where $Z_1, Z_2, Z_3 \in \mathbf{D}^\perp$ are defined on G. Easy computations show $\phi_k AU_k \in \mathbf{D}^\perp$, $k = 1,2,3$, which imply $AU_k \in \mathbf{D}^\perp$, $k = 1,2,3$, that is to say M is curvature-adapted. By Lemma 1, we can suppose without losing any generality that the vector fields $\{U_1, U_2, U_3\}$ are defined on G and principal with principal curvatures μ_k, $k = 1,2,3$, respectively. Note that the functions a_{kl} are independent of the basis of $\mathbf{D}$ used to compute them. Besides, if $k \neq l$, then a_{kl} vanishes. Indeed, if the basis $\{E_i : i = 1, \dots, 4m-4\}$ is such that $AE_i = \lambda_i E_i$ at some point of G for certain real numbers λ_i, $i = 1, \dots, 4m-4$, then $\sum_{i=1}^{4m-4} g(A\phi_k\phi_l E_i, E_i) = \sum_{i=1}^{4m-4} \lambda_i g(\phi_k\phi_l E_i, E_i) = 0$. On the other hand, if the basis $\{E_i : i = 1, \dots, 4m-4\}$ is such that $A\phi_l E_i = \lambda_i \phi_l E_i$ for certain real numbers λ_i, $i = 1, \dots, 4m-4$, then $\sum_{i=1}^{4m-4} g(\phi_k A\phi_l E_i, E_i) = \sum_{i=1}^{4m-4} \lambda_i g(\phi_k\phi_l E_i, E_i) = 0$. These two computations prove our assertion. From this, (7), (8), (9) and as M is curvature-adapted, (36) becomes

$$(37)\quad 0 = a_{ll}\phi_l Z + (8m-13)\phi_l AZ - 2A\phi_l Z - \phi_{l+1}A\phi_{l+2}Z + \phi_{l+2}A\phi_{l+1}$$

for any $Z \in \mathbf{D}$, $l = 1, 2, 3$. Now we apply $-\phi_l$ to (37) and, by (8),

$$(38)\quad 0 = a_{ll}Z + (8m-13)AZ + 2\phi_l A\phi_l Z + \phi_{l+1}A\phi_{l+1}Z + \phi_{l+2}A\phi_{l+2}Z$$

for any $Z \in \mathbf{D}$, $l = 1, 2, 3$. If we change Z by $-\phi_l Z$ in (37), by (8) then

$$(39)\quad 0 = a_{ll}Z - (8m-13)\phi_l A\phi_l Z - 2AZ + \phi_{l+1}A\phi_{l+1}Z + \phi_{l+2}A\phi_{l+2}Z$$

for any $Z \in \mathbf{D}$, $l = 1, 2, 3$. Subtracting (39) from (38) we obtain $0 = (8m-11)AZ + (8m-11)\phi_l A\phi_l Z$, that is to say, $A\phi_l Z = \phi_l AZ$ for any $Z \in \mathbf{D}$, $l = 1, 2, 3$. The rest of the proof is to check which real hypersurfaces in the list of Theorem 1 satisfy (34).

a) Horosphere, tube of radius $r > 0$ over a totally geodesic $\mathbf{Q}H^k$, $k = 0, \dots, m-1$. By Table 1 there is a real number λ such that $AX = \lambda X$ for any $X \in \mathbf{D}$. Therefore, (34) is satisfied.

b) Tube of radius $r > 0$ over a totally geodesic $\mathbf{Q}H^k$, $k = 1, \dots, m-1$. Table 1 shows $\mathbf{D} = V_{\lambda_1} \oplus V_{\lambda_2}$ where $\lambda_1 = \coth(r)$, $\lambda_2 = \tanh(r)$. Besides, V_{λ_i}, $i = 1, 2$, is quaternionic. Now take unit $X \in V_{\lambda_1}$, $Y = \phi_1 X$ and unit $Z \in V_{\lambda_2}$ and introduce them in (35), bearing in mind (8) and (9), $0 = -2\lambda_1\phi_1 Z + 2\lambda_2\phi_1 Z$, from which $\lambda_1 = \lambda_2$. But the equation $\coth(r) = \tanh(r)$ has no real solutions. This is a contradiction which finishes the proof. □

Corollary 1. *There are no real hypersurfaces in* $\mathbf{Q}H^m$, $m \geq 2$, *such that*

$$(40)\qquad (R(X,Y)A)Z + (R(Y,Z)A)X + (R(Z,X)A)Y = 0$$

for any X, Y, Z tangent to M.

Proof. Let M be a real hypersurface in $\mathbf{Q}H^m$, $m \geq 2$, which satisfies (40). Then M satisfies (33). So we only have to check which real hypersurfaces in the list of Theorem 4 satisfy (40). In each case there are two nonzero real numbers x, y such that

$$AX = xX + y\sum_{k=1}^{3} f_k(X)U_k \tag{41}$$

for any X tangent to M. The first Bianchi identity shows that (40) is equivalent to $R(X,Y)AZ + R(Y,Z)AX + R(Z,X)AY = 0$ for any X, Y, Z tangent to M. Now we substitute (41) in this last equation bearing in mind the first Bianchi identity and $y \neq 0$, and then $0 = \sum_{k=1}^{3}\{f_k(Z)R(X,Y)U_k + f_k(Y)R(Z,X)U_k + f_k(X)R(Y,Z)U_k\}$ for any $X, Y, Z \in TM$. Choose a unit $Y \in \mathbf{D}$, $Z = \phi_2 Y$, $X = U_1$ in this last equation. By (14) and (41), then $0 = R(Y, \phi_2 Y)U_1 = -2U_3$. This is a contradiction that concludes the proof. $\square$

Corollary 2. *There are no real hypersurfaces in $\mathbf{Q}H^m$, $m \geq 2$, such that $R \cdot A = 0$.*

Acknowledgments. The authors would like to thank the referee for some very useful comments which have improved the outcome of this paper.

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Departamento de Geometría y Topología, Facultad de Ciencias, Universidad de Granada, E-18071 Granada, Spain
E-mail address: miortega@ugr.es

Departamento de Geometría y Topología, Facultad de Ciencias, Universidad de Granada, E-18071 Granada, Spain
E-mail address: jdperez@ugr.es