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FACTOR MAPS, ENTROPY AND FIBER CARDINALITY FOR MARKOV SHIFTS

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ABSTRACT. It is well known that a factor map between transitive shifts of finite type either preserves entropy and is bounded-to-1 or it does not preserve entropy and is uncountable-to-1. In this paper we elucidate the relation between entropy and fiber cardinality for factor maps between transitive locally compact Markov shifts. We show that every countable-to-1 factor map increases the Gurevic entropy while every finite-to-1 factor map preserves Gurevic entropy. We study finite-to-1 proper factor maps and show that they additionally preserve positive and strongly positive recurrence. Then we investigate finite-to-1 proper factor maps between Markov shifts which have an expansive 1-point compactification. We conclude the paper with some examples showing that properly finite-to-1 and properly countable-to-1 factor maps exist between synchronized systems.

Introduction. Shifts of finite type (SFT), [**19**], [**20**], can be generalized in two ways. One can keep the compactness of the shift space but relax the Markov property which leads to synchronized systems and coded systems, [**1**], [**5**], [**10**], [**13**], [**14**]. Or one can keep the Markov property and relax the compactness which leads to locally compact Markov shifts, [**6**]–[**9**], [**12**]–[**15**], [**19**], [**20**]. Markov shifts and coded systems are strongly related. We quote two results to illuminate this relation. Coded systems are those compact subshifts which are the surjective factors of transitive Markov shifts [**14**], and a subshift compactification of a transitive locally compact Markov shift is always coded [**13**].

In this paper we study factor maps between locally compact Markov shifts and complete the results on factor maps between coded and synchronized systems obtained in [**5**].

A *subshift* is a shift invariant subset S of $\mathbf{N}^{\mathbf{Z}}$, endowed with the product topology of the discrete topology on $\mathbf{N} = \{1, 2, \dots\}$, together

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with the left shift map. To reduce notation we use the same symbol for the space and the shift map; which one is meant will always be clear from context. The topology on S is generated by the cylinder sets ${}_i[x_0 \dots x_n] := \{y \in S \mid y_i = x_0, \dots, y_{i+n} = x_n\}$. For a point $x = (x_i)_{i \in \mathbf{Z}} \in S$ and integers $n \leq m$, we use the notation $x[n,m]$ to denote the subblock $x_n x_{n+1} \dots x_{m-1} x_m$ of $x, x[n,\infty)$ to denote the right infinite ray $x_n x_{n+1} \dots$, and finally $x(-\infty, m]$ to denote the left infinite ray $\dots x_{m-1} x_m$.

Two subshifts are *conjugate* if there is a homeomorphism $f : S \to T$ which commutes with the shift maps. Any subshift T of $\mathbf{N}^{\mathbf{Z}}$ which is conjugate to S is called a *presentation* of S. A *locally compact subshift* is a subshift S where the space S is locally compact.

A *Markov shift* S is (by definition) a subshift which is conjugate to the set S_G of bi-infinite walks on the edges of a countable directed graph, G, with the left shift transformation acting on S_G. The subshift S_G is called a graph presentation of S. We consider only nondegenerate graphs G. The Markov shift S is locally compact if and only if G has finite in- and out-degree (at most finitely many in-coming and out-going edges at every vertex); for a discussion of the relation between the possible degrees of G and entropy, see [**9**]. The Markov shift is compact if and only if G is a finite graph if and only if it is a shift of finite type (SFT). Transitivity of S means strongly connectedness of G.

Standing assumption. Since we consider in this paper with very few exceptions (Lemma 1.1, Corollary 1.5, Examples 1.6 and 1.7) only locally compact transitive Markov shifts, if not stated otherwise, we always mean by "Markov shift" a locally compact transitive Markov shift.

The 1-*point compactification* S_0 of a locally compact subshift S is the compact metric dynamical system which consists of the Alexandroff 1-point compactification of the shift space together with the extended shift map, [**11**]. The Gurevic entropy of S, $h_G(S)$ is defined to be the topological entropy $h_{\mathrm{top}}(S_0)$, [**16**]. If S is compact, then the Gurevic entropy coincides with the topological entropy, $h_G(S) = h_{\mathrm{top}}(S)$. The Gurevic entropy $h_G(S)$ can be computed for a Markov shift S by loop counting, that is, if S_G is any graph presentation for S, v is any vertex in G and $C_v(n)$ is the number of loops of length n at vertex v, then $h_G(S) = \limsup_n 1/n \log C_v(n)$, (we always put $\log 0 := 0$ in this paper) [**16**], [**17**], [**19**]. Furthermore, $h_G(S) = \sup\{h_G(T) \mid T \subset S, T \text{ is an SFT}\}$.

We study the relation between entropies of the domain and range shifts and the fiber cardinalities of a factor map. Let S, T be subshifts. By a *factor map* $f : S \to T$ we mean a continuous shift commuting onto map. We call a factor map *entropy preserving* if $h_G(S) = h_G(T)$. The map f is *entropy decreasing* (*entropy increasing*) if and only if $h_G(S) \geq h_G(T)$ ($h_G(S) \leq h_G(T)$). Let $f : S \to T$ be a factor map. We call the preimage set $f^{-1}y$ of a point $y \in T$ the *fiber* of f over y. We say f is *bounded-to*-1 if there is some $M \in \mathbf{N}$ such that all fibers of f have cardinality at most M. The map f is *finite-to*-1 if all fibers are finite sets, and f is *countable-to*-1 if all fibers are countable sets, and finally f is *uncountable-to*-1 if there is a fiber which is not countable.

We now give an outline of the content of the paper. Let S, T be subshifts and $f : S \to T$ be a factor map.

If S is a compact subshift, then of course, T is compact and $h_{\text{top}}(S) \geq h_{\text{top}}(T)$. If S is an SFT, then either the map preserves entropy and is bounded-to-1 or it does not preserve entropy and is uncountable-to-1, [**20**]. This dichotomy remains true in the larger class of shifts with specification, [**5**]. If S is merely compact, then a countable-to-1 map preserves entropy, but an entropy preserving map need not be countable-to-1 even if S is synchronized, [**5**]. To complete this picture we shall show in Section 5 that if S is synchronized then properly finite-to-1 (meaning the map is finite-to-1, but not bounded-to-1) and properly countable-to-1 factor maps exist, Examples 5.1 and 5.2.

If S is a locally compact Markov shift, then T need not be locally compact, Examples 1.6, 1.7, and $h_G(S) < h_G(T)$ is possible, [**21**]. More generally, for every noncompact Markov shift S and every SFT T which satisfy a trivial periodic point condition, there is a factor map $f : S \to T$, [**15**]. Thus, in this case the entropy condition apparent in the compact setting completely vanishes. This is the starting point for us. We investigate the connection between entropies and fiber cardinalities for factor maps $f : S \to T$, where S and T are both locally compact transitive Markov shifts. We show that if f is countable-to-1, then f is entropy increasing, Theorem 2.2. We give examples that the entropy can strictly increase; thus, in contrast to the compact setting, the induced map on the set of invariant Borel probability measures need not be onto, though f is. This makes it impossible to adapt to the locally compact case the measure theoretical proof that if S is compact then a finite-to-1 map preserves entropy. Furthermore, it is not possible to adapt Bowen's proof [**3**] using (n, ε)-separated sets, since the Bowen-entropy of a Markov shift in graph presentation with respect to the standard metric is, in general, different from Gurevic entropy, and it is not even a conjugacy invariant [**9**]. We give a purely topological argument that finite-to-1 factor maps between locally compact Markov shifts are entropy preserving, Theorem 2.4. As a corollary of the proof we obtain that a factor map which is bounded-to-1 on periodic points is entropy preserving, Corollary 2.5. However, the borderline between entropy preserving and not entropy preserving is very fine; we give an example of a countable-to-1 map which is finite-to-1 on periodic points and not entropy preserving, Example 2.6. Thus, properly countable-to-1 factor maps between Markov shifts exist, since finite-to-1 maps preserve entropy by Theorem 2.4.

We then study factor maps between Markov shifts such that preimages of compact sets are compact. These factor maps are called proper, Definition 3.1. A factor map is proper if and only if it extends to a factor map between the 1-point compactifications, Lemma 3.2. A proper factor map is entropy decreasing, Lemma 3.3. Thus, a finite-to-1 proper factor map is entropy preserving. We study these maps in detail and show that they behave much like finite-to-1 factor maps between SFT's, Proposition 3.6, Lemma 4.6 and Lemma 4.7.

1. Factor maps defined on Markov shifts. In this section we study factor maps defined on locally compact Markov shifts which are not necessarily transitive now. If the domain is compact, then the range is compact as well. However, if the domain is merely locally compact, the range need not be locally compact, Examples 1.6 and 1.7. But, if the domain is transitive and the range is a Markov shift, then the range will be locally compact and transitive as well, Corollary 1.5. To see this we use the following characterization of nonlocally compact Markov shifts.

Lemma 1.1 [**15**, Observation 2.3.2]. *If S is a transitive Markov shift* *which is not locally compact, then the space of S is homeomorphic to* $\mathbf{N}^{\mathbf{Z}}$.

Proof. Let S be given in graph presentation S_G, and assume that the space of S is not homeomorphic to $\mathbf{N}^{\mathbf{Z}}$. Then there is a nonempty set C which is compact-open, [**18**]. For $x \in C$ there is thus some $n \geq 0$ such that the cylinder set ${}_{-n}[x_{-n}, \dots, x_n]$ is compact. Now let α be a vertex in G. Since S is transitive there is thus a path $p_1 \dots p_k$ in G which begins at the terminal vertex of x_n and ends at vertex α. Let $B := \{y \in S_G \mid y[-n, n+k] = x[-n, n]p_1 \dots p_k\}$. Let E denote the set of edges in G with initial vertex α. Then $\{y \in B \mid y_{n+k+1} = e\}$, $e \in E$ is an open cover of disjoint sets of B. Since $B \subset C$, B is compact and, thus, E is finite. Similarly the set of edges with terminal vertex α is finite. Thus S_G is locally compact. ◻

Lemma 1.2. *The set $\mathbf{N}^{\mathbf{Z}}$ is not σ-compact* (*i.e., not a countable union of compact sets*).

Proof. Let $A_1, A_2, \dots$ be a sequence of compact sets in $\mathbf{N}^{\mathbf{Z}}$. We construct a point $x \in \mathbf{N}^{\mathbf{Z}}$ which is not contained in the union of the A_i as follows. Since $A_1 \cap [1]_0$ is compact, there is some n_1 such that $x \in \mathbf{N}^{\mathbf{Z}}$, $x_0 = 1$ and $x_1 \geq n_1$ implies $x \notin A_1$. Since $A_2 \cap_0 [1n_1]_1$ is compact, there is some n_2 such that $x \in \mathbf{N}^{\mathbf{Z}}$, $x_0 = 1$, $x_1 = n_1$ and $x_2 \geq n_2$ implies $x \notin A_2$. Inductively, we obtain a sequence n_i such that for the point $x \in \mathbf{N}^{\mathbf{Z}}$ with $x_i = 1$ for all $i \leq 0$, and $x_i = n_i$ for all $i \geq 1$, it holds that $x \notin \cup_i A_i$. ◻

Corollary 1.3. *A transitive Markov shift is locally compact if and only if it is σ-compact.*

Example 1.4. A subshift can be transitive, σ-compact and not locally compact. For $n \geq 1$, let $c(n) := 0^n n$. Let $x \in \{0, 1, 2, \dots\}^{\mathbf{Z}}$. Then $x \in S$ if and only if $x = 0^\infty$ or x can be written as some bi-infinite concatenation $\dots c(n_{-1})c(n_0)c(n_1)\dots$ with $|n_i - n_{i+1}| \leq 1$ for all $i \in \mathbf{Z}$. Then S is transitive. Let $A_0 := \{0^\infty\}$ and, for $n \geq 1$, let $A_n := \{x \in S \mid 1 \leq x_i \leq n$ for some $-n \leq i \leq n\}$. Since $A_n \subset \{x \in \{0, 1, 2, \dots\}^{\mathbf{Z}} \mid x_i \leq |i| + n$ for all $i \in \mathbf{Z}\}$, A_n is compact.

If $x \in S$ and $x_k = n \geq 1$, then $x \in A_{\max(|k|,n)}$. Thus S is σ-compact. The point $0^\infty \in S$ has no compact neighborhood, since a cylinder set ${}_{-N}[0\ldots 0]_N$ contains for $n > 2N$ a point $y^n \in S$ which has $(y^n)_{N+1} = n$, and the sequence y^n has no convergent subsequence in S. $\square$

Corollary 1.5. *Let $f : S \to T$ be a factor map. Let S be a transitive, locally compact Markov shift and T a Markov shift. Then T is also locally compact and transitive.*

Proof. Factors of transitive systems are transitive, and factors of σ-compact systems are σ-compact. Apply Corollary 1.3. $\square$

Example 1.6 shows that the transitivity assumption on S in Corollary 1.5 was essential, while Example 1.7 below shows that the Markov shift assumption on T was essential.

Example 1.6. A locally compact Markov shift S factoring onto a nonlocally compact Markov shift T. The Markov shift S has symbols $\{(a,n),(b,n),n \mid n \in \mathbf{N}\}$ and S is the closure of the union of orbits $((a,n))^\infty n((b,n))^\infty$, while T is a Markov shift having symbols $\{a,b\}\cup\mathbf{N}$ and T is the closure of the union of orbits $a^\infty n b^\infty$. The factor map f is 1-block and drops the n in the symbols (a,n) and (b,n). $\square$

Example 1.7. A transitive locally compact Markov shift factoring onto a nonlocally compact subshift. Just take any coded, nonsynchronized system Y. Then there is a transitive locally compact Markov shift S and a continuous shift commuting map $g : S \to Y$ such that $g(S)$ is dense in Y, [**10**, Theorem 1.7]. Since Y is not synchronized, $g(S)$ is not open in Y, [**10**, Theorem 1.1]. Thus, $T := g(S)$ is a subshift, which is not locally compact [**4**, Theorem 6.5]. With $f : S \to T$, $fx = gx$, $x \in S$, we have the desired example. $\square$

2. Factor maps between Markov shifts: entropy and fiber cardinality. Now we consider factor maps between transitive locally compact Markov shifts. There are two trivial implications. A bounded-to-1 map is finite-to-1 and a finite-to-1 map is countable-to-1. We start with an easy example showing that a finite-to-1 map need not be bounded-to-1.

Example 2.1. A finite-to-1 factor map between Markov shifts which is not bounded-to-1. Let T be the Markov shift given by the graph with vertex set $\mathbf{Z}$, and there is an edge

(1) from vertex n to vertex $n+1$, for each $n \in \mathbf{Z}$,

(2) from $2n+1$ to $-(2n+1)$, for each $n \geq 1$,

(3) from $2n$ to $2n$, for each $n \geq 1$.

Let S be the Markov shift given by the graph with vertex set $\{(n,0) \mid n \in \mathbf{Z}\} \cup \{(2n,k) \in \mathbf{N} \times \mathbf{N} \mid 1 \leq k \leq n\}$, and there is an edge

(1) from vertex $(n,0)$ to vertex $(n+1,0)$, for each $n \in \mathbf{Z}$,

(2) from $(2n+1,0)$ to $(-(2n+1),0)$, for each $n \geq 1$,

(3)(a) from $(2n,k)$ to $(2n,k+1)$, for each $0 \leq k < n$,
(b) from $(2n,n)$ to $(2n,0)$, for each $n \geq 1$,
(c) from $(2n,k)$ to $(2n+1,0)$, for each $1 \leq k \leq n$.

Since in T for any pair of vertices n, m, there is at most one edge with initial vertex n and terminal vertex m, we can define a shift commuting and continuous map $f : S \to T$ by mapping a vertex (n,k) of S to the vertex n of T. Using (1) and (3a)–(3c), one sees that f is surjective.

The S-orbit of length $n+1$ which visits the vertex $(2n,n)$ is mapped to the fixpoint of T which visits the vertex $2n$. Thus, f is not bounded-to-1. As already observed, every fixpoint in T has a finite number of preimages. If $y \in T$ is not a fixpoint, then there is some $n \in \mathbf{Z}$ such that y_n is an edge with initial vertex $m \notin 2\mathbf{N}$. Thus, for $x \in f^{-1}y$ we get x_n has initial vertex $(m,0)$. Now $y[n,\infty)$ determines $x[n,\infty)$. If there is some $n' < n$ such that $y_{n'}$ is an edge with initial vertex $m' \notin 2\mathbf{N}$, then $y[n',n]$ determines $x[n',n]$. If, for all $n' < n$ the edge $y_{n'}$ starts in a vertex from $2\mathbf{N}$, then $m \geq 3$ and $x(-\infty,n]$ has exactly m possibilities. Thus, f is finite-to-1. $\square$

In Example 2.6 we shall also see that a countable-to-1 factor map need not be finite-to-1. Before this, we investigate the interplay between entropy and fiber cardinalities. First, we shall see that maps which are countable-to-1 on periodic points are entropy increasing. For a

Markov shift T, let $\operatorname{Per}(T)$ denote the set of periodic points in T, that is, $\operatorname{Per}(T) = \{y \in T \mid T^n y = y \text{ for some } n \in \mathbf{N}\}$.

Theorem 2.2. *Let S be a transitive locally compact Markov shift and T be a locally compact subshift. Let $f : S \to T$ be a factor map. Let $f^{-1}p$ be countable for each $p \in \operatorname{Per}(T)$. Then $h_G(S) \le h_G(T)$. In particular, if f is countable-to-1, then $h_G(S) \le h_G(T)$.*

Proof. Let $\varepsilon > 0$. Choose a transitive SFT R in S such that $h_{\text{top}}(R) > h_G(S) - \varepsilon$, [**16**], [**19**]. Then $f|_R : R \to f(R)$ is a factor map between compact subshifts which is countable-to-1 on periodic points. Since R is a transitive SFT, $f|_R$ is bounded-to-1, and thus $h_{\text{top}}(f(R)) = h_{\text{top}}(R)$. Since $f(R)$ is closed in T_0, we get $h_G(T) = h_{\text{top}}(T_0) \ge h_{\text{top}}(f(R)) = h_{\text{top}}(R) > h_G(S) - \varepsilon$. $\square$

Countable-to-1 maps which strictly increase entropy do exist, Example 2.6. In [**8**], the idea of Example 2.6 is used and countable-to-1 maps which strictly increase entropy are constructed in a much more general setting. Thus, contrary to the compact case, the induced map on the spaces of invariant Borel probability measures is not onto in general, since every factor map decreases measure entropy. This fact makes it impossible to use measures to prove that finite-to-1 maps are entropy preserving. We shall give here a purely topological argument, for which we prepare with the following lemma. It gives a sufficient condition for a factor map being entropy decreasing. In the proof of the main theorem of this section, Theorem 2.4, we shall verify that a finite-to-1 map satisfies this sufficient condition. However, Example 2.6 below shows how delicate the arguments in Theorem 2.4 have to be, since it presents an example of a countable-to-1 biclosing 1-block factor map which is finite-to-1 on periodic points but not entropy preserving.

Lemma 2.3. *Let S and T be locally compact transitive Markov shifts given in graph presentation and $f : S \to T$ be a factor map. Let $z \in \operatorname{Per}(T)$ which has only finitely many preimages, and such that for every T-block w with $z_- w z_+ \in T$, there is a finite set $E(w)$ of edges in S and a point $x = x^{(w)} \in S$ with $fx = z_- w z_+$ and $x_i \in E(w)$, for all $i \in \mathbf{Z}$. Then $h_G(S) \ge h_G(T)$.*

Proof. Let M be the set of points in S which are mapped to a point in the orbit of z. Since $f^{-1}z$ is finite, the set M is a finite set of periodic points. Thus, there is a finite set of edges, say E, in S such that a point in M has all edges in E. Let $p \geq 1$ be so large that $y, y' \in M$, and $y[0,p] = y'[0,p]$ implies $y = y'$. Let F be a finite set of edges in S containing E. Then there is some $N = N(F)$ such that whenever $x \in S$ with $fx[-N,N] = z[-N+i, i+N]$ for some $i \in \mathbf{Z}$ and $x_0 \in F$, then $x[-p,p] = y[-p,p]$ for a point $y \in M$. (Otherwise, by compactness of $\{x \mid x_0 \in F\}$, there would be a point $x \in M$ with $x_0 \in F$ and $x[-p,p] \neq y[-p,p]$ for all $y \in M$, a contradiction.) Thus, if $x \in S$ with $x_0 \in F$ and $fx(-\infty, N] = S^i z(-\infty, N]$ for some i, then for each $n \leq 0$, there is $y^n \in M$ with $x[-p+n, n+p] = y^n[-p+n, n+p]$. Then $y^n[0,p] = y^{n+1}[-1, p-1]$ for all $n < 0$, and thus, by the choice of p, $y^n = y^{n+1}$ for all $n < 0$. Thus, $x(-\infty, 0] = y(-\infty, 0]$ for some $y \in M$. Similarly, if $x \in S$ with $x_0 \in F$ and $fx[-N, \infty) = S^i z[-N, \infty)$ for some i, then $x[0,\infty) = y[0,\infty)$ for some $y \in M$. Now fix n and consider the set of T-blocks w of length n such that $u^{(w)} \in T$ where $u^{(w)}(-\infty, 0] = z_-$ and $u^{(w)}[1, \infty) = wz_+$. By assumption there is a finite set of edges $E(w)$ in S and a preimage $x^{(w)}$ of $u^{(w)}$ having all edges in $E(w)$. We may assume that $E(w)$ contains E. With $F = E(w)$ we thus obtain $x^{(w)}(-\infty, -N(E(w))]$ is a ray in M and $x^{(w)}[n+1+N(E(w)), \infty)$ is a ray in M. In particular, $x^{(w)}_{-N(E(w))}$ and $x^{(w)}_{n+1+N(E(w))}$ are edges in E. Thus we get that, by applying the above with $F = E$, $x^{(w)}(-\infty, -N(E)]$ and $x^{(w)}[n+1+N(E), \infty)$ are rays in M. Thus, if $w \neq w'$, then $x^{(w)}[-N(E)-p, n+1+N(E)+p] \neq x^{(w')}[-N(E)-p, n+1+N(E)+p]$. Thus, $\#\{S$-blocks of length $n + 2(N(E)+p) + 2$ which start and end with a symbol in $E\} \geq \#\{T$-blocks w of length n such that $z_- w z_+ \in T\}$. This holds for all n. Thus, $h_G(S) \geq h_G(T)$. ◻

The following theorem shows that finite-to-1 maps between Markov shifts preserve the Gurevic entropy. It is an open problem whether this is true or not in the larger class of locally compact transitive subshifts.

Theorem 2.4. *Let S and T be locally compact transitive Markov shifts and $f : S \to T$ a finite-to-1 factor map. Then $h_G(S) = h_G(T)$.*

Proof. Since f is countable-to-1, $h_G(S) \leq h_G(T)$ by Theorem 2.2.

To prove the other inequality, we may assume that S and T are in graph presentation, and we shall show that the assumptions of Lemma 2.3 are satisfied. Let $Z \subset T$ be a nonempty compact open set, and let $k \in \mathbf{N}$. We say "Z forces k preimages" if there are compact open sets $U_1, U_2, \dots, U_k$ such that $U_i \cap U_j = \varnothing$ for $i \neq j$ and $f^{-1}t \cap U_i \neq \varnothing$ for all $t \in Z$ and all $1 \leq i \leq k$.

Note that if Z forces k preimages, then for $t \in Z$ we have $k \leq \# f^{-1}t$. We may define a map $H : \{Z \subset T \mid Z \neq \varnothing \text{ and } Z \text{ compact open}\} \to \mathbf{N} \cup \{0\}$ where $H(Z) := 0$, if there is no $k \in \mathbf{N}$ such that Z forces k preimages and $H(Z) := \sup\{k \in \mathbf{N} \mid Z \text{ forces } k \text{ preimages}\}$ if there is such a k.

Since f is onto, the sets $f[e]_0$, e an edge in S, cover all of T. Thus there is an edge such that $f[e]_0$ is not nowhere dense. Let Z be a cylinder in T such that $f[e]_0 \cap Z$ is dense in Z. Since $[e]_0$ is compact and f is continuous, $f[e]_0$ is closed in T and $f[e]_0 \cap Z = Z$. So every point in Z has a preimage in the set $[e]_0$. Thus $H(Z) \geq 1$, which shows that $H \neq 0$.

We show now that H is bounded. Assume not. Then, for each $k \in \mathbf{N}$ there is a compact open $Z^k \subset T$ such that $H(Z^k) \geq k$. By transitivity of T there is a sequence $n(k) \to \infty$ and a point $t \in T$ such that $T^{n(k)}t \in Z^k$. Since f is finite-to-1, $\# f^{-1}t < \infty$. Thus there is a k with $\# f^{-1}t < k$. But $\# f^{-1}t = \# f^{-1}(T^{n(k)}t) \geq k$, a contradiction.

Let $d := \sup H$, and fix a compact open set $Z \subset T$ with $H(Z) = d$. Let $U_1, \dots, U_d \subset S$ be disjoint compact open sets such that for every $t \in Z$ and every $1 \leq i \leq d$ it holds that $f^{-1}t \cap U_i \neq \varnothing$. Let $U := \cup_{1 \leq i \leq d} U_i$. Now fix $z \in \operatorname{Per}(T) \cap Z$.

Let M be the set of points in S which are mapped to a point in the orbit of z. Since $f^{-1}z$ is finite and z is periodic, there is a finite set of edges, say E, in S such that a point in M has all edges in E. Let F be a finite set of edges in S containing E such that $x \in U$ implies $x_0 \in F$. Then there is some N such that whenever $x \in S$ with $fx[-N, N] = z[-N+i, i+N]$ for some $i \in \mathbf{Z}$ and $x_0 \in F$, then $x[-1,1] = y[-1,1]$ for a point $y \in M$. In particular, $x_{-1}, x_1 \in F$. Thus, if $x \in U$ and $fx(-\infty, N] = S^i z(-\infty, N]$ for some i, then $x_n \in F$ for all $n \leq 0$ and if $x \in U$ with $fx[-N, \infty) = S^i z[-N, \infty)$ for some i, then $x_n \in F$ for all $n \geq 0$.

Let w be a T-block such that $z_- w z_+ \in T$. Let $u \in T$ with $u(-\infty, 0] = z(-\infty, 0]$ and $u[1, \infty) = wz[1, \infty)$. Since $z \in \mathrm{Per}\,(T) \cap Z$, we can fix $n < -N$ and $m > |w| + N$ such that $S^n u, S^m u \in Z$. Now assume that there is no $x \in f^{-1}u$ with $S^n x \in U$ and $S^m x \in U$. Then, since $u \in T^{-n}Z \cap T^{-m}Z$ and f is continuous, there is a compact open set $Z' \subset T^{-n}Z \cap T^{-m}Z$ such that $u \in Z'$; and for every $u' \in Z'$ it holds if $x' \in f^{-1}u'$ with $S^n x' \in U$, then $S^m x' \notin U$. Thus, by the choice of the sets U_i, every point $u' \in Z'$ has preimages in each of the sets $S^{-n}U_i \cap S^{-m}U^c$ and $S^{-n}U^c \cap S^{-m}U_i$, $1 \leq i \leq d$. These are $2d$ disjoint compact open sets. Thus $H(Z') \geq 2d$, a contradiction. Thus we have shown that there is an $x \in f^{-1}u$ with $S^n x \in U$ and $S^m x \in U$. Thus, since $n < -N$ and $m > |w| + N$, we get $x_k \in F$ for all $k \leq n$ and all $k \geq m$. Thus $\#\{x_k \mid k \in \mathbf{Z}\} < \infty$, and thus z satisfies the conditions of Lemma 2.3. $\square$

Let S, T be transitive locally compact Markov shifts and $f : S \to T$ be a factor map. If f is bounded-to-1 on periodic points, then $h_G(S) \leq h_G(T)$, Theorem 2.2. The above proof shows, if f is bounded-to-1 on periodic points, then $h_G(S) \geq h_G(T)$. Thus,

Corollary 2.5. *If $f : S \to T$ is bounded-to-1 on periodic points, then $h_G(S) = h_G(T)$.*

Thus, a map which is bounded-to-1 on the periodic points is entropy preserving. The next example shows that a map which is merely finite-to-1 on periodic points need not be entropy preserving. Since finite-to-1 maps preserve entropy, Theorem 2.4, this example is also an example for a countable-to-1 nonfinite-to-1 factor map between Markov shifts. Furthermore, the factor map in the example is also biclosing, which means that if $x, y \in S$ with $x(-\infty, 0] = y(-\infty, 0]$ and $fx = fy$, then $x = y$, and if $x, y \in s$ with $x[0, \infty) = y[0, \infty)$ and $fx = fy$, then also $x = y$. Thus, the example also shows that in the noncompact setting biclosing factor maps need not be constant-to-1, which is true for factor maps between SFT's.

Example 2.6. A countable-to-1 factor map which is finite-to-1 on periodic points and does not preserve entropy.

We shall define a label map from a Markov shift S with $h_G(S) < \log 2$ onto the 2-shift $T = \{0,1\}^{\mathbf{Z}}$. We define S together with the map $f : S \to T$ inductively. The Markov shift S has a special vertex which we call α. In the nth step, we shall obtain a positive number $L(n)$ such that $2 + L(n)$ is a multiple of $(n+1)!$, a subset V_n of T-blocks of a certain fixed length, a finite labeled graph with special vertices $(+, u)$ and $(-, u)$, $u \in V_n$ such that the shortest paths from α to a vertex $(+, u)$ and the shortest paths from $(-, u)$ to α, $u \in V_n$, have a length being a multiple of $2 + L(n-1)$. The graph is chosen such that it defines an irreducible SFT $R(n)$ with $h_{\text{top}}(R(n)) < 1/2 \cdot \log 2$.

Let $L(0) = 1$. For $n = 1$ the construction is as follows. For $a \in \{0, 1\}$ there is an edge starting in α ending in vertex $(+, a)$ labeled a, and there is an edge starting in vertex $(-, a)$ ending in α labeled a. Let $V_1 = \{a \mid a \in \{0, 1\}\}$, that means $V_1 = \{0, 1\}$. Now fix $L(1) \geq 1$. For $a \in \{0, 1\}$, add a path of length $L(1)$ from vertex $(+, a)$ to vertex $(-, a)$ labeled $a^{L(1)}$. This finishes the first step of construction. We have obtained an irreducible SFT which we call $R(1)$. By enlarging $L(1)$ we may assume that $h_{\text{top}}(R(1)) < 1/2 \cdot \log 2$ and that $2 + L(1)$ is a multiple of $2!$.

Given $R(n-1)$ with its labeling, V_{n-1} and $L(n-1)$ we do the construction for n as follows. Let $u \in V_{n-1}$. For each T-block $w \neq u^{1+L^{(n-1)}}$ of length $|w| = |u| \cdot (1 + L(n-1))$ we create a path of length $|w|$ from vertex $(+, u)$ to vertex $(+, uw)$ labeled w and a path from vertex $(-, wu)$ to vertex $(-, u)$ of length $|w|$ labeled w. Let $V_n = \{v \text{ is a } T\text{-block } \mid |v| = |u| \cdot (2 + L(n-1)) \text{ and } v \neq u^{2+L(n-1)} \text{ for all } u \in V_{n-1}\}$, so the set of new special vertices we created in the nth step is $(+, u)$ and $(-, u)$, $u \in V_n$. Then a shortest path from α to a vertex $(+, u)$, $u \in V_n$, has a length which is a multiple of $2 + L(n-1)$ and thus a multiple of $n!$, by choice of $L(n-1)$. Then we fix $L(n)$ large and draw for each $u \in V_n$ a path of length $|u| \cdot L(n)$ from vertex $(+, u)$ to vertex $(-, u)$ labeled $u^{L(n)}$. We obtain an SFT $R(n)$. For $L(n)$ large enough $h_{\text{top}}(R(n)) < 1/2 \cdot \log 2$. By enlarging further we may assume that $2 + L(n)$ is a multiple of $(n+1)!$.

This finishes the construction of S and a continuous shift commuting map $f : S \to T$ given by $(fx)_i :=$ label of the edge x_i. Since each of the SFT's $R(n)$ has $h_{\text{top}}(R(n)) < 1/2 \cdot \log 2$, we get that $h_G(S) \leq 1/2 \cdot \log 2 < \log 2$. We show now that f is onto, countable-to-1 and that every periodic point in T has only finitely many preimages in

S. Let $y \in T$. We show that there is a unique path x_+ starting at α which is labeled $y[0,\infty)$. Let $m_1 = 1$ and, for $n \geq 2$, let m_n be the length of the shortest path from a vertex $(+, u)$ to a vertex $(+, uw)$, where $u \in V_{n-1}$ and $uw \in V_n$.

There is a unique edge e in S with initial vertex α and terminal vertex $(+, y_0)$. Let $x_0 = e$. Now consider the block $y[1, m_2]$. By construction of S, there is a unique path p starting from $(+, y_0)$ of length m_2 and having label $y[1, m_2]$. Let $x[1, m_2] = p$. If $y[1, m_2] = (y_0)^{1+L(1)}$, then $x[1, m_2]$ ends in α and we restart the argument. Otherwise, $x[1, m_2]$ ends in vertex $(+, y[0, m_2])$ and $y[0, m_2] \in V_2$. Then we consider the block $y[m_2+1, m_2+m_3]$. Again, by construction of S there is a unique path of length m_3 starting from vertex $(+, y[0, m_2])$ and having label $y[m_2+1, m_2+m_3]$. So we can define $x[m_2+1, m_2+m_3]$ to be this unique path. Repeating this argument over and over finally gives a unique path x_+ starting from α which is labeled $y[0,\infty)$. Since the incoming paths into α are constructed symmetrically, the analogue argument shows that there is a unique path x_- ending at α and having label $y(-\infty, -1]$. Thus, f is onto. Since every point in S visits the special vertex α at least once, this also shows that f is countable-to-1 and actually shows that f is biclosing.

We shall now show that a periodic point in T has only finitely many preimages in S. So let $y \in T$ be periodic with least period, say n. Now we shall show that for each $1 \leq i \leq n$, there is some $m(i) \geq 1$ such that there is a loop at α labeled $(y[i, i+n-1])^{m(i)}$. Since every preimage of y visits α at least once and there is at most one preimage of y being in α at time i, this will show that y has only finitely many periodic preimages.

Let N be the length of a shortest path from α to a vertex $(+, u)$, $u \in V_n$. Then N is a multiple of $n!$ and thus there is an $m \geq 1$ such that $n \cdot m = N$. Let $x \in S$ be a preimage of y such that x_i starts in vertex α. Let $w = y[i, i+n-1]$. If $w^m \in V^n$, then $x[i, i+N-1]$ ends at vertex $(+, w^m)$ and there is a path from vertex $(+, w^m)$ to $(-, w^m)$ of length $N \cdot L(n-1)$ labeled $w^{mL(n-1)}$, and a path from $(-, w^m)$ to α labeled w^m. Thus, in this case $x[i, i+nm(2+L(n-1))-1]$ is a loop at α labeled $w^{m(2+L(n-1))}$. If $w^m \notin V_n$, then $x[i, i+N-1]$ visits α at least once. Thus, in either case we have shown that there is some $L \leq nm(2+L(n-1))$ such that $x[i, i+L-1]$ is a first return loop at α. Thus, there are indices $i = i_1 < i_2 < \cdots$ with

$i_k - i_{k-1} \leq nm(2 + L(n-1))$ such that $x[i_{k-1}, i_k - 1]$ is a first return loop at α for all $k \geq 2$. Thus, there are some $k < k'$ such that $y[i_k, \infty) = y[i_{k'}, \infty)$, since the set $\{y[j, \infty) \mid j \in \mathbf{Z}\}$ is finite. Since y has least period n it thus follows that $i_{k'} - i_k$ is a multiple of n. Since the map is biclosing as shown above, it follows thus that $x = (x[i_k, i_{k'}])^\infty$. Thus, $x[i, i + i_{k'} - i_k - 1]$ is a loop at α which has label $y[i, i+n-1]^p$ for some $p \geq 1$. $\square$

3. Finite-to-1 proper factor maps. We have seen that the behavior of factor maps between general Markov shifts is quite "unusual"; they can increase entropy, even if they are countable-to-1 and finite-to-1 on periodic points! So one might be interested in finding a natural property of a factor map, which ensures that the map is entropy decreasing. (Another point of view is that one can try to find subclasses of Markov shifts on which factor maps are entropy decreasing; one possible such class is presented in [**7**, Theorem 3.18]). Since the entropy we are considering is the topological entropy of the 1-point compactification, it is quite natural to consider factor maps which extend to a continuous factor map between the 1-point compactifications. This is possible if and only if the factor map is uniformly continuous with respect to the Gurevic-metric (on both Markov shifts). To avoid technical complications we restrict ourselves here to factor maps $f : S \to T$, where both S and T are noncompact. It is easy to extend the results presented below to the general case; we leave this to the reader.

Standing assumption for the rest of this section. All Markov shifts considered are noncompact.

By the 1-point compactification S_0 of a Markov shift S we mean the dynamical system which consists of the Alexandroff 1-point compactification of the shift space together with the extended shift map, see [**11**]. The extended shift map is a homeomorphism which fixes the unique point in S_0 which does not lie in S. We denote this point by ∞. Since the Markov shift S is locally compact, S_0 is compact metric. A metric d_S on S such that the completion of S with respect to this metric is S_0, is called a Gurevic metric. The metric is unique up to uniform equivalence. If S is given in graph presentation with edge set $E = \mathbf{N}$, then an explicit formula for the metric is given by, see [**11**, p. 627],

$$d_S(x,y) = \sum_{n\in\mathbf{Z}} \frac{1}{2^{|n|}} \cdot \left|\frac{1}{x_n} - \frac{1}{y_n}\right|, \quad x,y \in S.$$

Definition 3.1 [**2**]. Let S,T be locally compact transitive Markov shifts. A factor map $f : S \to T$ is proper if $f^{-1}K$ is compact for every compact set $K \subset T$.

Thus, if S and T are locally compact Markov shifts given in graph presentation, then a factor map $f : S \to T$ is proper if and only if for every edge e in T, the number of edges e' of S, such that there is a point $x \in S$ with $x_0 = e'$ and $(fx)_0 = e$, is finite.

The next lemma characterizes proper maps in terms of the Gurevic metric.

Lemma 3.2 [**2**]. *Let S,T be locally compact transitive Markov shifts. Let d_S, respectively d_T, denote the Gurevic metric on S, respectively T. Let $f : S \to T$ be a factor map. Then f is proper if and only if f is a uniformly continuous map $(S,d_S) \to (T,d_T)$.*

Proof. Let $f : (S,d_S) \to (T,d_T)$ be uniformly continuous. It suffices to show that $f^{-1}K \subset S$ is compact for any $K \subset T$ compact open. Since f is continuous, $f^{-1}K$ is closed and open. Let $\varepsilon := d(K,K^c)$. Since K is closed and open, $\varepsilon > 0$. Since f is uniformly continuous, there is thus some $\delta > 0$ such that $d_S(x,y) < \delta$ implies $d_T(fx,fy) < \varepsilon$. By the definition of d_S there is some compact open set $E \subset S$ such that $x,y \in S$, $x \notin E$ and $y \notin E$ implies $d_S(x,y) < \delta$. Thus, either $fE^c \subset K$ or fE^c is disjoint from K. But $fE^c \subset K$ implies $fS \subset fE \cup K$ is compact, since f is continuous and E is compact, and thus, since T is by assumption not compact, f would not be onto, a contradiction. Thus $f^{-1}K \subset E$, and is thus compact.

Now assume that f is proper. Let $\varepsilon > 0$, and fix a compact open set $K \subset T$ such that $x,y \in T - K$ implies $d_T(x,y) < \varepsilon$. By assumption $f^{-1}K \subset S$ is compact open. Since $f^{-1}K$ is compact, there is $\delta > 0$ so that $x,y \in f^{-1}K$ and $d_S(x,y) < \delta$ implies $d_T(fx,fy) < \varepsilon$. Since $f^{-1}K$ is closed and open, if $\delta > 0$ is small enough then a ball with radius δ is either contained in $f^{-1}K$ or in $f^{-1}K^c$. Thus, for all $x, y \in S$, it holds that $d_S(x,y) < \delta$ implies $d_T(fx, fy) < \varepsilon$. $\square$

Thus, a factor map is proper if and only if it extends to a factor map $f_0 : S_0 \to T_0$ between the 1-point compactifications by mapping ∞ to ∞. Every conjugacy is a proper map, and thus the 1-point compactification is conjugacy invariant of the Markov shift.

Lemma 3.3. *Let S, T be locally compact subshifts and $f : S \to T$ a proper factor map. Then $h_G(S) \geq h_G(T)$. If f is countable-to-1 proper, then $h_G(S) = h_G(T)$.*

Proof. Since f is proper it extends to a factor map $S_0 \to T_0$. These are compact metric dynamical systems, thus $h_{\text{top}}(S_0) \geq h_{\text{top}}(T_0)$. Since measures lift under factor maps on compact metric spaces and countable-to-1 maps preserve measure entropy, the variational principle [**23**] implies that $h_G(T_0) = h_G(S_0)$ if f is countable-to-1. Since $h_G(S) = h_{\text{top}}(S_0)$, the lemma is proved. $\square$

We shall now see that finite-to-1 proper maps between Markov shifts behave much like finite-to-1 factor maps between SFT's. Indeed, merely proper maps are more like factor maps between SFT's. We mention, without proof, that for example, whenever $f : S \to T$ is a proper factor map, then $\# f^{-1}(p) < \infty$ for all $p \in \text{Per}\,(T)$ if and only if $f^{-1}(p)$ countable for all $p \in \text{Per}\,(T)$ and in this case there is a doubly transitive point $y \in T$ with $\# f^{-1}(y) = \min\{\# f^{-1}(p) \mid p \in \text{Per}\,(T)\}$.

Definition 3.4 [**15**, Definition 4.1.1]. A mixing locally compact Markov shift T with $h_G(T) < \infty$ is strongly positive recurrent if $h_G(S) < h_G(T)$ for every closed proper subsystem $S \subset T$, (proper means $S \neq \varnothing$ and $S \neq T$).

Lemma 3.5 [**15**, Lemma 4.1.2]. *Let T be a mixing locally compact Markov shift with $h_G(T) < \infty$. Then T is strongly positive recurrent if and only if $h_G(S) < h_G(T)$ for every proper closed mixing locally compact Markov shift $S \subset T$.*

Proposition 3.6. *Let S, T be mixing locally compact Markov shifts with finite Gurevic entropy. Let $f : S \to T$ be a countable-to-1 proper factor map. Then $h_G(S) = h_G(T)$ and*

(a) *The shift S is positive recurrent if and only if T is positive recurrent.*

(b) *The shift S is strongly positive recurrent if and only if T is strongly positive recurrent.*

Proof. Since f is countable-to-1, $h_G(S) \leq h_G(T)$, Theorem 2.2, and since f is proper, $h_G(S) \geq h_G(T)$, Lemma 3.3, and thus $h_G(S) = h_G(T)$.

A mixing Markov shift is positive recurrent if and only if it has a measure of maximal entropy, [**19**, Proposition 7.2.13]. A countable-to-1 proper factor map extends to a countable-to-1 factor map on the one-point-compactifications. Since measures lift under factor maps on compact metric spaces and countable-to-1 maps preserve measure entropy, statement (a) follows.

Let S be strongly positive recurrent. Let Y be a proper closed subsystem in T. If Y is compact, then $h_G(Y) < h_G(T)$. Now assume that Y is not compact. Then $f^{-1}(Y)$ is a proper closed subshift of S, and thus $h_G(f^{-1}(Y)) < h_G(S)$. Then f restricts to a proper map from $f^{-1}(Y) \to Y$. Thus, by Lemma 3.3, $h_G(Y) \leq h_G(f^{-1}(Y)) < h_G(S) = h_G(T)$ and T is strongly positive recurrent.

Now let T be strongly positive recurrent. Let N be a proper closed subshift in S. We show that $f(N)$ is a closed subshift in T. For that, let $x^n \in N$ and $y \in T$ be such that $f(x^n) \to y$. We may assume that all $f(x^n)$ and y are contained in some compact open set $K \subset T$. Since f is proper, $f^{-1}K \subset S$ is compact open and contains all x^n. Thus, a subsequence is convergent to some $x \in f^{-1}K$. Since all $x^n \in N$ and N is closed, thus $x \in N$ and, by continuity of f, $fx = y$. Thus, $f(N)$ is a closed subshift in T and thus $f(N)$ is a locally compact subshift of T. Since $f|N : N \to f(N)$ is countable-to-1 and proper, $h_G(f(N)) = h_G(N)$, Lemma 3.3. We show that $f(N)$ is a proper subshift of T, and then, since T is strongly positive recurrent, it follows that $h_G(N) < h_G(S)$. Assume that $f(N) = T$. Then consider $(f|_N)_0 : N_0 \to T_0$. Since T is strongly positive recurrent, T is positive recurrent, [**15**, Corollary 4.1.15]. Thus, the extended map

$(f|_N)_0 : N_0 \to T_0$ lifts the maximal measure of T to a measure μ on N which has entropy $h_G(T)$. But since $h_G(T) = h_G(S)$ and S has a unique maximal measure, we get that μ is the maximal measure on S and μ has full support, [**17**]. Thus S is the support of $\mu \subset N$, a contradiction. This proves (b). $\square$

The next two examples show that the properties finite-to-1 and proper are independent.

Example 3.7. A finite-to-1 proper factor map between Markov shifts preserves entropy, by Theorem 2.4. However, a proper entropy preserving factor map between Markov shifts need not be countable-to-1.

Let S be the Markov shift given by the graph with vertex set $\mathbf{Z} \times \{0,1\} - \{(0,1)\}$ and there

(1) is one edge from vertex $(0,0)$ to vertex $(1,i)$, for each $i \in \{0,1\}$,

(2) is one edge from vertex $(2^m, i)$ to vertex $(2^m + 1, i)$, for each $m \geq 1$, $i \in \{0,1\}$,

(3) is one edge from $(2^m, i)$ to $(-2^m, i)$, for each $m \geq 1$, $i \in \{0,1\}$,

(4) are two edges from $(n,0)$ to $(n+1,0)$, for each $n \in \mathbf{Z} - \{-1,0\} - \{2^m \mid m \geq 1\}$,

(5) are two edges from $(-1,0)$ to $(0,0)$,

(6) are two edges from $(n,1)$ to $(n+1,1)$, for each $n \in \mathbf{Z} - \{-1,0\} - \{2^m \mid m \geq 1\}$,

(7) are two edges from $(-1,1)$ to $(0,0)$.

Let T be the Markov shift given by the graph with vertex set $\mathbf{Z} \times \{0,1\} - \{(0,1)\}$ and the edges are according to the same rules (1)–(5) and, additionally,

(6a) is *one* edge from $(n,1)$ to $(n+1,1)$, for each $n \in \mathbf{Z} - \{-1,0\} - \{2^m \mid m \geq 1\}$,

(7a) is *one* edge from $(-1,1)$ to $(0,0)$.

Note that S and T are transitive. For vertices α and β let $E_S(\alpha,\beta)$ denote the set of edges in S from vertex α to vertex β. Similarly, $E_T(\alpha,\beta)$. Note that always $2 \geq \#E_S(\alpha,\beta) \geq \#E_T(\alpha,\beta)$ and that

$\#E_S(\alpha,\beta) = \#E_T(\alpha,\beta)$ for each pair of vertices $\alpha,\beta \in \mathbf{Z}\times\{0\}$. Thus we can choose a surjective map $\pi : E_S \to E_T$ such that $\pi(e) \in E_T(\alpha,\beta)$ if and only if $e \in E_S(\alpha,\beta)$. Then $f : S \to T$ where $(fx)_i := \pi x_i$, $i \in \mathbf{Z}$, defines a factor map from S onto T. The point $t \in T$ where t_n begins in vertex $(n,1)$, $n \neq 0$, and t_0 begins in vertex $(0,0)$ has uncountably many preimages, by (4). Thus, f is uncountable-to-1. The map f is proper, since $\#\pi^{-1}(e) \leq 2$ for all $e \in E_T$. We shall now see that $h_G(S) = h_G(T) = \log 2$. The graph of S has outdegree 2; thus, $h_G(S) \leq \log 2$. By rules (1)–(5) there are at least $2^{2^m}\cdot(2^{2^m-m})$ paths in T of length $2\cdot 2^m+1$ through the vertices $(0,0),(1,0),\dots,(2^m,0)$, $(-2^m,0),\dots,(-1,0),(0,0)$. Thus, $h_G(T) \geq \lim_m 1/(2^{m+1}+1)\cdot\log 2^{2^m}\cdot(2^{2^m-m}) = \log 2$. Since $T\subset S$, we thus have $\log 2 \leq h_G(T) \leq h_G(S) \leq \log 2$, and thus the factor map f is entropy preserving. $\square$

Example 3.8. A bounded-to-1 factor map need not be proper. The vertex set for the graph of S is $\{n, n' \mid n \in \mathbf{Z}\}$, and there is an edge

(1) e_n from vertex n to vertex $n+1$, for each $n \in \mathbf{Z}$,

(2) b_n from vertex n to vertex $-n$, for each $n \geq 1$,

(3) e'_n from vertex n' to vertex $(n+1)'$, for each $n \in \mathbf{Z} - \{-1,0\}$,

(4) e'_0 from vertex 0 to vertex $1'$,

(5) e'_{-1} from vertex $(-1)'$ to vertex 0,

(6) b'_n from vertex n' to vertex $(-n)'$, for each $n \geq 1$,

(7) a'_n from vertex $(n+1)'$ to vertex $(-n)'$, for each $n \geq 1$,

(8) c from vertex 0 to vertex $0'$,

(9) a from vertex $0'$ to vertex $0'$,

(10) d from vertex $0'$ to vertex 0.

Let T be given by the graph with vertex set $\mathbf{Z}\cup\{1',2'\}$, and there is an edge

(1) e_n from vertex n to vertex $n+1$, for each $n \in \mathbf{Z}$,

(2) b_n from vertex n to vertex $-n$, for each $n \geq 1$,

(3) e'_0 from vertex 0 to vertex $1'$,

(4) e'_1 from vertex $1'$ to vertex $2'$,

(5) c from vertex 0 to vertex $2'$,

(6) a from vertex $2'$ to vertex $2'$,

(7) d from vertex $2'$ to vertex 0.

(The subgraphs of S, respectively T, which use only vertices from $\mathbf{Z}$ are there only for the reason to make T noncompact). Define a shift commuting map $f : S \to T$ by defining $f(x)_0$ for $x \in S$ as follows.

(1) If $x_0 = b'_1$, then let $f(x) = e'_1$.

(2) If $x_0 = e'_{-1}$, then let $f(x)_0 = d$.

(3) If $x_0 \in \{e'_n \mid n \in \mathbf{Z} - \{-1, 0, 1\}\} \cup \{a'_n, b'_n \mid n \geq 2\}$, then let $f(x)_0 = a$.

(4) In all other cases, let $f(x)_0 = x_0$.

Then f is a factor map. The map f is not proper since $f^{-1}\{y \in T \mid y_0 = a\} = \{x \in S \mid x_0 \in \{e'_n \mid n \in \mathbf{Z} - \{-1, 0, 1\}\} \cup \{a'_n, b'_n \mid n \geq 2\} \cup \{a\}\}$ and thus is not compact. The map f is bounded-to-1, since every point $y \in T$ for which $y(-\infty, n] \neq a^\infty d$ for all $n \in \mathbf{Z}$ has a unique preimage. A point $y \in T$ such that for some $n \in \mathbf{Z}$, $y(-\infty, n] = a^\infty d$ has two preimages. □

4. Expansiveness of the 1-point compactification. So far we have seen that a finite-to-1 proper factor map between Markov shifts preserves several properties of the 1-point compactification. An important property is expansiveness. The 1-point compactification S_0 of a Markov shift S can be expansive, as for the shifts in Example 3.8, or can be nonexpansive, as for the shifts in Example 2.1. We show that the expansiveness of the 1-point compactification lifts under finite-to-1 proper factor maps. For that we give a characterization of expansiveness of S_0 in terms of a graph presentation of S.

Lemma 4.1. *Let S be a noncompact locally compact transitive Markov shift given in graph presentation. Then S_0 is expansive if and only if there is a finite set of edges, say E, such that*

(1) *any point $x \in S$ sees an edge from E,*

(2) *for any pair of edges, say e, f, and any n, there is at most one path p of length n which has all edges outside of E and such that epf is a path in S,*

(3) *for every edge e there is at most one ray $x[0,\infty)$ with $x_0 = e$ and $x_n \notin E$ for all $n \geq 1$, and there is at most one ray $x(-\infty, 0]$ with $x_0 = e$ and $x_n \notin E$ for all $n \leq -1$.*

Proof. Let d denote the metric on S_0 such that d restricted to $S \times S$ is the Gurevic metric d_S, as in the formula just before Definition 3.1.

First, let S_0 be expansive, and let $\varepsilon > 0$ be such that for all $x, y \in S_0$ with $x \neq y$ there is some $n \in \mathbf{Z}$ such that $d((S_0)^n x, (S_0)^n y) > \varepsilon$. Then let F be a finite set of edges of S such that $x \in S$ and $x_0 \notin F$ implies $d(x, \infty) < \varepsilon/2$. Since $S_0(\infty) = \infty$, (1) holds for the set F.

(a) For $x, y \in S$, $x_0, y_0 \notin F$ implies $d(x, y) < \varepsilon$.

Now let N be so large that

(b) for $x, y \in S$ with $x[-N, N] = y[-N, N]$, it holds that $d(x, y) < \varepsilon$. Then fix a finite set of edges E such that $F \subset E$ and

(c) for $x \in S$, $x_0 \notin E$ and $x_n \in F$ implies $|n| > N$.

Let $x, y \in S$ and $n \geq 0$ with $x_i = y_i$ for all $i \notin \{0, \ldots, n\}$ and $x_i, y_i \notin E$ for all $0 \leq i \leq n$. Then (c) implies that $x_i, y_i \notin F$ for $-N \leq i \leq -1$ and for $n+1 \leq i \leq n+N$. Since $F \subset E$, $x_i, y_i \notin F$ for $0 \leq i \leq n$. Thus (a) implies $d(S^i x, S^i y) < \varepsilon$ for $-N \leq i \leq n + N$. Since $x_i = y_i$ for all $i \leq -1$ and for all $i \geq n + 1$, (b) implies $d(S^i x, S^i y) < \varepsilon$ for $i \leq N - 1$ and for $i \geq N + n + 1$. Thus, $d(S^i x, S^i y) < \varepsilon$ for all i, and thus $x = y$ and (2) holds for the set E. In the same way, (3) is shown to hold for the set E. Thus, (1)–(3) holds for the set E.

Now let E be a finite set of edges such that (1)–(3) holds. We show that S_0 is expansive. Let $\varepsilon > 0$ such that for $x \in S$ with $x_0 \in E$ it holds that $d(x, y) > \varepsilon$ for $y \in S$ with $x_0 \neq y_0$ and also for $y = \infty$. Now let $x, y \in S_0$. If $y = \infty$ and $x \in S$, then by (1) there is some n such that $x_n \in E$ and thus $d((S_0)^n x, (S_0)^n y) > \varepsilon$. Let $x, y \in S$, $x \neq y$. By (1) there are $n \in \mathbf{Z}$ such that $x_n \in E$ or $y_n \in E$. If for all those n we would have $x_n = y_n$, then (2) and (3) imply $x = y$, a contradiction. Thus, there is some n with $x_n \neq y_n$ and $x_n \in E$ or $y_n \in E$, and then by the choice of ε, $d(S^n x, S^n y) > \varepsilon$. Thus, ε is an expansiveness constant for S_0. ◻

The 1-point compactification S_0 is *finitely expansive* if there is an $\varepsilon > 0$ such that for all $x \in S_0$ there is a finite subset $M(x) \subset S_0$

such that $d((S_0)^n y, (S_0)^n x) < \varepsilon$ for all $n \in \mathbf{Z}$ implies $y \in M(x)$. Thus, if S_0 is expansive, then S_0 is finitely expansive. In general the notion of finite expansiveness is more general than expansiveness. The following lemma, however, shows that for the 1-point compactification of a Markov shift, the converse implication is true. This will be used in the proof of Lemma 4.4.

Lemma 4.2. *Let S be a locally compact transitive Markov shift such that S_0 is finitely expansive. Then S_0 is expansive.*

Proof. We shall show that there is a finite set E which satisfies conditions (1)–(3) from Lemma 4.1. Let d denote the metric on S_0 induced by the Gurevic metric. Since S_0 is finitely expansive, there is an $\varepsilon > 0$ such that for all $x \in S_0$ there is a finite subset $M(x) \subset S_0$ such that whenever $d((S_0)^n y, (S_0)^n x) < \varepsilon$ for all n, then $y \in M(x)$.

There is a finite set E_1 which satisfies (1). Assume not. Then, for each finite edge set E there is a point x such that $x_n \notin E$ for all n. Let $E(0) = E$ where E is so large that whenever $x, y \in S$ with $x_0, y_0 \notin E$, then $d(x, y) < \varepsilon$. And let x^0 be any point in S, such that $x_n^0 \notin E(0)$ for all n. Then define inductively $E(k+1) = E(k) \cup \{x_0^k\}$, and choose a point x^{k+1} such that $x_n^{k+1} \notin E(k+1)$ for all n. By the choice of E, we have that $d(S^n x^k, S^n x^m) < \varepsilon$ for all k, m, n in contradiction to the fact that S_0 is finitely expansive.

There is a finite set E_2 which satisfies (2). Choose a finite set of edges E and an integer N such that:

(a) If $x, y \in S$ with $x_0, y_0 \notin E$, then $d(x, y) < \varepsilon$.

(b) If $x, y \in S$ with $x[-N, N] = y[-N, N]$, then $d(x, y) < \varepsilon$.

Let E' be a finite set of edges containing E and such that $x \in S$ with $x_0 \in E$ and $x_n \notin E'$ implies $|n| > N+1$ and such that $S_{E'}$ is irreducible. Then, for x, $y \in S$ such that $x_n, y_n \notin E'$ whenever $x_n \neq y_n$, it holds that $d(S^m x, S^m y) < \varepsilon$ for all m. Thus, if there would be two distinct paths p, q of the same length having first and last edge in E' and all other edges outside E', then, since $S_{E'}$ is irreducible, one would find an uncountable set A of points for which $x, y \in A$ implies $d(S^n x, S^n y) < \varepsilon$ for all n. This is impossible, since S_0 is finitely expansive. Thus E' satisfies (2).

There is a finite set E_3 which satisfies (3). Assume for every finite set of edges, say E, there are infinitely many vertices v such that for each v there are two infinite paths starting at v which run the whole time outside of E and have distinct first edge. Then choose E so large that

(a) If $x, y \in S$ with $x_0, y_0 \notin E$, then $d(x, y) < \varepsilon$.

By enlarging E we may assume that the SFT S_E is irreducible. Determine N such that

(b) If $x, y \in S$ with $x[-N, N] = y[-N, N]$, then $d(x, y) < \varepsilon$.

Enumerate those vertices from which two paths are starting with distinct first edge and running outside of E the whole time as $v(1), v(2), \ldots$. Let V be the set of vertices which is an initial vertex of an edge in E. For each $v(k)$ choose a shortest path from a vertex in V to $v(k)$ which has only edges outside of E. Call this path w^k. By taking a suitable subsequence we may assume that all the paths w^k start with the same block w of length $N + 1$, and that the lengths of the w^k are strictly increasing. Since S_E is irreducible there is a point p such that all $p_n \in E$ and the initial vertex of p_0 is the initial vertex of w. Let x^1 be a point with $x^1(-\infty, -1] = p(-\infty, -1]$, $x^1[0, \infty)$ starts with w^1 and such that $x^1_n \notin E$ for all $n \geq 0$.

If there are increasing sequences n_k, i_k such that the initial vertex of $x^1_{n_k}$ is $v(i_k)$, then since from each vertex $v(k)$ there are two paths starting, each having edges outside E, we can choose points x^k for $k \geq 2$ such that $x^k(-\infty, n_k - 1] = x(-\infty, n_k - 1]$, $x^k_{n_k} \neq x^1_{n_k}$ and $x^k_n \notin E$ for all $n \geq 0$. Thus we have an infinite set A of points x which have $x(-\infty, N] = p(-\infty, -1]w$ and $x_n \notin E$ for all $n \geq 0$.

If x^1 visits only finitely many of the vertices $v(k)$, then choose k so large that x^1 never visits the vertex $v(k)$. Then let x^2 be a point with $x^2(-\infty, -1] = p(-\infty, -1]$, $x^2[0, \infty)$ starts with w^k and such that $x^2_n \notin E$ for all $n \geq 0$. Then, again, if x^2 visits infinitely many of the vertices $v(k)$ we can find an infinite set A of points x which have $x(-\infty, N] = p(-\infty, -1]w$ and $x_n \notin E$ for all $n \geq 0$. Otherwise, there is a vertex $v(k)$ which is not visited by x^1 and not by x^2. Then let x^3 be a point with $x^3(-\infty, -1] = p(-\infty, -1]$, $x^3[0, \infty)$ starts with w^k and such that $x^3_n \notin E$ for all $n \geq 0$. Repeating this procedure gives an infinite set A of points x which have $x(-\infty, N] = p(-\infty, -1]w$ and $x_n \notin E$ for all $n \geq 0$. But now, by (a) and (b) for any pair $x, y \in A$ we have $d(S^n x, S^n y) < \varepsilon$ for all n, contradicting the fact that S_0 is finitely expansive. A symmetric argument for rays ending in E shows that there is a finite set E_3 which satisfies (3).

The union of the sets E_1, E_2 and E_3 is a finite set which satisfies (1)–(3). $\square$

Example 4.3. The 1-point compactification S_0 can be countably expansive without being expansive. Let S be given by the following graph. The vertex set is $\{n \in \mathbf{Z} \mid n \leq 1\} \cup \{(n,k) \mid n > k \geq 0\}$. The edges are as follows. There is an edge from vertex

(1) $n-1$ to vertex n, for each $n \leq 1$,

(2) 1 to vertex $(1,0)$,

(3) (n,k) to vertex $(n+1,k)$ for each $n > k \geq 0$,

(4) $(k+1,k)$ to vertex $(k+2,k+1)$ for each $k \geq 0$,

(5) $(k+\sum_{i=1}^m i, k)$ to vertex $-(k+\sum_{i=1}^m i)$ for $0 \leq k \leq m$, $m \geq 1$.

Note that every point visits the edge starting at vertex 1, and for each $n \geq 1$, there is exactly one first return loop of length $2n+2$ to vertex 1. Thus there is a finite set of edges which satisfies (1) and (2) of Lemma 4.1. But there is no finite set which satisfies (3), since for a finite set E of edges there is a $k_0 \geq 0$ such that $n > k \geq k_0$ implies the edges starting from vertex (n,k) do not belong to E; thus, starting from vertex (k_0+1, k_0), there are infinitely many rays which never visit E, namely for each $k \geq k_0$, there is a unique ray through the vertices $(k_0+1,k_0), (k_0+2,k_0+1), \ldots, (k+1,k), (k+2,k), (k+3,k), \ldots$. Thus, S_0 is not expansive. However, for each finite set E of edges containing the edge from -1 to 0, there are only countably many rays which have all their edges outside of E; thus, S_0 is countably expansive. $\square$

Lemma 4.4. *Let S and T be noncompact locally compact transitive Markov shifts. Let $f : S \to T$ be a finite-to-1 proper factor map. Then T_0 expansive implies S_0 expansive. The converse is not true.*

Proof. Since f is finite-to-1 proper, $f_0 : S_0 \to T_0$ is finite-to-1, continuous. Let $\varepsilon > 0$ be an expansiveness constant for T_0. Choose $\delta > 0$ such that $d(x,y) < \delta$ implies $d(f_0x, f_0y) < \varepsilon$. Then, if $x, y \in S_0$ with $d((S_0)^n x, (S_0)^n y) < \delta$ for all n, then $d((T_0)^n f_0 x, (T_0)^n f_0 y) < \varepsilon$

for all n, and thus $f_0x = f_0y$. Thus, S_0 is finitely expansive, since f_0 is finite-to-1. Thus, by Lemma 4.2, S_0 is expansive.

To see that the converse is not true, consider the following Markov shifts. The graph for S has vertex set $\mathbf{Z}$ and the edges are as follows.

(1) There are two edges a and b from vertex 0 to vertex 1.

(2) There is an edge e_n from vertex n to vertex $n+1$, for each $n \in \mathbf{Z} - \{0\}$.

(3) There is an edge c_n from vertex n to vertex $-n$, for each $n \geq 1$.

The graph for T also has vertex set $\mathbf{Z}$, and the edges are as follows.

(1) There is an edge e_n from vertex n to vertex $n+1$ for each $n \in \mathbf{Z}$.

(2) There are two edges a_n and b_n from vertex n to vertex $-n$, for each $n \geq 1$.

Using the characterization in Lemma 4.1, one sees that S_0 is expansive (use the set $E = \{a, b\}$), and T_0 is not expansive, since the paths $e_{n-1}a_ne_{-n}$ and $e_{n-1}b_ne_{-n}$ violate (2) of Lemma 4.1 for large enough n. Define a factor map $f : S \to T$ by defining $f(x)_0$ for $x \in S$ as follows.

(1) If $x_0 \in \{a, b\}$, then let $f(x)_0 = e_0$.

(2) If $x_0 = e_n$ for some $n \in \mathbf{Z} - \{0\}$, then let $f(x)_0 = x_0 = e_n$.

(3) If $x_0 = c_n$ for some $n \geq 1$, then let $[f(x)_0 = a_n$ if $x_{-n} = a]$, and let $[f(x)_0 = b_n$ if $x_{-n} = b]$.

This map is proper, onto and at most 2-to-1. $\square$

In Lemma 4.4, it is essential that the factor map be finite-to-1. An example for a countable-to-1 proper factor map $f : S \to T$, with T_0 expansive and S_0 not expansive, is obtained by taking for S the Markov shift from Example 4.3 and letting T be the Markov shift given by the graph with vertex set $\mathbf{Z}$ and then for each $n \in \mathbf{Z}$, there is an edge from vertex n to vertex $n+1$, and for each $n \geq 1$, there is an edge from vertex $n+1$ to vertex $-n$. So then the edge set consisting of the unique T-edge which starts in vertex 1 satisfies (1)–(3) of Lemma 4.1, and thus T_0 is expansive. Note that, for each pair of T-vertices α, β, there is at most one T-edge from α to β. Thus we can define a factor map $f : S \to T$ by mapping an S-vertex $n \leq 1$ to the T-vertex n

and an S-vertex (n, k), $n > k \geq 0$ to the T-vertex $n+1$. The map f is proper, since each T-vertex has only finitely many S-vertices as preimages. We show that the map is countable-to-1. Since in S there is for each n exactly one first return loop of length $2n+2$ at vertex 1, which is mapped to the unique first return loop in T at vertex 1 of the same length, we get that every $y \in T$, such that y_n starts in vertex 1 for infinitely many $n > 0$, has a unique preimage. Let $y \in T$, $x \in f^{-1}y$ such that there is some n such that y_n starts in vertex 1, and y_k does not start in vertex 1 for $k > n$, then $y(-\infty, n]$ uniquely determines $x(-\infty, n]$, and for each $i \geq 1$ we get that x_{n+i} is an edge starting in a vertex $(i+1, k_i)$. By the definition of S there are only countably many possible choices for the sequences $(k_i)_{i \geq 1}$. Thus $f^{-1}y$ is countable.

Let S be a subshift and m a block of S. Then m is a synchronizing block if $x, y \in S$, $x[1, |m|] = y[1, |m|] = m$ implies that there is a $z \in S$ with $z(-\infty, 0] = x(-\infty, 0]$ and $z[1, \infty) = y[1, \infty)$. A compact subshift S is synchronizing if S is transitive and has a synchronizing block, [**1**], [**10**]. For compact subshifts S the synchronizing part $\mathrm{SYN}(S) := \{x \in S \mid x \text{ sees a synchronizing block}\}$ is a conjugacy invariant. If S is an SFT, then $\mathrm{SYN}(S) = S$. The next lemma shows that the 1-point compactification of a mixing Markov shift has the property that all except at most one point are in the synchronizing part.

Lemma 4.5. *Let S be a mixing locally compact Markov shift with S_0 expansive. Then S_0 is a synchronized system with $S \subset \mathrm{SYN}(S_0)$ and S is strongly positive recurrent.*

Proof. Let S be given in a graph presentation. Since S_0 is expansive, it is synchronized, [**15**, Theorem 7.3.4], [**13**, Lemma 1.10]; we give a simplified proof here. Since S_0 is expansive, there is a finite set E of edges in S which satisfy (1)–(3) from Lemma 4.1. By renaming the edges, we may assume that $0 \notin E$. Let $i : S \to (E \cup \{0\})^{\mathbf{Z}}$ be a shift commuting map defined by $(ix)_0 = x_0$ if $x_0 \in E$ and $(ix)_0 = 0$ otherwise. Properties (1)–(3) imply that i is an injective map and $0^\infty \notin i(S)$. The map i extends to a continuous map $S_0 \to i(S) \cup \{0^\infty\}$ by $i(\infty) := 0^\infty$. Thus, i is a conjugacy to the subshift $T := i(S) \cup \{0^\infty\}$ where every T-symbol $\neq 0$ is synchronizing.

Since S_0 is a subshift, $h_G(S) = h_{\text{top}}(S_0) < \infty$. To show that S is strongly positive recurrent, we verify (c) of [**15**, Theorem 4.1.13]; we show that there is some vertex v such that the growth rate of the number of first return loops at v is strictly less than the growth rate of the number of loops at v.

Let $1 < \lambda < \infty$ with $h_G(S) = \log\lambda$. We may assume that the SFT S_E is mixing. Now fix a vertex, say v, in S_E. By enlarging E if necessary, we may assume additionally that every edge e which starts or ends in v belongs to E. Let $C(n)$ denote the first return loops of length n at vertex v. We want to define an injective map from $C(n)$, into the set of loops at v of length n, of a certain SFT $S_{E'}$ of S not dependent on n. Then, since $h_{\text{top}}(S_{E'}) < h_G(S)$, we get that $\limsup_n 1/n\cdot\log\#C(n) < \log\lambda$, and by [**15**, Theorem 4.1.13], it follows that S is strongly positive recurrent.

Fix for every pair of edges $e, f \in E$ such that there is a path of length ≥ 1, from terminal vertex of e to initial vertex of f, which only has edges outside of E, a shortest such path; call it $p(e,f)$. Let $M = \max|p(e,f)|$. Since S_E is mixing, there is some N such that, for every pair of edges e, f and every $n \geq N$ there is a path in S_E of length n from e to f. Since S is locally compact, we can find a finite set E' with $E \subset E'$ such that

(a) if p is a path starting and ending in E and having an edge outside E', then $|p| > 2M + 4N$.

Let $B(n)$ be the set of loops of length n in $S_{E'}$ at vertex v. We shall now define an injective map $i : C(n) \to B(n)$ for all $n > 2M + 4N$. For that, fix for each edge $f \in E$ a shortest path in E starting with f and ending at vertex v, say t_f, and a shortest path in E from vertex v which ends with f, say s_f. By choice of N these have length $\leq N$. Fix for every $n \geq N$ a loop l_n in E at vertex v of length n. Consider a first return loop $p = p_1 \dots p_n \in C(n)$. If p is a path in $S_{E'}$, then $i(p) := p$. If p is not a path in $S_{E'}$, then p leaves the set E' at least once. So we can consider all the time intervals $[i,j]$ for which $p_k \notin E$ for $i < k < j$ and $p_i \in E$, $p_j \in E$, and there is some $i < k < j$ with $p_k \notin E'$. Let $e = p_i, f = p_j$. We replace the part $p[i,j]$ of the path p by $ep(e,f)t_f l_m s_e p(e,f)f$, where m is so that this part has the same length as $p[i,j]$. This is possible, by (a). We do so for all those subpaths of p. This gives $i(p)$. We claim that the map is injective; we show that $i(p)$ determines p uniquely. If $i(p)$ is a first return loop at v, then $i(p) = p$. If $i(p)$ is not a first return loop at v, then consider the first return at v. Since p is a first return loop, by construction of $i(p)$ we have that before the first return of v we see $ep(e, f)t_f$ for suitable e, f. Since t_f is a path in E and $p(e, f)$ is a path outside E, we can recognize this uniquely. Now go from the first return to v further to the right such that we see the first occurrence of $s_e p(e, f)f$. Again we can recognize this uniquely, since $p(e, f)$ is a path outside of E. Thus, we have recognized a maximal subpart of p which was replaced. By (2), the original part is uniquely determined by e, f and the length of the subpath. Thus, $i(p)$ determines p uniquely and thus, i is injective. The lemma is thus proved. $\square$

Lemma 4.6. *Let S, T be mixing locally compact Markov shifts with expansive 1-point compactifications. Let $f : S \to T$ be a finite-to-1 proper factor map. Then f has a degree, that is, every doubly transitive point has the same finite number of preimages all of which are doubly transitive.*

Proof. The map f extends to a factor map $f_0 : S_0 \to T_0$. By Lemma 4.5, S_0 and T_0 are synchronized systems. Let $D(S_0)$ denote the doubly transitive points in S_0; $D(T_0)$ those in T_0. Clearly $f_0(D(S_0)) \subset D(T_0)$, thus $D(S_0) \subset f_0^{-1}(D(T_0))$. Now let $x \in S_0$ such that $f_0 x \in D(T_0)$. Assume $x \notin D(S_0)$. Then there is a block, say m, such that x_+ or x_- does not see m. Since S_0 is compact, we can thus construct a point $y \in S_0$ such that $f_0 y \in D(T_0)$ and y never sees the block m. Let R be the orbit closure of y. Since y never sees m, R is a proper subshift of S_0. Since $f_0 y \in D(T_0)$, the map f_0 restricts to a factor map from R onto T_0. Thus, $h_{\text{top}}(R) \geq h_{\text{top}}(T_0)$, and thus $h_{\text{top}}(R) = h_{\text{top}}(S_0)$. This contradicts the fact that S is strongly positively recurrent, by Lemma 4.5. Thus, $D(S_0) = f_0^{-1}(D(T_0))$. By [**5**, Theorem 3.3], f_0 has a degree, and thus f, too. $\square$

Lemma 4.7. *Let S, T be mixing locally compact Markov shifts with expansive 1-point compactifications. Let $f : S \to T$ be a finite-to-1, proper factor map. Then there is a finite-to-1 proper factor map $g : S \to T$ with degree 1.*

Proof. By Lemma 3.2, f extends to a finite-to-1 factor map $f_0 : S_0 \to T_0$ and S_0 and T_0 are synchronized systems by Lemma 4.5. By [**5**, Theorem 4.4], there is a finite-to-1 factor map $g_0 : S_0 \to T_0$ of degree 1. By construction of g_0 we have $g_0^{-1}(\infty) = \infty$, and thus g_0 restricts to a finite-to-1 proper factor map $g : S \to T$ with degree 1. □

5. Properly finite-to-1 and countable-to-1 maps for synchronized systems. We conclude this paper with two examples which show that for synchronized systems, finite-to-1 factor maps which are not bounded-to-1, and countable-to-1 maps which are not finite-to-1, do exist. Thus, it is not necessary to drop the compactness assumption to obtain such phenomena.

Example 5.1. Synchronized systems S and T and a countable-to-1 factor map $f : S \to T$ which is not finite-to-1. The idea is to "embed" the factor map which collapses the orbit $0^\infty 10^\infty$ to the fixed point 1^∞ into a factor map between synchronized systems.

Let $S \subset \{0,1,2\}^{\mathbf{Z}}$ be the coded system given by the code $C := \{20^n 10^n \mid n \geq 0\}$, [**1**], [**10**]. Then the symbol 2 is synchronizing for S, since it occurs only as the first symbol in any code word. Let f be the 1-block map given by $f(x)_0 = 1$ if $x_0 \leq 1$, and $f(x)_0 = x_0$ if $x_0 = 2$. Then the image $T \subset \{1,2\}^{\mathbf{Z}}$ is a sofic shift, since it is the shift which has an odd number of 1's between two consecutive 2's and 22 is not allowed.

We show first that f is not finite-to-1. Since $0^n 10^n$ is an S-block for each n, we have that $x = 0^\infty 10^\infty \in S$, with $x_0 = 1$, say. Then $fx = 1^\infty$. Thus, 1^∞ has infinitely many preimages, namely $S^n x$, $n \in \mathbf{Z}$.

We show that f is countable-to-1. Since f fixes the symbols 2 and maps $0, 1$ to 1, any point in T which sees the symbol 2 has a unique preimage. The point 1^∞ has the orbit $0^\infty 10^\infty$ as its preimage set. □

Example 5.2. Synchronized systems S and T and a finite-to-1 factor map $f : S \to T$ which is not bounded-to-1. The idea is similar to the previous example. We embed a finite-to-1, nonbounded-to-1, map between nontransitive subshifts into a factor map between synchronized systems, but it is more delicate to keep the map finite-to-1.

For $k \in \mathbf{N}$, let

• $W_k := \{5^k a_1 5^k a_2 \dots 5^k a_k 5^k \in \{0,1,5\}^{k(k+2)} \mid a_i \in \{0,1\}$ for all $1 \le i \le k\}$.

and fix an enumeration, say $w_{k,i}$, $1 \le i \le 2^k$, of the set W_k. Then let

• $u_{n,k,i} := 2^n w_{k,i} 2^n 4 \in \{0,1,2,4,5\}^{k(k+2)+2n+1}$, $1 \le i \le 2^k$, $n \ge k$, $k \in \mathbf{N}$.

And, finally, let $m(k) := 2^k$ and

• $c_{n,k} := 3u_{n,k,1}u_{n,k,2}\dots u_{n,k,m(k)} \in \{0,1,2,3,4,5\}^{1+m(k)\cdot(k(k+2)+2n+1)}$.

Let $S \subset \{0,1,2,3,4,5\}^{\mathbf{Z}}$ be the coded system given by the code $C := \{c_{n,k} \mid n \ge k \ge 1\}$. The symbol 3 is synchronizing for S, since it occurs only as the first symbol in any code word. Let f be the 1-block map given by $f(x)_0 = 1$ if $x_0 \le 1$, and $f(x)_0 = x_0$ if $x_0 \ge 2$. Then the image $T \subset \{1,2,3,4,5\}^{\mathbf{Z}}$ is a coded system, [**1**]. The subshift T is in fact synchronized with synchronizing symbol 3.

We show first that f is not bounded-to-1. Fix $k \in \mathbf{N}$. Then, for each $1 \le i \le 2^k$, the block $u_{n,k,i}$ is an S-block for all $n \ge k$. Thus, there is an $x^i \in S$ with $x^i(-\infty,-1] = 2^\infty$ and $x[0,\infty) = w_{k,i}2^\infty$. Then $fx^i(-\infty,-1] = 2^\infty$ and $fx^i[0,\infty) = 5^k 1 5^k 1 \dots 5^k 1 5^k 2^\infty$. Thus $y := fx^i$ has at least 2^k preimages. Since k was arbitrary this shows that f is not bounded-to-1.

We show that f is finite-to-1. Let $y \in T$. Since f fixes the symbols $2,3,4,5$ and maps $0,1$ to 1, if $\#\{n \in \mathbf{Z} \mid y_n = 1\} = m < \infty$ then $\#f^{-1}y \le 2^m < \infty$. Now assume that $\#\{n \in \mathbf{Z} \mid y_n = 1\} = \infty$. Let $x \in f^{-1}y$. Let $m \in \mathbf{Z}$ with $y_m = 1$. Since $\#\{n \in \mathbf{Z} \mid y_n = 1\} = \infty$ there is a largest $q < m$ such that $y_q = 3$. Thus, $y[q,m]$ begins with a block $32^n 5^k 1$ for some $n \ge k \ge 1$. But then $x[q, q + |c_{n,k}| - 1] = c_{n,k}$ and $q + |c_{n,k}| - 1 > m$. Thus x_m is uniquely determined by $y[q,m]$. The argument works for every m such that $y_m = 1$, thus y has a unique preimage. This proves that f is finite-to-1. $\square$

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