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SYMPLECTIC GEOMETRY OF VECTOR BUNDLE MAPS OF TANGENT BUNDLES

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ABSTRACT. If (M, g) is a Riemannian manifold, then TM has a canonical almost Kähler structure. The derivative of a map of Riemannian manifolds rarely preserves the Kähler forms of the tangent bundles, even up to conformality. Thus we define a weakening of symplectomorphism, called H-isotropic map and study the H-isotropy of vector bundle maps.

1. Introduction and notation. If L is a submanifold of an almost Hermitian manifold (N, J, g, ω), $\omega = g(J\cdot,\cdot)$, then the normal bundle $L^\perp$ of L also possesses an almost Hermitian structure $(\hat{J}, \hat{g}, \hat{\omega})$. Here $\hat{\omega}$ is called the canonical almost symplectic structure of $L^\perp$ (cf. [**4**]). An interesting problem in symplectic geometry is: when are ω and $\hat{\omega}$ isomorphic? (Cf. [**6**], [**4**].) A job relevant to this problem is to study vector bundle maps between two such bundles $L_1^\perp$ and $L_2^\perp$ (e.g., [**4**, Theorem 4.1]). The tangent bundle of a Riemannian manifold can be thought of as a special case of a normal bundle of an almost Hermitian manifold [**4**]. Moreover, the almost symplectic form on TM is in fact just a pull-back of the canonical symplectic form on T^*M. Thus we are motivated to study the symplectic geometry of vector bundle maps of tangent bundles of Riemannian manifolds.

Suppose (M, g) is a Riemannian manifold. Then TM is equipped with Sasaki metric $\hat{g}$ [**8**], [**2**]. If $X \in \Gamma(TM)$, then we use X^H and X^V to denote its horizontal and vertical lifts to TM, respectively. An almost complex structure J for TM compatible with $\hat{g}$ is defined as follows: $J(X_\xi^H + Y_\xi^V) = X_\xi^V - Y_\xi^H$ [**2**]. The 2-form $\omega := \hat{g}(J\cdot,\cdot)$ is exactly $D^*(\omega_c)$ where $D : TM \to T^*M$ is the dual map induced by g and ω_c is the canonical symplectic form on T^*M [**2**]. Thus we call $(J, \hat{g}, \omega)$ the *canonical almost Kähler structure* of TM. While $\hat{g}$ has been studied extensively, little seems to have been done about ω.

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Given a map $f : (N_1, \omega_1) \to (N_2, \omega_2)$ of symplectic manifolds, we say that f is *symplectically conformal*, respectively *symplectically homothetic, symplectic*, if c exists, a nonvanishing real-valued function, respectively nonzero real constant, $c = 1$, on N_1 such that $f^*(\omega_2) = c\omega_1$. Conformal, homothetic, and isometric maps between Riemannian manifolds are similarly defined. (Notice that we do not assume the dimensions of N_1 and N_2 coincide.)

There is a $\hat{g}$-orthogonal decomposition $TTM = \mathcal{H} \oplus \mathcal{V}$ of TTM into the horizontal subbundles $\mathcal{H} = \mathcal{H}TM$ and vertical subbundle $\mathcal{V} = \mathcal{V}TM$ of TTM, where $\mathcal{H}$, respectively $\mathcal{V}$, is the collection of all the X_ξ^H, respectively X_ξ^V. If (M, g), respectively (M', g'), is a Riemannian manifold, then we always use $(J, \hat{g}, \omega)$, respectively $J', \hat{g}', \omega'$, to denote the canonical almost Kähler structure of TM, respectively, TM'. For $f : (M, g) \to (M', g')$, we frequently write $\hat{f}$ for f_* to emphasize that it is a map from $(TM, J\hat{g}, \omega)$ to $(TM', J', \hat{g}', \omega')$ and to avoid awkwardness of certain notations such as $(f_*)^*$. $\hat{f}$ is rarely symplectically conformal (cf. Proposition 2.4 and (ii) of Theorem 4.1). Thus we are content if $\hat{f}$ has some weaker symplectic properties. Since $\hat{f}_*(\mathcal{V}_\xi)$ is always isotropic, with respect to ω', for all $\xi \in TM$, $\mathcal{V}_\xi = (\mathcal{V}TM)_\xi$, we are naturally led to the following

Definition 1.1. Suppose $F : (TM, J, \hat{g}, \omega) \to (TM', J', \hat{g}', \omega')$ is a map between two tangent bundles of Riemannian manifolds equipped with their canonical almost Kähler structures. Then we say F is *H-isotropic* if $F_*(\mathcal{H}_\xi)$ is an isotropic subspace of $T_{F(\xi)}TM'$, with respect to ω' for all $\xi \in TM$, where $H_\xi = (\mathcal{H}TM)_\xi$.

We usually abbreviate "vector bundle map" to VBM. This paper deals with the H-isotropy of an arbitrary VBM $F : TM \to TM'$ over an arbitrary C^∞ map f: $(M, g) \to (M', g')$ of Riemannian manifolds. In Section 2 we introduce some basic tools such as the covariant derivative B^F of F and use the expression of F_* by B^F to derive some basic properties of H-isotropic VBMs. In Section 3 we obtain some sufficient conditions (Theorems 3.2 and 3.4) and some restriction (Proposition 3.5) for generating H-isotropic VBMs. In Section 4 we derive a rigidity result of induced H-isotropic maps (Theorem 4.1) and a sufficient condition for induced H-isotropy (Theorem 4.2). Examples are given when appropriate in most sections and especially in Section 5.

Given manifolds and maps are assumed to be C^∞, and given manifolds are assumed to be connected. If M is an open subset of $\mathbf{R}^n$, unless otherwise indicated, we will assume M carries the metric induced from the usual metric on $\mathbf{R}^n$ and use the usual natural coordinate system $\{x, y, z, \dots\}$ for M and the corresponding frame field $\{(\partial/\partial x), (\partial/\partial y), (\partial/\partial z), \dots\}$ for TM. We also usually write x^1 for x and x^2 for y, etc., without explicit mention. The summation convention will be used, although sometimes we still write $\sum$ explicitly for clarity.

2. Preliminaries and basic properties of H-isotopic VBMs. Suppose $F : TM \to TM'$ is a VBM over $f : (M, g) \to (M', g')$. F can be canonically viewed as a section of $T^*M \otimes f^{-1}TM'$, which is equipped with the connection $D_1 \otimes D_2$. Here D_1 is the connection on T^*M induced by the Levi-Civita connection ∇^M of (M, g), and D_2 is the pullback of the Levi-Civita connection $\nabla^{M'}$ of (M', g') to the pullback bundle $f^{-1}TM$. (When there is no risk of confusion, we just use the symbol ∇ to denote $D_1 \otimes D_2$.) The covariant derivative of F will be denoted by B^F, i.e.,

$$B^F(X, Y) = (\nabla_X F)(Y) = \nabla^f_X(F(Y)) - F(\nabla^M_X Y)$$

for all $X, Y \in \Gamma(TM)$. F is called *parallel* if $B^F = 0$. The *torsion* T^F of F is defined by

$$T^F(X, Y) = B^F(X, Y) - B^F(Y, X)$$

for all $X, Y \in \Gamma(TM)$. F is called *torsionless* if $T^F = 0$. A covariant 3-tensor field A^F on M is defined by

$$A^F(X, Y, Z) = g'(f_* X, B^F(Y, Z))$$

for all $X, Y, Z \in \Gamma(TM)$. We also use the symbols $\beta^f := B^{\hat{f}}$ and $\alpha^f := A^{\hat{f}}$. Notice that β^f is just the second fundamental form of f, and $\hat{f}$ is always torsionless.

If $\{e_i\}$ and $\{E_r\}$ are local frame fields of TM and TM', respectively, and Γ^k_{ij} and $\overline{\Gamma}^t_{rs}$ denote the corresponding Christoffel symbols, then we usually write B^F_{ij} for $B^F(e_i, e_j)$ and A^F_{kij} for $A^F(e_k, e_i, e_j)$, and an easy calculation yields

$$B^F_{ij} = (e_i F_{rj} + \hat{f}_{si} F_{tj} \overline{\Gamma}^r_{st} - F_{rk} \Gamma^k_{ij}) E_r, \tag{1}$$

where F_{rj} and $\hat{f}_{rj}$ are (and will be through out this paper) respectively defined by $F(e_j) = F_{rj}E_r$ and $\hat{f}(e_j) = \hat{f}_{rj}E_r$. In particular, if $\{x^i\}$, respectively, $\{y^r\}$, is a local coordinate system for M, respectively M', then

$$B^F\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \left(\frac{\partial F_{rj}}{\partial x^i} + \frac{\partial f^s}{\partial x^i}F_{tj}\overline{\Gamma}^r_{st} - F_{rk}\Gamma^k_{ij}\right)\frac{\partial}{\partial y^r}. \tag{2}$$

Hence, if $\overline{\Gamma}^r_{st}(f(p)) = \Gamma^k_{ij}(p) = 0$ for all r, s, t, k, i, j and some p, then

$$T^F\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\bigg|_p = \left(\frac{\partial F_{rj}}{\partial x^i} - \frac{\partial F_{ri}}{\partial x^j}\right)\frac{\partial}{\partial y^r}\bigg|_p. \tag{3}$$

If $G : TM' \to TM''$ is another VBM of tangent bundles of Riemannian manifolds, then we can easily show

$$B^{G\circ F} = B^G(F\cdot, F\cdot) + G \circ B^F. \tag{4}$$

We refer the reader to [**3**], [**9**] for the derivative of the formulas similar to (1), (2) and (4).

Lemma 2.1. *Let* $F : TM \to TM'$ *be a* VBM *over* $f : (M, g) \to (M', g')$. *Then*

$$F_*(X^V) = F(X)^V, F_*(X^H_\xi) = (f_*X)^H_{F(\xi)} + B^F(X, \xi)^V_{F(\xi)} \tag{5}$$

for all $X, \xi \in \Gamma(TM)$. *In particular, we have the formula in* [**7**]:

$$\hat{f}_*(X^V_\xi) = (f_*X)^V_{\hat{f}(\xi)}, \hat{f}_*(X^H_\xi) = (f_*X)^H_{\hat{f}(\xi)} + \beta^f(X, \xi)^V_{\hat{f}(\xi)}. \tag{6}$$

Proof. The proof is straightforward and we only sketch it. The first equation of (5) is trivial. Suppose $\xi, X \in T_pM$ and $\gamma : [0, 1] \to M$ is a curve such that $\gamma'(0) = X$. Let $\{e_i\}$ be a parallel orthonormal frame field of TM along γ and $\{e^i\}$ its dual. Let $\{E_r\}$ be a parallel orthonormal frame field of TM' along $f \circ \gamma$. Let P_t, respectively Q_t, be the parallel translation from $\gamma(0)$ to $\gamma(t)$ along Γ, respectively from $f \circ \gamma(0)$ to $f \circ \gamma(t)$ along $f \circ \gamma$. Without loss of generality, we assume $\xi = e_1(0)$.

Suppose $F|T_{\gamma(t)}M = \sum F_{ri}(t)E_r \otimes e^i$. Then

$$\begin{aligned} B^F(X,\xi) &= \frac{d}{dt}\Big|_0 ((Q_t^{-1} \circ F \circ P_t)(\xi)) \\ &= \frac{d}{dt}\Big|_0 \Big(Q^{-1}\Big(\sum F_{r1}(t)E_r\Big)\Big) = \sum F'_{r1}(0)E_r(0), \end{aligned}$$

and thus

$$\begin{aligned} F_*(X^H_\xi) = \frac{d}{dt}\Big|_0 (F(e_1)) &= \frac{d}{dt}\Big|_0 \Big(\sum F_{r1}(t)E_r(t)\Big) \\ &= (f_*X)^H_{F(\xi)} + \sum F'_{r1}(0)E_r(0)^V = (f_*X)^H_{F(\xi)} + B^F(X,\xi)^V_{F(\xi)}. \end{aligned}$$

This proves the second equation of (5). $\square$

Remark 2.2. This lemma implies (i) F is almost complex if and only if $F = \hat{f}$ and f is totally geodesic; (ii) F is isometric if and only if F is *fiberwise isometric* (i.e., $g'(F(X), F(X)) = g(X,X)$ for all $X \in \Gamma(TM)$) and parallel, and f is isometric; (iii) in particular, $\hat{f}$ is isometric $\Leftrightarrow$ f is isometric and totally geodesic $\Leftrightarrow$ $\hat{f}$ is isometric and almost complex.

For convenience, we usually write $X \oplus_\xi Y$ for $(X^H + Y^V)_\xi$ for all $X, Y, \xi \in \Gamma(TM)$. As a corollary of Lemma 2.1, we easily see

$$\begin{aligned} F^*\omega'(X \oplus_\xi X', Y \oplus_\xi Y') = A^F&(X,Y,\xi) - A^F(Y,X,\xi) \\ &+ g'(f_*X, F(Y')) - g'(f_*X', F(Y)) \end{aligned} \tag{7}$$

for all $X, X', Y, Y', \xi \in \Gamma(TM)$. In particular, we have the following characterization of H-isotropy:

Proposition 2.3. *Let* $F : TM \to TM'$ *be a* VBM *over* $f : (M,g) \to (M',g')$*. Then*

(i) F *is* H*-isotropic if and only if* A^F *is symmetric in the first two slots.*

(ii) *Suppose* $g'(f_*X, F(Y)) = g'(f_*(Y), F(X))$ *for all* $X, Y \in \Gamma(TM)$*. Then* F *is* H*-isotropic* $\Leftrightarrow$ *if* $\xi \in TM$ *and* Q *is a subspace of* $T_\xi TM$*.*

Then $\hat{f}_*(Q)$ *is isotropic if and only if* $\hat{f}_*(JQ)$ *is* $\Leftrightarrow$ $\hat{f}^*\omega'$ *is a* $(1,1)$ *-form on* TM *(i.e.,* $\hat{f}^*\omega'(J\cdot, J\cdot) = \hat{f}^*\omega'$*).*

Proof. Part (i) follows directly from (7). Part (ii) follows from (7) and the fact that $\hat{f}_*(\mathcal{V}_\xi)$ is isotropic and $J\mathcal{V}_\xi = H_\xi$. $\square$

Let W^f be the covariant 2-tensor field on TM defined by $W^f = (\hat{f}^*\omega')(\cdot, J\cdot)$. We easily see by the proposition that if $\hat{f}$ is H-isotropic, then W^f is symmetric and positive semi-definite, and $W^f = W^f(J\cdot, J\cdot)$. In fact, if f is isometric and totally geodesic, then $W^f = \hat{f}^*\hat{g}' = \hat{g}$.

The following important fact will be used several times.

Proposition 2.4. *For* $f : (M, g) \to (M', g')$*,* f *is isometric if and only if* $\hat{f}$ *is symplectic.*

Proof. The backward direction of the proposition is trivial because of (6). Thus we assume now f is isometric. By the elementary theory of harmonic maps, $\alpha^f = 0$. Thus, by (7) and Proposition 2.3,

$$\begin{aligned}\omega(X \oplus_\xi X', Y \oplus_\xi Y') &= g(X, Y') - g(X', Y)\\ &= g'(f_*X, f_*Y') - g'(f_*X', f_*Y)\\ &= \hat{f}^*\omega'(X \oplus_\xi X', Y \oplus_\xi Y')\end{aligned}$$

for all X, X', Y, Y', $\xi \in \Gamma(TM)$. $\square$

Notice that this proposition can also be proved by the technique of Liouville vector fields as used in [**4**].

Corollary 2.5. (i) *Suppose* $F : TM \to TM'$ *is an* H*-isotropic* VBM *over a submersion* $f : (M, g) \to (M', g')$*. Then for all* $\xi \in TM$*,* $\dim(F_*(\mathcal{H}_\xi)) = \operatorname{rank} f$.

(ii) *Suppose* $f : (M, g) \to (M', g')$ *has constant rank, and* $\hat{f}$ *is* H*-isotropic. Then for every* $\xi \in TM$*,* $\dim(\hat{f}_*(\mathcal{H}_\xi)) = \operatorname{rank} f$.

Proof. By (i) of Proposition 2.3, $B^F(X,\xi)=0$ whenever $\xi \in T_xM$, $X \in \operatorname{kernel}(F_x)$ and $x \in M$. Part (i) then follows from the second equation of (5). Part (ii) follows from part (i), Proposition 2.4, and the fact that locally f maps M to a $\operatorname{rank}(f)$-dimensional submanifold of M'. $\square$

The following example illustrates several cases in which $\dim(F_*(\mathcal{H}_\xi))$ may not equal $\operatorname{rank} f_x$ for $\xi \in T_xM$.

Example 2.6. (i) (Equip $\mathbf{R}^2$ with the usual natural coordinates $\{x,y\}$.) Let $M=(0,\infty)\times(0,\infty)$, $M'=\mathbf{R}^2$, $f: M \to M'$ be defined by $f(x,y)=x$, and VBM $F: TM \to TM'$ over f defined by

$$F=\begin{pmatrix} x, 1 \\ 0, (y-1)^2 \end{pmatrix}$$

(with respect to the frame field $\{(\partial/\partial x),(\partial/\partial y)\}$). By (2), (3) and (i) of Proposition 2.3, we can easily check that F is torsionless and H-isotropic, $F|T_xM : T_xM \to T_{f(x)}M'$ is bijective for all $x \in M$, and f has constant rank 1. An easy calculation yields $B^F[(\partial/\partial y),(\partial/\partial x)]=0$ and $B^F[(\partial/\partial y),(\partial/\partial y)]=2(y-1)[(\partial/\partial y)]$. Thus, by (5), $\dim(F_*(\mathcal{H}_\xi))=1$ if $\xi \in T_{(x,1)}M$; for $y \neq 1$, $\dim(F_*(\mathcal{H}_\xi))=1$ if $\xi=(\partial/\partial x)_{(x,y)}$ and $\dim(F_*(\mathcal{H}_\xi))=2$ if $\xi=(\partial/\partial y)_{(x,y)}$.

(ii) Let $f: \mathbf{R} \to \mathbf{R}$ be defined by $f(x)=x^2$. By (2), $\beta^f[(\partial/\partial x),(\partial/\partial x)]=2(\partial/\partial x)$. Thus, by (6), $\dim(\hat{f}_*(\mathcal{H}_\xi))=1$ if $\xi=(\partial/\partial x)_0$. But $\operatorname{rank} f_*(0)=0$.

(iii) Let $M=(0,1)\times(0,1)$. Let $f: M \to \mathbf{R}$ be defined by $f(x,y)=xy+y$. We easily see that f has constant rank 1 and, by Proposition 2.3 and (2), $\hat{f}$ is not $\hat{H}$-isotropic. Since every 0- or 1-dimensional subspace of a symplectic vector space is isotropic, there exists a $\xi \in TM$ such that $\dim(\hat{f}_*(\mathcal{H}_\xi))=2$. $\square$

3. Conditions and restrictions for obtaining H-isotropic maps. In this section we obtain some sufficient conditions for H-isotropic VBMs and see how an H-isotropic VBM prevents us from getting another one.

The following lemma is interesting itself:

Lemma 3.1. *Let $F : TM \to TM'$ be a torsionless, fiberwise isometric* VBM *over a map $f : (M, g) \to (M', g')$. Suppose* $\dim M = \dim M'$. *Then F is parallel.*

Proof. Fix $p \in M$ and $q = f(p)$. Choose local orthonormal frame fields $\{e_i\}$ and $\{E_i\}$ around p and $f(p)$, respectively, such that $\nabla^M_{e_i} e_j|_p = \nabla M'_{E_i} E_j|_q = 0$ for all i, j. Without loss of generality, we assume $F_{ij}|_p = \delta_{ij}$. By (1), $B^F(e_i, e_j)|_p = (e_i F_{kj}) E_k|_p$. Thus, $e_i F_{kj}|_p = e_j F_{ki}|_p$. But we also have $e_k F_{ij}|_p = -e_k F_{ji}|_p$ since $F_{rj} F_{ri} = \delta_{ij}$. Hence $e_i F_{kj}|_p = 0$ for all i, j, k, and thus $B^F = 0$. □

The following theorem is the first of our three theorems for obtaining H-isotropic maps:

Theorem 3.2. *Let $F : TM \to TM'$ be a torsionless, fiberwise isometric* VBM *over f: $(M, g) \to (M', g')$. Suppose there exists a $\dim(M)$-dimensional submanifold M'' of M' such that $F(TM) \subset TM''$. Then F is H-isotropic.*

Proof. Locally F can be written as $F = G \circ H$ where H is the map F with codomain changed to TM'', and G is the derivative of the isometric immersion from M'' to M'. By (4), $B^F(X, Y) = B^G(H(X), H(Y)) + (G \circ B^H)(X, Y)$ for all $X, Y \in \Gamma(TM)$. Thus, H is torsionless. Thus, H is parallel by Lemma 3.1. The theorem then follows from Propositions 2.3 and 2.4. □

Compare the following example with the previous theorem.

Example 3.3. Let $M = \{(x, y) \in \mathbf{R}^2 : (x + y)^2 < (1/2)\}$, $M' = \mathbf{R}^3$, $f : M \to M'$ defined by $f(x, y) = (x, 0, 0)$ and $F : TM \to TM'$ defined by

$$F = \begin{pmatrix} x + y, x + y \\ \sqrt{(1/2) - (x + y)^2}, \sqrt{(1/2) - (x + y)^2} \\ (1/\sqrt{2}), -(1/\sqrt{2}) \end{pmatrix}.$$

By (3), we easily see that F is torsionless and fiberwise isometric. But an easy calculation yields $A^F_{121}(= A^F((\partial/\partial x), (\partial/\partial y), (\partial/\partial x))) = 1$ and $A^F_{211} = 0$. Thus F is not H-isotropic by Proposition 2.3. □

The following theorem provides another condition for H-isotropy.

Theorem 3.4. *Suppose* $\dim M \geq 2$ *and* $f : (M, g) \to (M', g')$ *is a map of Riemannian manifolds. Suppose there exists a positively-valued function* c *on* M *such that* $f^*g' = cg$. *Let* $F = (c'/c)\hat{f}$ *for some real-valued function* c' *on* M. *Then* F *is* H*-isotropic if and only if* c' *is a constant. In particular, for every nonzero constant* c'', $(c''/c)\hat{f}$ *is symplectically homothetic.*

Proof. By Proposition 2.4, we can assume without loss of generality that $M = M'$ and $f = \mathrm{Id}$, (but $g \neq g'$ in general). Fix $p \in M$. Let $\{x^i\}$ be local normal coordinates of (M, g) around p. As before, we use Γ^k_{ij} and $\overline{\Gamma}^k_{ij}$ to denote the corresponding Christoffel symbols for (M, g) and (M, g'), respectively. By (2),

$$A^F_{kij}|_p = c\left(\frac{\partial F_{kj}}{\partial x^i} + \frac{c'}{c}\overline{\Gamma}^k_{ij}\right)\bigg|_p.$$

Thus F is H-isotropic if and only if

$$\frac{\partial F_{kj}}{\partial x^i} + \frac{c'}{c}\overline{\Gamma}^k_{ij}\bigg|_p = \frac{\partial F_{ij}}{\partial x^k} + \frac{c'}{c}\overline{\Gamma}^i_{kj}\bigg|_p \tag{8}$$

for all i, j, k, p.

By the usual formulas for the Christoffel symbols

$$\Gamma^k_{ij} = \frac{1}{2}\sum_m g^{km}\left(\frac{\partial g_{jm}}{\partial w_i} + \frac{\partial g_{im}}{\partial w_j} - \frac{\partial g_{ij}}{\partial w_m}\right),$$

we obtain

$$\overline{\Gamma}^k_{ij}(p) = \frac{1}{2c}\left(\frac{\partial c}{\partial x^i}\delta_{jk} + \frac{\partial c}{\partial x^j}\delta_{ik} - \frac{\partial c}{\partial x^k}\delta_{ij}\right)\bigg|_p.$$

In particular, if $k \neq i \neq j$, then $\overline{\Gamma}_{ij}^k(p) = (1/2c)(\partial c/\partial x^i)\delta_{jk}|_p$; if $i \neq j$, then $\overline{\Gamma}_{ij}^j(p) = (1/2c)(\partial c/\partial x^i)|_p$ (no summation) and $\overline{\Gamma}_{jj}^i(p) = -(1/2c)(\partial c/\partial x^i)|_p$ (no summation). Therefore, we can consider the following three cases:

(i) Suppose $k \neq i$, $i \neq j$, $j \neq k$. Then (8) is trivially true.

(ii) Suppose $k \neq i$, $i \neq j$, $j = k$. Then (8) can be rewritten as

$$\left(\frac{\partial}{\partial x^i}\,\frac{c'}{c} + \frac{c'}{2c^2}\,\frac{\partial}{\partial x^i}\right)\bigg|_p = -\,\frac{c'}{2c^2}\,\frac{\partial}{\partial x^i}\bigg|_p. \tag{9}$$

(iii) Suppose $k \neq i$, $i = j$. Then (8) can be rewritten as

$$-\,\frac{c'}{2c^2}\,\frac{\partial}{\partial x^k}\bigg|_p = \left(\frac{\partial}{\partial x^k}\,\frac{c'}{c} + \frac{c'}{2c^2}\,\frac{\partial}{\partial x^k}\right)\bigg|_p.$$

Hence, F is H-isotropic if and only if (9) is true for all i and p. The latter is clearly equivalent to $(\partial c'/\partial x^i)|_p = 0$ for all i and p. This proves the first conclusion of the theorem. The second conclusion of the theorem then follows directly from (7). $\square$

From the previous theorem, we suspect that if F is an H-isotropic VBM, then cF is probably not H-isotropic unless c is a constant. The following proposition essentially confirms this suspicion and thus puts some restriction on getting an H-isotropic map from a known one.

Proposition 3.5. *Let $F : TM \to TM'$ be an H-isotropic* VBM *over $f : (M, g) \to (M', g')$. Suppose $\dim(F(T_xM) \cap f_*(T_xM)) \geq 2$ for all $x \in M$, and c is a real-valued function on M. Then cF is H-isotropic if and only if c is a constant.*

Proof. The backward direction of this proposition is trivial by Proposition 2.3. Suppose cF is H-isotropic. By (1) we easily derive $B^{cF}(Y,Z) = (Yc)F(Z) + cB^F(Y,Z)$ for all $Y, Z \in \Gamma(T_pM)$. Thus $g'(f_*X, (Yc)F(Z)) = g'(f_*Y, (Xc)F(Z))$ for all $X, Y, Z \in \Gamma(TM)$. Now fix a $p \in M$. Suppose $Y \in T_pM$. We can choose $X, Z \in T_pM$ such that $f_*Y \perp F(Z)$ and $g'(f_*X, F(Z)) = 1$. Then

$$Yc = g'(f_*X, (Yc)F(Z)) = g'(f_*Y, (Xc)F(Z)) = 0. \qquad \square$$

If $F^*\omega' = G^*\omega'$ for $F, G : TM \to TM'$, then F is H-isotropic if and only if G is. Thus, the following observation, which follows directly from (7) and Proposition 2.3, can also be viewed as a restriction of getting an H-isotropic map from a known one. Suppose $F, G : TM \to TM'$ are VBMs over $f : (M, g) \to (M', g')$, $F(TM), G(TM) \subset f_*(TM)$ and F is H-isotropic. If $F^*\omega' = G^*\omega'$, then $F = G$.

4. Induced maps of tangent bundles. In this section we obtain a rigidity result of induced H-isotropic maps and obtain a sufficient condition for obtaining induced H-isotropic maps.

If $f : (M, g) \to (M', g')$ is an immersion and $\dim M = 1$, then $\hat{f}$ is symplectically conformal. When $\dim M \geq 2$, the story is quite different. This can be seen from the following rigidity result of H-isotropic induced maps of tangent bundles.

Theorem 4.1. *Suppose $f : (M, g) \to (M', g')$ and $\dim M \geq 2$. Then*

(i) *If $\hat{f}$ is H-isotropic, and, for every $x \in M$, there exist $\xi(x) \in T_xM$ and $c(x) \in \mathbf{R} - \{0\}$ such that $\hat{f}^*(\omega'_{\hat{f}(\xi(x))}) = c(x)\omega_{\xi(x)}$, then $\hat{f}$ is symplectically homothetic.*

(ii) *$\hat{f}$ is symplectically conformal $\Leftrightarrow$ $\hat{f}$ is symplectically homothetic $\Leftrightarrow$ f is homothetic $\Leftrightarrow$ f is conformal and $\hat{f}$ is H-isotropic.*

Proof. Suppose the assumption of part (i) holds. By (7) we have

$$\begin{aligned} g(X, Y') &= \omega(X^H_{\xi(x)}, (Y')^V_{\xi(x)}) = \hat{f}^*\omega'(X^H_{f_*(\xi(x))}, (Y')^V_{f_*(\xi(x))}) \\ &= g'(f_*X, f_*Y') \end{aligned}$$

for all $X, Y' \in \Gamma(T_xM)$ and $x \in M$. Hence, f is conformal. By Theorem 3.4, we then easily see that f is homothetic. Hence $\beta^f = 0$, and thus $\hat{f}$ is symplectically homothetic by (7). This concludes the proof of part (i).

If f is homothetic, then $\alpha^f = 0$ by the elementary theory of harmonic maps. Thus part (ii) follows directly from the conclusion and proof of part (i). □

The following theorem provides a handy sufficient condition for induced H-isotropic maps.

Theorem 4.2. *Suppose $f : (M, g) \to (M', g')$ is a diffeomorphism such that f preserves geodesics up to parameterization. Then $\hat{f}$ is H-isotropic.*

Proof. We can assume that $M = M'$ and $f = \mathrm{Id}$ (but $g \neq g'$ in general). Fix $p \in M$. Let $(x^1, \dots, x^n)$ be a normal coordinate system around p for (M, g) and $(u^1, \dots, u^n; v^1, \dots, v^n)$ the associated coordinates for TM. (That is, the element $\sum v^i (\partial/\partial x^i)|_{x^1,\dots,x^n)}$ in TM is represented by $(u^1, \dots, u^n; v^1, \dots, v^n)$, where $u^i = x^i$.) Notice that span $\{(\partial/\partial u^1)|_\xi, \dots, (\partial/\partial u^n)|_\xi\}$ is $\mathcal{H}_\xi$ with respect to $(TM, \hat{g})$ for all $\xi \in T_pM$.

For any $i = 1, \dots, n$, there exists an $\mathbf{R}$-valued function c_i defined on some interval $(-\varepsilon, \varepsilon)$ such that the curve γ defined by

$$\gamma(t) = (0, \dots, 0, t, 0, \dots, 0; 0, \dots, 0, c_i(t), 0, \dots, 0)$$

is a horizontal curve in $(TM, \hat{g}')$, where on the right side of the equation t and $c_i(t)$ occur at the ith and $(n+i)$th places, respectively. Thus $(\partial/\partial x^i)^{H'} = (\partial/\partial u^i) + (\partial c_i/\partial x_i)(\partial/\partial v^i)$ (no summation), where H' denotes the horizontal lift with respect to g'. An easy calculation then yields that span $\{(\partial/\partial u^1)|_\xi, \dots, (\partial/\partial u^n)|_\xi\}$ is isotropic with respect to ω' if $\xi \in T_pM$. □

By Theorems 4.1 and 4.2, we have

Corollary 4.3. *Suppose $\dim M \geq 2$ and $f : (M, g) \to (M', g')$ is a diffeomorphism such that f preserves geodesics up to parameterization and angles. Then f is homothetic.*

5. Examples. Since the concept of H-isotropic maps is introduced in this very paper, we would like to see some more examples and counterexamples.

It is easy to construct a VBM which is symplectically conformal but not symplectically homothetic (cf. Theorem 4.1):

Example 5.1. Let $M = \{(x, y) \in \mathbf{R}^2 : (x-1)^2 + (y-1)^2 < 1\}$, $M' = \mathbf{R}^2$, $f : M \to M'$ be defined by $f(x, y) = [x, (-x^2 + y^2/2)]$ and

$F : TM \to TM'$ be represented by $F = \begin{pmatrix} y,x \\ 0,1 \end{pmatrix}$. We can easily verify that A^F is symmetric in the first two slots and $y\langle X, Y\rangle = \langle f_* X, F(Y)\rangle$ for all $X, Y \in T_{(x,y)}M$, $(x, y) \in M$. Thus, by (7), F is symplectically conformal but not symplectically homothetic. □

The following example is a straightforward application of our developed theory to the case of real hyperbolic spaces.

Example 5.2. Suppose $n \geq 2$. Let (B^n, g) be the usual unit ball with flat metric and (B^n, g_1), respectively (B^n, g_2), the Poincaré, respectively Klein, disk model for the real hyperbolic space $\mathbf{R}H^n$ (e.g. [**5**]). The identity map $\mathrm{Id}_1 : (B^n, g) \to (B^n, g_1)$ is conformal but not homothetic. Thus $\hat{\mathrm{Id}}_1$ is not H-isotropic by (ii) of Theorem 4.1. The identity map $\mathrm{Id}_2 : (B^n, g) \to (B^N, g_2)$ preserves geodesics up to parameterization. Hence, $\hat{\mathrm{Id}}_2$ is H-isotropic by Theorem 4.2. But $\hat{\mathrm{Id}}_2$ is not symplectically conformal by (ii) of Theorem 4.1. □

Theorem 4.1 implies that if $f : M \to M'$ is a biholomorphism between Kähler manifolds of real dimension 2, then $\hat{f}$ is H-isotropic $\Leftrightarrow$ $\hat{f}^{-1}$ is H-isotropic $\Leftrightarrow$ $\hat{f}$ is symplectically conformal. The following example deals with the case when f is a diffeomorphsm but not a biholomorphism.

Example 5.3. Let M and M' be open subsets $\mathbf{R}^2$ and x^1, x^2 the usual natural coordinates of $\mathbf{R}^2$. Fix a map $f : M \to M'$. By (2) we have $\alpha^f[(\partial/\partial x^k), (\partial/\partial x^i), (\partial/\partial x^j)] = \sum(\partial f^m/\partial x^k)(\partial^2 f^m/\partial x^i \partial x^j)$. Thus, by Proposition 2.3, $\hat{f}$ is H-isotropic if and only if the following two equations hold:

$$\frac{\partial f^1}{\partial x^1}\frac{\partial^2 f^1}{\partial x^2 \partial x^2} + \frac{\partial f^2}{\partial x^1}\frac{\partial^2 f^2}{\partial x^2 \partial x^2} = \frac{\partial f^1}{\partial x^2}\frac{\partial^2 f^1}{\partial x^1 \partial x^2} + \frac{\partial f^2}{\partial x^2}\frac{\partial^2 f^2}{\partial x^1 \partial x^2},$$

$$\frac{\partial f^1}{\partial x^1}\frac{\partial^2 f^1}{\partial x^2 \partial x^1} + \frac{\partial f^2}{\partial x^1}\frac{\partial^2 f^2}{\partial x^2 \partial x^1} = \frac{\partial f^1}{\partial x^2}\frac{\partial^2 f^1}{\partial x^1 \partial x^1} + \frac{\partial f^2}{\partial x^2}\frac{\partial^2 f^2}{\partial x^1 \partial x^1}.$$

Therefore, we can easily check that each of the following two claims is true for suitable M and M':

(a) Suppose $f(x, y) = (x + 2y, (x + y)^2)$ and thus $f^{-1}(x, y) = (-x + 2\sqrt{y}, x - \sqrt{y})$. Then $\hat{f}$ is H-isotropic, but $\hat{f}^{-1}$ is not H-isotropic.

(b) Suppose $f(x,y) = (x, y^2)$ and thus $f^{-1}(x,y) = (x, \sqrt{y})$. Then both $\hat{f}$ and $\hat{f}^{-1}$ are H-isotropic, but neither $\hat{f}$ nor $\hat{f}^{-1}$ is symplectically conformal. □

Example 5.4. Let $f : M \to M'$ be a Riemannian submersion. Then $\hat{f}$ is H-isotropic if and only if f is totally geodesic.

The backward direction of the claim follows from Proposition 2.3. Now suppose $\hat{f}$ is H-isotropic. We will use [**3**, Lemma 1.5]. Let $\beta = \beta^f$ and $T^H(M)$, respectively $T^V(M)$, denote the horizontal, respectively vertical, distribution on M associated with f. We have $\beta|T^H(M) \times T^H(M) = 0$. But α^f is totally symmetric by Proposition 2.3. Hence $\beta|T^V(M) \times T^V(M) = \beta|T^H(M) \times T^V(M) = \beta|T^V(M) \times T^H(M) = 0$. This equation implies that f is totally geodesic and the distribution $T^H(M)$ is integrable.

In particular, if f is the canonical projection from TN to a Riemannian manifold N, or if f is the canonical projection from the normal bundle $L^\perp$ to a submanifold L of a Riemannian manifold, $L^\perp$ is equipped with the Sasaki metric [**1**], then $\hat{f}$ is H-isotropic if and only if N, respectively the normal connection on $L^\perp$, is flat. □

By (4) and (i) of Proposition 2.3, if $F : TM \to TM'$ is a parallel VBM and $G : TM' \to TM''$ is an H-isotropic VBM, then $G \circ F$ is H-isotropic. We can use this to construct many other examples of H-isotropic VBMs.

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