A GENERALIZATION OF KUMMER'S IDENTITY

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ABSTRACT. The well-known formula of Kummer evaluates the hypergeometric series $_2F_1\left(\begin{smallmatrix}A,B\\C\end{smallmatrix}\,\middle|\,-1\right)$ when the relation $C - A + B = 1$ holds. This paper deals with the evaluation of $_2F_1(-1)$ series in the case when $C - A + B$ is an integer. Such a series is expressed as a sum of two Γ-terms multiplied by terminating $_3F_2(1)$ series. A few such formulas were essentially known to Whipple in the 1920s. Here we give a simpler and more complete overview of this type of evaluation. Additionally, algorithmic aspects of evaluating hypergeometric series are considered. We illustrate Zeilberger's method and discuss its applicability to nonterminating series and present a couple of similar generalizations of other known formulas.

1. The generalization. The subject of this paper is a generalization of Kummer's identity (see [**11**], [**2**, Section 2.3] or [**1**, Corollary 3.1.2]):

$$(1) \qquad _2F_1\left(\begin{matrix}a,b\\1+a-b\end{matrix}\,\middle|\,-1\right) = \frac{\Gamma(1+a-b)\Gamma(1+\frac{a}{2})}{\Gamma(1+a)\Gamma(1+\frac{a}{2}-b)}.$$

The hypergeometric series on the left is defined if $a - b$ is not a negative integer, and it is absolutely convergent for $\mathrm{Re}\,(b) < 1/2$. After analytic continuation of $_2F_1\left(\begin{smallmatrix}a,b\\1+a-b\end{smallmatrix}\,\middle|\,z\right)$ on $\mathbf{C}\setminus[1,\infty)$, and after division of both sides by $\Gamma(1 + a - b)$ the formula has meaning and is correct for all complex a, b. In this paper, whenever $_2F_1\left(\begin{smallmatrix}A,B\\C\end{smallmatrix}\,\middle|\,z\right)$ denotes a well-defined hypergeometric series, it also denotes its analytic continuation on $\mathbf{C}\setminus[1,\infty)$.

The generalization to be considered evaluates the hypergeometric series $_2F_1\left(\begin{smallmatrix}A,B\\C\end{smallmatrix}\,\middle|\,-1\right)$ whenever $C - A + B$ is any integer. In the terminology of [**1**], our generalization applies to $_2F_1(-1)$ series that are *contiguous* to a series for Kummer's formula (1). As is known (see [**1**,

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Section 2.5]), the 15 classical Gauss contiguity relations can be iterated to produce a linear relation between any three contiguous $_2F_1(z)$ series, with coefficients that are rational functions in the parameters of those series. This also applies to their analytic extensions. The generalized formula is such a relation in explicit form between contiguous $_2F_1\left(\begin{smallmatrix} a+n,b \\ a-b \end{smallmatrix} \middle| -1\right)$, $_2F_1\left(\begin{smallmatrix} a,b \\ 1+a-b \end{smallmatrix} \middle| -1\right)$ and $_2F_1\left(\begin{smallmatrix} a-1,b \\ a-b \end{smallmatrix} \middle| -1\right)$, where n is an integer, and the last two series are evaluated using Kummer's identity (1). The coefficient of the first series cannot be the zero function because the quotient of the other two series is not in $\mathbf{C}(a,b,n)$. In the generalized formula, these coefficients are written as terminating $_3F_2(1)$ series.

We write the generalization in the form

$$(2) \quad _2F_1\left(\begin{matrix} a+n,b \\ a-b \end{matrix} \middle| -1\right) = P(n)\frac{\Gamma(a-b)\Gamma(\frac{a+1}{2})}{\Gamma(a)\Gamma(\frac{a+1}{2}-b)} + Q(n)\frac{\Gamma(a-b)\Gamma(\frac{a}{2})}{\Gamma(a)\Gamma(\frac{a}{2}-b)}.$$

Here, the two Γ-terms are equal (respectively) to $_2F_1\left(\begin{smallmatrix} a-1,b \\ a-b \end{smallmatrix} \middle| -1\right)$ and $\frac{a-b}{a-2b}{_2F_1}\left(\begin{smallmatrix} a,b \\ 1+a-b \end{smallmatrix} \middle| -1\right)$, and $P(n), Q(n)$ are rational functions in a, b for every integer n. The most convenient expressions for $P(n)$ and $Q(n)$ are summarized in the three theorems below. In fact, expressions of $_2F_1(-1)$ series in (2), in terms of terminating series and Γ-functions, were known to Whipple [17]. His formulas (8.3) and (8.41) would express the $_2F_1(-1)$ series in (2) in terminating series for negative or positive n, respectively. Whipple's formulas (11.5) and (11.51) form the statement of Theorem 1 below. Whipple derived them as a consequence of transformations of $_3F_2(1)$ series allied to general $_2F_1(-1)$ series, and from [4, formulas (2.6), (2.7)], where some $_2F_1(1/2)$ series are expressed in terms of terminating series. However, Whipple's main concern was the relations of general $_2F_1(-1)$ and $_3F_2(1)$ series. As we will see, his approach is not convenient when some of those series terminate.

In this paper we strive for a clear overview of possible expressions for $P(n)$ and $Q(n)$ in terms of terminating $_3F_2(1)$ series, with simpler proofs. Another aim is to consider algorithmic aspects of evaluating hypergeometric series. In particular, we specialize formula (2) to two-term identities, which however seem to go beyond Zeilberger's approach. Also, a few evaluations similar to (2) are presented; specifically, these are evaluations of hypergeometric series contiguous to the

$_2F_1(1/4)$ and $_3F_2(1)$ series in Gosper's and Dixon's identities; see (35), (36).

In the following theorems, we summarize the most convenient expressions for $P(n)$ and $Q(n)$. A few more such expressions are presented in (16)–(19).

Theorem 1. *Suppose that n is a nonnegative integer, or -1, and a, b are complex numbers such that $(a)_n \neq 0$ and $a - b$ is not zero or a negative integer. Then the coefficients $P(n)$ and $Q(n)$ in formula (2) can be written as*:

$$(3) \qquad P(n) = \frac{1}{2^{n+1}} {}_3F_2 \left(\begin{matrix} -\frac{n}{2}, -\frac{n+1}{2}, \frac{a}{2} - b \\ \frac{1}{2}, \frac{a}{2} \end{matrix} \right),$$

$$(4) \qquad Q(n) = \frac{n+1}{2^{n+1}} {}_3F_2 \left(\begin{matrix} -\frac{n-1}{2}, -\frac{n}{2}, \frac{a+1}{2} - b \\ \frac{3}{2}, \frac{a+1}{2} \end{matrix} \right).$$

Theorem 2. *Suppose that n is a nonnegative integer and a, b are complex such that $(a)_n \neq 0$ and $a - b$ is not zero or a negative integer. Then the coefficients $P(n)$ and $Q(n)$ in formula (2) can be written as*:

$$(5) \quad P(n) = \frac{1}{2} {}_3F_2 \left(\begin{matrix} -\frac{n}{2}, -\frac{n+1}{2}, b \\ -n, \frac{a}{2} \end{matrix} \right), \quad Q(n) = \frac{1}{2} {}_3F_2 \left(\begin{matrix} -\frac{n-1}{2}, -\frac{n}{2}, b \\ -n, \frac{a+1}{2} \end{matrix} \right).$$

The hypergeometric sums should be interpreted as terminating series with, up to ± 1, $\lfloor \frac{n}{2} \rfloor$ terms.

Theorem 3. *Let $P(n, a, b)$ and $Q(n, a, b)$ denote the coefficients $P(n)$ and $Q(n)$ in (2) as functions of a, b as well. If n is a nonnegative integer and $a, b \notin \{0, 1, \dots, n\}$, then*

$$(6) \qquad P(-n-1, a, b) = 2^{2n} \frac{(1 - \frac{a}{2})_n}{(1 - b)_n} P(n - 1, a - 2n, b - n),$$

$$(7) \qquad Q(-n-1, a, b) = -2^{2n} \frac{(\frac{1-a}{2})_n}{(1 - b)_n} Q(n - 1, a - 2n, b - n).$$

Because of the last theorem, we do not give expressions for $P(n)$ and $Q(n)$ for a negative n, except (13), (14) in the proof of Theorem 3.

These theorems are proved in Section 2. There we also give an overview of transformations between other expressions for $P(n)$ and $Q(n)$ and give a survey of Whipple's approach in [**17**]. In Section 3, Theorem 2 is proved using the more universal Zeilberger's method. The key observation is that the sequences $P(n)$ and $Q(n)$ satisfy the same recurrence relation as the lefthand side of (2). Theorem 1 can also be proven in this way. Notice that any different expressions for $P(n)$ and $Q(n)$ must represent the same rational functions in a, b for every n, because the quotient of the Γ-terms in (2) is not in $\mathbf{C}(a, b)$. Section 4 is devoted to algorithmic aspects of evaluation of hypergeometric series with similar generalizations of Dixon's and Gosper's identities.

2. Classical proof. We assume here that $\mathrm{Re}\,(a/2) > \mathrm{Re}\,(b) > 0$. One can simply check that Theorems 1 and 2 hold for the analytic continuation of the $_2F_1(-1)$ series as well.

To prove Theorem 1, we recall Whipple's identity [**17**, formula (8.41)]

$$
(8) \quad {}_2F_1\left(\begin{array}{c}A, B\\C\end{array}\middle|-1\right) = \frac{\Gamma(C)}{2 \cdot \Gamma(A)} \times
$$

$$
\sum_{k=0}^{\infty}(-1)^k\frac{(C-A+B-1)_k}{k!}\frac{\Gamma(\frac{A}{2}+\frac{k}{2})}{\Gamma(C-\frac{A}{2}+\frac{k}{2})}.
$$

As was communicated by Askey, this identity can be proven easily using Euler's integral representation [**3**, formula 2.12(1)] for the $_2F_1(z)$ series. One has to rearrange the integrand as

$$
(9) \quad t^{A-1}(1-t)^{C-A-1}(1+t)^{-B} = t^{A-1}(1+t)^{-C+A-B+1}(1-t^2)^{C-A-1},
$$

expand $(1+t)^{-C+A-B+1}$ as a series, interchange integration and summation, change the variable $t \mapsto \sqrt{s}$ and recognize the beta-integral [**3**, formula 1.5(1)].

We apply[1] formula (8) to the righthand side of the identity [**3**, formula 2.9(2)]:

$$
(10) \quad {}_2F_1\left(\begin{array}{c}a+n,b\\a-b\end{array}\middle|-1\right) = 2^{-2b-n}{}_2F_1\left(\begin{array}{c}a-2b,-b-n\\a-b\end{array}\middle|-1\right).
$$

After this, we sum up the terms with even and odd indexes separately, transform the Γ-factors slightly and get formula (2) with $P(n), Q(n)$ defined by (3), (4).

Theorem 2 follows from Theorem 1 by the following transformation of a terminating $_3F_2(1)$ series (see [**1**, Proof of Corollary 3.3.4]):

$$(11) \qquad _3F_2\left(\begin{matrix} -m, A, B \\ E, F \end{matrix}\right) = \frac{(E-A)_m}{(E)_m} \, _3F_2\left(\begin{matrix} -m, A, F-B \\ 1+A-E-m, F \end{matrix}\right),$$

where m must be a nonnegative integer. To make sure that the interpretation of the ill-defined hypergeometric series in (5) is correct for this transformation, one may specialize A to $-\nu/2$ or $-(\nu \pm 1)/2$ with complex ν, instead of $-n/2$, etc., and take the limit $\nu \to n$.

To prove Theorem 3 we use Euler's integral again. After rearranging the integrand in (9) as $t^{A-1}(1-t)^{C-A+B-1}(1-t^2)^{-B}$ and expanding $(1-t)^{C-A+B-1}$, we eventually get the formula:

$$(12)$$
$$_2F_1\left(\begin{matrix} A, B \\ C \end{matrix}\,\middle|\, -1\right) = \frac{1}{2}\frac{\Gamma(C)\Gamma(1-B)}{\Gamma(A)\Gamma(C-A)}$$
$$\times \sum_{k=0}^{\infty} \frac{(A-B-C+1)_k}{k!} \frac{\Gamma(\frac{A}{2}+\frac{k}{2})}{\Gamma(\frac{A}{2}+\frac{k}{2}+1-B)}.$$

As in the proof of Theorem 1, we apply this formula to $_2F_1\left(\begin{matrix} a-n-1,b \\ a-b \end{matrix}\,\middle|\, -1\right)$ transformed by (10), and add the terms with even and odd indices separately. The result is:

$$(13)$$
$$P(-n-1) = 2^n \frac{(1-\frac{a}{2})_n}{(1-b)_n} \, _3F_2\left(\begin{matrix} -\frac{n}{2}, -\frac{n-1}{2}, \frac{a}{2}-b \\ \frac{1}{2}, \frac{a}{2}-n \end{matrix}\right),$$

$$(14)$$
$$Q(-n-1) = -n 2^n \frac{(\frac{1-a}{2})_n}{(1-b)_n} \, _3F_2\left(\begin{matrix} -\frac{n-1}{2}, -\frac{n-2}{2}, \frac{a+1}{2}-b \\ \frac{3}{2}, \frac{a+1}{2}-n \end{matrix}\right).$$

Comparing these expressions with (3), (4) gives Theorem 3. □

To get more expressions for $P(n)$ and $Q(n)$ one can use standard transformations of terminating $_3F_2(1)$ series. For example, one may repeatedly apply (11) or rewrite a terminating series in the reverse order. In general, a terminating $_3F_2(1)$ series can be transformed to 17

other terminating $_3F_2(1)$ series; see [**16**, Section 8] and [**2**, Section 3.9]. To give these transformations a group structure, one has to consider transpositions of the two lower and two upper parameters as nontrivial transformations. Then one gets a group of 72 elements which acts on the set of 18 hypergeometric series; see [**13**]. The action of this group can be summarized as follows: Let $y_0, \ldots, y_5$ be six parameters satisfying $y_0 + y_1 + y_2 = y_3 + y_4 + y_5 = 1 - m$. Then the expression

$$(15) \qquad (y_0 + y_4)_m (y_0 + y_5)_m {}_3F_2\left(\begin{array}{c} -m, y_0 + y_1 - y_3, y_0 + y_2 - y_3 \\ y_0 + y_4, y_0 + y_5 \end{array}\right)$$

is invariant under the permutations within the sets $\{y_0, y_1, y_2\}$ and $\{y_3, y_4, y_5\}$, and gets multiplied by $(-1)^m$ when these two sets are interchanged. For instance, formula (11) corresponds to the permutation $y_0 \leftrightarrow y_5$, $y_1 \leftrightarrow y_4$, $y_2 \leftrightarrow y_3$.

Application of these transformations to the series (3), (4) or (5) gives eight sets of 18 terminating $_3F_2(1)$ series, one set for a choice of $P(n)$ or $Q(n)$, positive or negative and even or odd n. The number of different hypergeometric series turns out to be 96. Here we summarize a few interesting expressions for $n \geq 0$:

$$(16)$$
$$P(n) = \frac{\lfloor \frac{n}{2} \rfloor!}{2 \cdot n!}\left(\frac{1-a}{2} + b\right)_{\lceil n/2 \rceil} {}_3F_2\left(\begin{array}{c} -\lceil \frac{n}{2} \rceil, \frac{a+1}{2} + \lfloor \frac{n}{2} \rfloor, \frac{a}{2} - b \\ \frac{1}{2}, 1 - b - \lceil \frac{n}{2} \rceil \end{array}\right)$$

$$(17) \qquad = \frac{1}{2^{n+1}} \frac{(b)_{\lceil n/2 \rceil}}{(\frac{a}{2})_{\lceil n/2 \rceil}} {}_3F_2\left(\begin{array}{c} -\lceil \frac{n}{2} \rceil, 1 + \lfloor \frac{n}{2} \rfloor, \frac{a}{2} - b \\ \frac{a}{2}, \frac{a+1}{2} - \lceil \frac{n}{2} \rceil - b \end{array}\right),$$

$$(18)$$
$$Q(n) = \frac{\lceil \frac{n}{2} \rceil!}{2 \cdot n!}\left(1 - \frac{a}{2} + b\right)_{\lfloor n/2 \rfloor} {}_3F_2\left(\begin{array}{c} -\lfloor \frac{n}{2} \rfloor, \frac{a}{2} + \lceil \frac{n}{2} \rceil, \frac{a+1}{2} - b \\ \frac{a+1}{2}, \frac{a}{2} - \lfloor \frac{n}{2} \rfloor - b \end{array}\right)$$

$$(19) \qquad = \frac{n+1}{2^{n+1}} \frac{(b)_{\lfloor n/2 \rfloor}}{(\frac{a+1}{2})_{\lfloor n/2 \rfloor}} {}_3F_2\left(\begin{array}{c} -\lfloor \frac{n}{2} \rfloor, 1 + \lceil \frac{n}{2} \rceil, \frac{a+1}{2} - b \\ \frac{3}{2}, 1 - b - \lfloor \frac{n}{2} \rfloor \end{array}\right).$$

To get expressions for negative n, one may use Theorem 3. Notice that the series in (17) and (19) terminate for both positive and negative n.

In the rest of this section, we follow Whipple's approach in [**17**], where transformations of not necessarily terminating $_3F_2(1)$ series are used to derive various identities with general $_2F_1(-1)$ series. We concentrate

on the $_2F_1(-1)$ series which are contiguous to the series in Kummer's formula (1). Notice that proofs of Theorems 1 and 3 are valid for any complex value of n, so that formula (2) with $P(n)$ and $Q(n)$ defined by (3)–(4) or (13)–(14) is true for any complex n. Formula (2) with $P(n), Q(n)$ defined by (5) is also true for all n; see Whipple's formulas (23)–(24) below. But one may check that, in general, these $P(n)$ and $Q(n)$ are not the same.

Transformations of general $_3F_2(1)$ series were first derived by Thomae [14]. Whipple introduced notation (see [16] and [2, Section 3.5-7]), which gives a group-theoretical insight into those formulas. To begin with, there is an action of the symmetric group S_5 on $_3F_2(1)$s. Hardy described it in the notes to Lecture VII in [7] by saying that the function

$$(20) \qquad \frac{1}{\Gamma(E)\Gamma(F)\Gamma(E+F-A-B-C)} {}_3F_2\left(\begin{matrix} A, B, C \\ E, F \end{matrix}\right)$$

is invariant under the permutations of $E, F, E+F-B-C, E+F-A-C$ and $E + F - A - B$. For example, we have (see [1, Corollary 3.3.5]):

$$(21) \qquad {}_3F_2\left(\begin{matrix} A, B, C \\ E, F \end{matrix}\right) = \frac{\Gamma(F)\Gamma(E+F-A-B-C)}{\Gamma(F-C)\Gamma(E+F-A-B)}$$

$$\times {}_3F_2\left(\begin{matrix} E-A, E-B, C \\ E, E+F-A-B \end{matrix}\right).$$

An orbit of the general $_3F_2(1)$ consists of 10 different series. Note that the series in (20) converge when $\mathrm{Re}\,(E + F - A - B - C) > 0$, and the whole expression is well defined and analytic for any parameters under this condition. The function (20) can be continued analytically to the region in the parameter space where at least one of the 10 series converges.

Further, a general S_5 orbit of $_3F_2(1)$s is associated to 11 other orbits so that we get sets of 120 *allied* $_3F_2(1)$ series, see [16]. For example,[2] the series in (20) is allied to

$$(22)$$
$$ {}_3F_2\left(\begin{matrix} A, 1 + A - E, 1 + A - F \\ 1 + A - B, 1 + A - C \end{matrix}\right) \quad \text{and} \quad {}_3F_2\left(\begin{matrix} E - A, E - B, E - C \\ E, 1 + F - E \end{matrix}\right).$$

In general, two allied series are not related by a two-term identity like (21). But for any three allied series, there is a linear relation between them, with coefficients being Γ-terms. This also gives three-term relations for the 12 functions of type (20), and even defines their analytic continuation to the whole space of parameters. Indeed, if the series in (20) diverges, then its ally $_3F_2\left(\begin{smallmatrix} 1-A,1-B,1-C \\ 2-D,2-E \end{smallmatrix}\right)$ converges; for the third term one can take convergent series from a similar pair of functions from other S_5-orbits. Besides, all allied series converge in a neighborhood of $A = B = C = 1/2$, $E = F = 1$.

In [**17**], Whipple applies the relations of allied series to a general $_2F_1(-1)$ series by expressing it as a $_3F_2(1)$ series and considering it as a member of the corresponding allied family. In particular, his formulas (3.1) and (3.51) read as follows:

(23)
$$_2F_1\left(\begin{matrix} a+\nu, b \\ a-b \end{matrix}\,\middle|\,-1\right) = \frac{\Gamma(a-b)\Gamma(\frac{a}{2})}{\Gamma(a)\Gamma(\frac{a}{2}-b)}\,_3F_2\left(\begin{matrix} -\frac{\nu-1}{2},-\frac{\nu}{2},b \\ -\nu,\frac{a+1}{2} \end{matrix}\right)$$

(24)
$$= \frac{\Gamma(a-b)\Gamma(\frac{a+1}{2})}{\Gamma(a)\Gamma(\frac{a+1}{2}-b)}\,_3F_2\left(\begin{matrix} -\frac{\nu}{2},-\frac{\nu+1}{2},b \\ -\nu,\frac{a}{2} \end{matrix}\right).$$

If $\nu \notin \{0,1,2,\dots\}$, we may relate the $_2F_1(-1)$ series to the S_5-orbit of the $_3F_2(1)$ series in (23)–(24) and get many two- and three-term relations with $_2F_1(-1)$ and $_3F_2(1)$ series. Some of these identities make sense and are correct even if ν is a nonnegative integer, because singular Γ-factors cancel. For instance, formula (2) with $P(n)$ and $Q(n)$ defined by (3)–(4) is a three-term identity between allied series; see the last paragraph of [**17**]. Similarly, (potentially) terminating series in Whipple's formulas (8.3) and (8.41) are derived from three-term identities of allied series.

On the other hand, the $_3F_2(1)$ series in (23)–(24) cannot be identified with the terminating series in the expressions in (5). One has to

compute:

$$\lim_{\nu \to n} {}_3F_2\left(\begin{matrix} -\frac{\nu}{2}, \ -\frac{\nu+1}{2}, \ b \\ -\nu, \ \frac{a}{2} \end{matrix}\right)$$

$$= 2P(n) - \frac{1}{4^{n+1}} \frac{(b)_{n+1}}{\left(\frac{a}{2}\right)_{n+1}} {}_3F_2\left(\begin{matrix} \frac{n+2}{2}, \ \frac{n+1}{2}, b+n+1 \\ n+2, \ \frac{a}{2}+n+1 \end{matrix}\right),$$

$$\lim_{\nu \to n} {}_3F_2\left(\begin{matrix} -\frac{\nu-1}{2}, \ -\frac{\nu}{2}, \ b \\ -\nu, \ \frac{a+1}{2} \end{matrix}\right)$$

$$= 2Q(n) + \frac{1}{4^{n+1}} \frac{(b)_{n+1}}{\left(\frac{a+1}{2}\right)_{n+1}} {}_3F_2\left(\begin{matrix} \frac{n+3}{2}, \ \frac{n+2}{2}, b+n+1 \\ n+2, \ \frac{a+1}{2}+n+1 \end{matrix}\right).$$

In the sum of these two inequalities the non-terminating ${}_3F_2(1)$ series on the righthand side cancel, since they are connected by transformation (21). In this way identities (23)–(24) prove Theorem 2.

Moreover, the ${}_3F_2(1)$ series (23)–(24) can be transformed by S_5 to four series which are well defined and terminate when ν is an (odd or even) positive integer n. Those terminating series are presented in formulas (16) and (18). However, this does not give expressions for ${}_2F_1(1)\left(\begin{matrix} a+n, b \\ a-b \end{matrix} \middle| -1\right)$ in terms of one termination ${}_3F_2(1)$ series, because the four series mentioned diverge for $\nu > 1/2$ (except when they terminate), and we cannot use the S_5-invariance of the corresponding function in (20). Notice, for example, that (21) implies a wrong relation between the ${}_3F_2(1)$ series in (16) and (18). As we see, Whipple's approach in [17] gets complicated in the case ν in (23)–(24) is an integer, and does not directly explain various expressions for our $P(n)$ and $Q(n)$.

3. A proof by Zeilberger's method. Here we prove Theorem 2 only. Theorem 1 can be proved in the same way.

Let us define $S(n) = {}_2F_1\left(\begin{matrix} a+n, b \\ a-b \end{matrix} \middle| -1\right)$. The contiguity relation [3, 2.8 (28)] between ${}_2F_1\left(\begin{matrix} A+1, B \\ C \end{matrix} \middle| z\right)$, ${}_2F_1\left(\begin{matrix} A-1, B \\ C \end{matrix} \middle| z\right)$ and ${}_2F_1\left(\begin{matrix} A, B \\ C \end{matrix} \middle| z\right)$ gives the following recurrence relation:

(25) $2(n+a)S(n+1) - (3n+2a)S(n) + (n+b)S(n-1) = 0.$

We claim that the sequences $P(n)$ and $Q(n)$ satisfy the same recurrence relation. Following the "creative telescoping" method of Zeilberger (see

[**12**], [**8**]), let

(26) $$p(n,k) = \frac{(-1)^k}{2 \cdot 4^k} \frac{(n+1)(n-k)!}{(n-2k+1)!k!} \frac{(b)_k}{\left(\frac{a}{2}\right)_k}$$

be the kth summand of $P(n)$ in (5). We set $p(n,k) = 0$ for $k > \lceil \frac{n}{2} \rceil$. Also define

$$r_1(n,k) = -\frac{2k(n-k+1)(a+2k-2)}{(n-2k+2)(n-2k+3)}, \quad R_1(n,k) = r_1(n,k)p(n,k).$$

One can check that

$$2(n+a)p(n+1,k) - (3n+2a)p(n,k) + (n+b)p(n-1,k)$$
$$= R_1(n,k+1) - R_1(n,k),$$

so

(27)
$$2(n+a)P(n+1) - (3n+2a)P(n) + (n+b)P(n-1) =$$
$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}(R_1(n,k+1) - (R_1(n,k))) - R_1\left(n, \lceil \frac{n+1}{2} \rceil\right) = 0.$$

Although this looks like an artificial trick, we follow the standard Wilf-Zeilberger method of proving combinatorial identities; see [**8, 12**]. The expression $r_1(n,k)$ is the *certificate* of our standardized proof. Given $p(n,k)$, the recurrence relation for $P(n)$ and the certificate $r_1(n,k)$ can be found by Zeilberger's algorithm. This algorithm is implemented in the computer algebra packages EKHAD (see [**18**, command `ct`]) and `hsum.mpl` see [**9**, command `sumrecursion` with option `certificate=true`]. Also check [**15**] for a link to a Maple worksheet for this proof. The finite sums in this proof require some attention, since they are not natural according to [**8**].

In the same way,

(28)
$$2(n+a)Q(n+1) - (3n+2a)Q(n) + (n+b)Q(n-1) =$$
$$\sum_{k=0}^{\lfloor (n-1)/2 \rfloor}(R_2(n,k+1) - R_2(n,k)) - R_2\left(n, \lceil \frac{n}{2} \rceil\right) = 0,$$

where

$$(29) \quad R_2(n,k) = \frac{2k(n-k+1)(a+2k-1)}{(n-2k+1)(n-2k+2)} \cdot \frac{(-1)^k}{2 \cdot 4^k} \binom{n-k}{k} \frac{(b)_k}{(\frac{a+1}{2})_k}$$

is the kth summand of $Q(n)$ in (5) multiplied by the corresponding certificate.

Note that the condition $(a)_n \neq 0$ ensures that the recurrence relation (25) does not degenerate to a first order relation until we evaluate $P(n)$ and $Q(n)$. It remains to check that formula (2) holds for two initial values of n. Kummer's identity (1) suggests $P(-1) = 1$ and $Q(-1) = 0$, which fits into the recurrence relation. We may use Gauss's contiguity relation [**3**, formula 2.8 (38)] between

$$_2F_1\left(\left.\begin{matrix}A, B\\C+1\end{matrix}\right|z\right), \ _2F_1\left(\left.\begin{matrix}A, B\\C\end{matrix}\right|z\right) \text{ and } _2F_1\left(\left.\begin{matrix}A-1, B\\C\end{matrix}\right|z\right)$$

to obtain

$$(30) \quad (a-2b)\frac{\Gamma(1+a-b)\Gamma(1+\frac{a}{2})}{\Gamma(1+a)\Gamma(1+\frac{a}{2}-b)} - 2(a-b)S(0) + (a-b)S(-1) = 0.$$

This implies the correct values $P(0) = 1/2$ and $Q(0) = 1/2$ and completes the proof.

Note that the Gauss contiguity relations hold for analytic extensions of hypergeometric functions on $\mathbf{C} \setminus [1, \infty)$. Therefore, this proof does not require convergence of the $_2F_1(-1)$ series. $\square$

In fact, the sequences $P(n)$ and $Q(n)$ satisfy the recurrence relation (25) for all n. The recurrence can be verified directly for $n = -2, -1, 0$. The values of $P(n)$ and $Q(n)$ for $n = -3, -2, -1, 0, 1$ are

$$\frac{2(a-2)(a-b-2)}{(b-1)(b-2)}, \quad \frac{a-2}{b-1}, \quad 1, \quad \frac{1}{2}, \quad \frac{a-b}{2a}$$

and

$$-\frac{2(a-1)(a-3)}{(b-1)(b-2)}, \quad -\frac{a-1}{b-1}, \quad 0, \quad \frac{1}{2}, \quad \frac{1}{2},$$

respectively. To compute the same recurrence relation for negative n, one can use Theorem 1. Alternatively, one may choose an expression for $P(n)$ and $Q(n)$ for negative n, say (13)–(14), and compute the recurrence relation with Zeilberger's algorithm.

To show equalities like (16)–(18) by Zeilberger's method, one would have to compute the recurrences for odd and even integers separately. The recurrence relation (25) for any such expression and for all n can be computed using contiguity relations for the $_3F_2(1)$ series. As is known (see [1, Section 3.7]), contiguous $_3F_2(1)$ series satisfy three-term relations (the coefficients being rational functions in the parameters of those series), just like the contiguous $_2F_1(z)$ series.

4. Algorithmic aspects. The generalized formula (2) can be specialized so that $P(n)$ or $Q(n)$ vanishes, giving an evaluation of $_2F_1(-1)$ series with a single Γ-term. For example,

$$(31) \qquad Q(-4) = -4\frac{(a-1)(a-3)(2a-b-7)}{(b-1)(b-2)(b-3)},$$

so if $b = 2a - 7$, then $Q(-4) = 0$, which implies

$$(32) \qquad _2F_1\left(\begin{array}{c} 3-c, 7-2c \\ c \end{array}\middle|-1\right) = \frac{3}{4}\frac{\Gamma(c)\Gamma(3-\frac{c}{2})}{\Gamma(5-c)\Gamma(\frac{3c}{2}-2)}.$$

Further, $P(-5) = 0$ if $2a^2 - 4ab + b^2 - 12a + 17b + 12 = 0$. Parameterizing the curve given by this equation, we get

$$(33)$$
$$_2F_1\left(\begin{array}{c} -\frac{2t^2-7t+6}{t^2-2}, \frac{t^2+4t-8}{t^2-2} \\ \frac{2t^2+3t-8}{t^2-2} \end{array}\middle|-1\right) = \frac{t^2+3t-6}{t(t-1)}\frac{\Gamma(\frac{3t-4}{t^2-2})\Gamma(\frac{t^2+7t-12}{2(t^2-2)})}{\Gamma(\frac{7t-10}{t^2-2})\Gamma(\frac{t(t-1)}{2(t^2-2)})}.$$

It could be expected that formulas like (32) can be proved automatically by current computer algebra algorithms, say by the Wilf-Zeilberger method. As is demonstrated in [10], this method or Zeilberger's algorithm can be adapted to nonterminating hypergeometric series if one can justify the "creative telescoping" trick by dominated convergence, and the hypergeometric series can be evaluated in the limit $n \to \infty$, where n is a discrete parameter. In general, a nonterminating hypergeometric series is given without a discrete parameter,

so it must be introduced by an algorithm. For example, after the substitution $a \mapsto a + 2n$, one can prove Kummer's formula (1) by the Wilf-Zeilberger method, see [**5**].

In the case of equation (32), we may substitute $c \mapsto c + n$ and apply Zeilberger's algorithm to get the right first order difference equation. However, we cannot evaluate the hypergeometric series, neither in the limit $n \to \infty$ nor for a finite value of n. What we can do is to combine explicitly Gauss's contiguity relations in such a way that we "accidentally" get a two-term relation where one of the terms can be evaluated by Kummer's formula. For example, the relation between contiguous $_2F_1\left(\begin{smallmatrix} A,B \\ C \end{smallmatrix} \middle| z\right)$, $_2F_1\left(\begin{smallmatrix} A+1,B-2 \\ C \end{smallmatrix} \middle| z\right)$ and, say, $_2F_1\left(\begin{smallmatrix} A,B-1 \\ C \end{smallmatrix} \middle| z\right)$, after the specialization $(A, B, C, z) \mapsto (3 - c, 7 - 2c, c, -1)$ becomes

$$(34) \qquad {}_2F_1\left(\begin{matrix} 3 - c, 7 - 2c \\ c \end{matrix} \middle| -1\right) = \frac{3}{4} {}_2F_1\left(\begin{matrix} 4 - c, 5 - 2c \\ c \end{matrix} \middle| -1\right).$$

In this way even the exotic (33) can be proved.

This shows that relations between contiguous hypergeometric series can be useful for finding new "nonstandard" evaluations of $_2F_1$ series. One may take such a relation and try to find families of its two term specializations with a discrete parameter n. This would give a first order recurrence relation, and if the series can be evaluated in the limit $n \to \infty$, one gets a (perhaps) new formula! Relations between contiguous series also give a way, alternative to Zeilberger's algorithm, to compute recurrence relations.

In [**15**], there is a link to Maple routines, which for three given integer vectors (k_i, l_i, m_i) for $i = 1, 2, 3$, derive a $\mathbf{C}(A, B, C, z)$-linear relation between three contiguous functions $_2F_1\left(\begin{smallmatrix} A+k_i,B+l_i \\ C+m_i \end{smallmatrix} \middle| z\right)$. Computer experiments found many first order recurrence relations for some values $z = 1/4,\ 1/3,\ 1/9,\ \exp(i\pi/3),\ 3 - 2\sqrt{2}, \ldots$; some of them can be successfully solved. It is an interesting question which $_2F_1(z)$ series can be evaluated in terms of Γ-functions. The evaluations produced so far can be obtained using classical quadratic or cubic transformations.

Here we generalize a few known formulas of the same type as (2). They were obtained by considering relations between three contiguous hypergeometric series where two of them can be evaluated by a known formula, and trying to express the coefficients in these relations as

hypergeometric series. This was done by considering partial fraction decomposition of these coefficients empirically. The formulas can be proved by showing that all three terms in a formula satisfy the same recurrence relation by Zeilberger's algorithm, and checking the identity for a couple of values of the discrete parameter.

We start with a generalization of Gosper's "nonstandard" evaluations of $_2F_1(1/4)$ series, see [**6**, formula 1/4.1-2]. A generalization is

$$(35) \quad {}_2F_1\left(\begin{array}{c} -a, \frac{1}{2} \\ 2a + \frac{3}{2} + n \end{array}\middle| \frac{1}{4}\right)$$
$$= \frac{2^{n+3/2}}{3^{n+1}} \frac{\Gamma(a + \frac{5}{4} + \frac{n}{2})\Gamma(a + \frac{3}{4} + \frac{n}{2})\Gamma(a + \frac{1}{2})}{\Gamma(a + \frac{7}{6} + \frac{n}{3})\Gamma(a + \frac{5}{6} + \frac{n}{3})\Gamma(a + \frac{1}{2} + \frac{n}{3})} K(n)$$
$$- (-3)^{n-2} 2^{3/2} \frac{\Gamma(a + \frac{5}{4} + \frac{n}{2})\Gamma(a + \frac{3}{4} + \frac{n}{2})\Gamma(a + 1)}{\Gamma(a + \frac{3}{2})\Gamma(a + \frac{1}{2} + \frac{n}{2})\Gamma(a + 1 + \frac{n}{2})} L(n),$$

where

$$K(1) = L(0) = 0, \quad K(0) = L(1) = 1,$$

for $n > 1$:

$$K(n) = (-1)^n \sum_{k=\lceil n/3 \rceil}^{\lfloor \frac{n}{2} \rfloor} \frac{27^k}{4^k} \frac{n(k-1)!}{(n-2k)!(3k-n)!} \frac{(a + \frac{1}{2})_k}{(a+1)_k},$$

$$L(n) = {}_4F_3\left(\begin{array}{c} -\frac{n-1}{3}, -\frac{n-2}{3}, -\frac{n-3}{3}, a+1 \\ -\frac{n-2}{2}, -\frac{n-3}{2}, a + \frac{3}{2} \end{array}\right),$$

and for $-n < 0$:

$$K(-n) = {}_4F_3\left(\begin{array}{c} -\frac{n}{3} - \frac{n-1}{3}, -\frac{n-2}{3}, -a \\ -\frac{n-1}{2}, -\frac{n-2}{2}, -a + \frac{1}{2} \end{array}\right)$$
$$= \sum_{k=0}^{\lfloor n/3 \rfloor} \frac{(-4)^k}{27^k} \frac{n(n-2k-1)!}{(n-3k)!k!} \frac{(-a)_k}{(-a + \frac{1}{2})_k},$$

$$L(-n) = (-1)^n \sum_{k=\lceil (n+1)/3 \rceil}^{\lfloor (n+1)/2 \rfloor} \frac{27^k}{4^k}$$
$$\times \frac{(n+1)(k-1)!}{(n-2k+1)!(3k-n-1)!} \frac{(-a - \frac{1}{2})_k}{(-a)_k}.$$

Gosper has found the special cases $n = 0, 1$. The Γ-factors to $K(n)$ and $L(n)$ are $\mathbf{C}(a)$-multiples of these two Gosper's evaluations, respectively, for each n. All three terms in (35) satisfy the recurrence relation

$$2(n + 2a + 1)(2n + 6a + 3)S(n + 1) + (2n + 4a + 3)(4n + 6a + 1)S(n)$$
$$- 3(2n + 4a + 1)(2n + 4a + 3)S(n - 1) = 0.$$

Next we recall the classical Dixon's identity which evaluates well-poised $_3F_2(1)$ series, see [**2**, Section 3.1]. We generalize it as follows:
(36)
$$_3F_2\left(\begin{array}{c} a + n, b, c \\ a - b, a - c \end{array}\right) = \frac{\tilde{P}(n)}{2} \frac{\Gamma(\frac{a+1}{2})\Gamma(a - b)\Gamma(a - c)\Gamma(\frac{a+1}{2} - b - c)}{\Gamma(a)\Gamma(\frac{a+1}{2} - b)\Gamma(\frac{a+1}{2} - c)\Gamma(a - b - c)}$$
$$+ \frac{\tilde{Q}(n)}{2} \frac{\Gamma(\frac{a}{2})\Gamma(a - b)\Gamma(a - c)\Gamma(\frac{a}{2} - b - c)}{\Gamma(a)\Gamma(\frac{a}{2} - b)\Gamma(\frac{a}{2} - c)\Gamma(a - b - c)},$$

where $\tilde{P}(-1) = 1$, $\tilde{Q}(-1) = 0$, then for $n \geq 0$:

$$\tilde{P}(n) = {_4F_3}\left(\begin{array}{c} -\frac{n}{2}, -\frac{n+1}{2}, b, c \\ -n, \frac{a}{2}, \frac{1-a}{2} + b + c \end{array}\right), \quad \tilde{Q}(n) = {_4F_3}\left(\begin{array}{c} -\frac{n-1}{2}, -\frac{n}{2}, b, c \\ -n, \frac{1+a}{2}, 1 - \frac{a}{2} + b + c \end{array}\right),$$

and for $-n < 0$:

$$\tilde{P}(-n - 1) = 2^{2n} \frac{(\frac{1-a}{2})_n(\frac{1+a}{2} - b - c)_n}{(1 - b)_n(1 - c)_n} \times$$
$$_4F_3\left(\begin{array}{c} -\frac{n}{2}, -\frac{n-1}{2}, b - n, c - n \\ 1 - n, \frac{a}{2} - n, \frac{1-a}{2} + b + c - n \end{array}\right),$$
$$\tilde{Q}(-n - 1) = -2^{2n} \frac{(\frac{1-a}{2})_n(\frac{a}{2} - b - c)_n}{(1 - b)_n(1 - c)_n} \times$$
$$_4F_3\left(\begin{array}{c} -\frac{n-1}{2}, -\frac{n-2}{2}, b - n, c - n \\ 1 - n, \frac{1+a}{2} - n, 1 - \frac{a}{2} + b + c - n \end{array}\right).$$

Dixon's identity is the special case $n = -1$. This generalized formula is a relation between contiguous $_3F_2(1)$ series in explicit form. For positive n it is strikingly similar to the generalization (2), (5) of Kummer's identity. In fact, the generalization in Theorem 2 is the limiting case $c \to \infty$ of (36), just as Kummer's formula is the limiting

case of Dixon's identity. The recurrence relation for the three terms in (36) is:

$$(n + a)(n - a + 2b + 2c + 1)S(n + 1) + (n + b)(n + c)S(n - 1)$$
$$- (2n^2 + 3bn + 3cn + n - a^2 + 2ab + 2ac+)S(n) = 0.$$

More evaluations of the same type can be obtained using standard transformations of $_2F_1(z)$ series to $_2F_1(z/(z-1))$ series, see [**3**, formulas 2.9(3)-(4)]. Applying them to the generalized Kummer's formula (2) gives evaluations of $_2F_1(1/2)$ which generalize classical formulas of Gauss and Bailey, see [**2**, Section 2.4]. The same transformation of (35) gives evaluations of $_2F_1(-1/3)$. Similarly, one can apply (21) to identity (36) and get generalizations of Watson's and Whipple's formulas [**2**, Sections 3.3-4].

All these formulas evaluate hypergeometric series which are contiguous to a series for which an evaluation is known. In order to find these formulas automatically, one needs an algorithm which would find the solutions of a recurrence relation in the form of terminating hypergeometric series.

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ENDNOTES

1. The same could be done directly to $_2F_1\left({a+n,b \atop a-b} \big| -1\right)$, of course. We would get the less-convenient formula

$$_2F_1\left({a+n,b \atop a-b}\Big| -1\right) = \frac{1}{2}\frac{\Gamma(a-b)\Gamma(\frac{a+n}{2})}{\Gamma(a+n)\Gamma(\frac{a-n}{2}-b)} {}_3F_2\left({-\frac{n}{2}, \quad -\frac{n+1}{2}, \quad \frac{a+n}{2} \atop \frac{1}{2}, \qquad \frac{a-n}{2}-b}\right)$$

$$+\frac{n+1}{2}\frac{\Gamma(a-b)\Gamma(\frac{a+n+1}{2})}{\Gamma(a+n)\Gamma(\frac{a-n+1}{2}-b)} {}_3F_2\left({-\frac{n-1}{2}, \quad -\frac{n}{2}, \quad \frac{a+n+1}{2} \atop \frac{3}{2}, \qquad \frac{a-n+1}{2}-b}\right).$$

Here for each positive integer n the two Γ-terms are $\mathbf{C}(a,b)$-multiples of the Γ-terms in (2), so the coefficients $P(n), Q(n)$ are equal to $\mathbf{C}(a,b)$-multiples of the

$_3F_2(1)$ series in this formula. But the correspondence depends on whether n is even or odd.

2. Whipple introduced for the $_3F_2(1)$ series six parameters $r_0, \ldots, r_5$ related by the condition $\sum r_i = 0$ so that: all allied series can be obtained by permutations of the six parameters and/or changing the sign of them all; an S_5-orbit is determined by fixing a parameter and an element of the set $\{+, -\}$ and S_5 permutes the remaining five parameters. Specifically, one may choose that the S_5 action on (20) fixes r_0 and take $E = 1 + r_4 - r_0$.

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