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SOME q-ORTHOGONAL POLYNOMIALS AND RELATED HANKEL DETERMINANTS

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1. Introduction. This paper grew out of some experiments using the computer algebra MAPLE. Let the function $f(t)$ have the Taylor series development

$$f(t) = \sum_{n=0}^{\infty} f_n t^n,$$

which we assume converges in a neighborhood of the origin. The coefficients f_n may be interpreted as the moments of a suitable function, actually the complex moments

$$f_n = L(z^n) = \frac{1}{2\pi i} \oint z^n \left(\frac{f(1/z)}{z} \right) dz,$$

where the path of integration is, say, a circle centered at the origin with a suitably large radius. Using the construction given in [**1**] and the Gram determinants

$$G_N = \begin{vmatrix} f_0 & f_1 & \cdots & f_N \\ f_1 & f_2 & \cdots & f_{N+1} \\ \vdots & \vdots & \ddots & \vdots \\ f_N & f_{N+1} & \cdots & f_{2N} \end{vmatrix},$$

one may construct the monic polynomials, call them $P_N(x)$, that are orthogonal to the distribution which gives these moments.

$$P_N(x) = \frac{1}{G_{N-1}} \begin{vmatrix} f_0 & f_1 & \cdots & f_N \\ f_1 & f_2 & \cdots & f_{N+1} \\ \vdots & \vdots & \ddots & \vdots \\ f_{N-1} & f_N & \cdots & f_{2N-1} \\ 1 & x & \cdots & x^N \end{vmatrix}.$$

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These polynomials satisfy a three term recurrence relation

$$P_{N+1}(x) = (x + B_N)P_N(x) - C_N P_{N-1}(x).$$

The coefficients B_N and C_N can be found from the above determinental expression for the polynomials. In particular,

$$C_N = \frac{G_N G_{N-2}}{G_{N-1}^2}.$$

Conversely, by taking products in this expression, a formula for the determinant G_N can be recovered

$$G_N = f_0 \prod_{j=1}^{N} C_j^{N+1-j}.$$

As usual, let

$$(A;q)_n = \prod_{j=0}^{n-1}(1 - Aq^j), \quad N > 0, (A;q)_0 = 1;$$
$$\begin{bmatrix} n \\ j \end{bmatrix} = \frac{(q;q)_n}{(q;q)_j (q,q)_{n-j}}.$$

MAPLE experiments indicate that when one makes the choice

$$f(t) = \frac{t}{\sum_{n=0}^{\infty} \frac{t^n (a;q)_n}{(b;q)_n} - 1} = \frac{1}{\sum_{n=0}^{\infty} \frac{t^n (a;q)_{n+1}}{(b;q)_{n+1}}},$$

then the formulas for the coefficients B_N and C_N turned out to be very simple. We were led to this choice for $f(t)$ by previous work, which had shown that when $f(t)$ was the generating function for the Bernoulli numbers, the B_N and C_N turned out to be simple. This paper presents a proof of the formulas suggested by MAPLE. There are many interesting special cases of our results. We have not yet been able to extend this analysis to other choices of $f(t)$.

2. Preliminary results. We prove here two preliminary results dealing with the evaluation of certain Hankel-type determinants.

Theorem 1. *Suppose that we have two sequences s_n and t_n satisfying*

$$t_0 = 1, \quad s_0 \neq 0$$

and

$$\sum_{j=0}^{N} t_j s_{N-j} = \begin{cases} s_0 & N = 0; \\ 0 & N > 0. \end{cases}$$

Let

$$S_N = \begin{vmatrix} s_0 & s_1 & \cdots & s_N \\ s_1 & s_2 & \cdots & s_{N+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_N & s_{N+1} & \cdots & s_{2N} \end{vmatrix}.$$

Let $T_0 = 1$ and, for $N > 0$,

$$T_N = \begin{vmatrix} t_2 & t_3 & \cdots & t_{N+1} \\ t_3 & t_4 & \cdots & t_{N+2} \\ \vdots & \vdots & \ddots & \vdots \\ t_{N+1} & t_{N+2} & \cdots & t_{2N} \end{vmatrix}.$$

Then

$$S_N = (-1)^N s_0^{N+1} T_N.$$

Proof. When $N = 0$, the assertion reduces to $s_0 = s_0$. When $N = 1$, we use the fact that $t_0 = 1$ to get

$$\begin{vmatrix} s_0 & s_1 \\ s_1 & s_2 \end{vmatrix} = \begin{vmatrix} s_0 & t_1 s_0 + t_0 s_1 \\ s_1 & t_1 s_1 + t_0 s_2 \end{vmatrix} = \begin{vmatrix} s_0 & 0 \\ s_1 & -t_2 s_0 \end{vmatrix} = -s_0^2 t_2 = -s_0^2 T_1.$$

We shall show how this same process, which we call "O-reduction", owing to the orthogonality of the sequences s_n and t_n, works in general. We combine columns of the S_N determinant using the orthogonality relation:

$$S_N = \begin{vmatrix} s_0 & t_1s_0+t_0s_1 & t_2s_0+t_1s_1+t_0s_2 & \cdots & \sum_{j=0}^{N} t_{N-j}s_j \\ s_1 & t_1s_1+t_0s_2 & t_2s_1+t_1s_2+t_0s_3 & \cdots & \sum_{j=0}^{N} t_{N-j}s_{j+1} \\ s_2 & t_1s_2+t_0s_3 & t_2s_2+t_1s_3+t_0s_4 & \cdots & \sum_{j=0}^{N} t_{N-j}s_{j+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_N & t_1s_N+t_0s_{N+1} & t_2s_N+t_1s_{N+1}+t_0s_{N+2} & \cdots & \sum_{j=0}^{N} t_{N-j}s_{j+N} \end{vmatrix}$$

$$= \begin{vmatrix} s_0 & 0 & 0 & \cdots & 0 \\ s_1 & -t_2s_0 & -t_3s_0 & \cdots & -t_{N+1}s_0 \\ s_2 & -t_3s_0-t_2s_1 & -t_4s_0-t_3s_1 & \cdots & -t_{N+2}s_0-t_{N+1}s_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_N & -\sum_{j=0}^{N-1} t_{N+1-j}s_j & -\sum_{j=0}^{N-1} t_{N+2-j}s_j & \cdots & -\sum_{j=0}^{N-1} t_{2N-j}s_j \end{vmatrix}.$$

We now multiply the second row by (s_i/s_0) and subtract it from the $(i+2)$nd row for $1 \le i \le N-1$:

$$S_N = \begin{vmatrix} s_0 & 0 & 0 & \cdots & 0 \\ s_1 & -t_2s_0 & -t_3s_0 & \cdots & -t_{N+1}s_0 \\ s_2-\frac{s_1^2}{s_0} & -t_3s_0 & -t_4s_0 & \cdots & -t_{N+2}s_0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ s_N-\frac{s_Ns_1}{s_0} & -\sum_{j=0}^{N-1} t_{N+1-j}s_j & -\sum_{j=0}^{N-2} t_{N+2-j}s_j & \cdots & -\sum_{j=0}^{N-2} t_{2N-j}s_j \end{vmatrix}.$$

We continue this process. We multiply the third row by (s_i/s_0) and subtract it from the $(i+3)$rd row for $1 \le i \le N-2$. Then we multiply the resulting fourth row by (s_i/s_0) and subtract it from the $(i+4)$th row for $1 \le i \le N-3$, etc. We find that

$$S_N = \begin{vmatrix} s_0 & 0 & 0 & \cdots & 0 \\ (.) & -t_2s_0 & -t_3s_0 & \cdots & -t_{N+1}s_0 \\ (.) & -t_3s_0 & -t_4s_0 & \cdots & -t_{N+2}s_0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (.) & -t_{N+1}s_0 & -t_{N+2}s_0 & \cdot & -t_{2N}s_0 \end{vmatrix} = s_0^{N+1}(-1)^N T_N. \qquad \square$$

Theorem 2. *Let S_n, T_n and the sequences s_n and t_n be as in the* *previous theorem and define*

$$\mathcal{S}_N(x) = \frac{1}{S_{N-1}} \begin{vmatrix} s_0 & s_1 & \cdots & s_N \\ s_1 & s_2 & \cdots & s_{N+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_{N-1} & s_N & \cdots & s_{2N-1} \\ 1 & x & \cdots & x^N \end{vmatrix}.$$

Let

$$\tau_n(x) = \sum_{j=0}^{n} t_{n-j} x^j.$$

Then

$$\mathcal{S}_N(x) = \frac{1}{T_{N-1}} \begin{vmatrix} t_2 & t_3 & \cdots & t_{N+1} \\ t_3 & t_4 & \cdots & t_{N+2} \\ \vdots & \vdots & \ddots & \vdots \\ t_N & t_{N+1} & \cdots & t_{2N-1} \\ \tau_1(x) & \tau_2(x) & \cdots & \tau_N(x) \end{vmatrix}.$$

Proof. We proceed with the O-reduction, as in Theorem 1. We carry out the column operations first:

$$\mathcal{S}_N(x) = \frac{1}{S_{N-1}} \begin{vmatrix} s_0 & t_1 s_0 + t_0 s_1 & \cdots & \sum_{j=0}^N t_{N-j} s_j \\ s_1 & t_1 s_1 + t_0 s_2 & \cdots & \sum_{j=0}^N t_{N-j} s_{j+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_{N-1} & t_1 s_{N-1} + t_0 s_N & \cdots & \sum_{j=0}^N t_{N-j} s_{j+N-1} \\ 1 & t_1 + x & \cdots & \sum_{j=0}^N t_{N-j} x^j \end{vmatrix}.$$

We now modify the row operations in the O-reduction by leaving the last row unaltered. We only treat the first N rows. We find

$$
\begin{aligned}
\mathcal{S}_N(x) &= \frac{1}{S_{N-1}}\begin{vmatrix} s_0 & 0 & \cdots & 0 \\ (.) & -t_2 s_0 & \cdots & -t_{N+1}s_0 \\ \vdots & \vdots & \ddots & \vdots \\ (.) & -t_N s_0 & \cdots & -t_{2N-1}s_0 \\ (.) & \tau_1(x) & \cdots & \tau_N(x) \end{vmatrix} \\
&= \frac{1}{S_{N-1}} s_0^N (-1)^{N-1} \begin{vmatrix} t_2 & t_3 & \cdots & t_{N+1} \\ t_3 & t_4 & \cdots & t_{N+2} \\ \vdots & \vdots & \ddots & \vdots \\ t_N & t_{N+1} & \cdots & t_{2N-1} \\ \tau_1(x) & \tau_2(x) & \cdots & \tau_N(x) \end{vmatrix} \\
&= \frac{1}{T_{N-1}} \begin{vmatrix} t_2 & t_3 & \cdots & t_{N+1} \\ t_3 & t_4 & \cdots & t_{N+2} \\ \vdots & \vdots & \ddots & \vdots \\ t_N & t_{N+1} & \cdots & t_{2N-1} \\ \tau_1(x) & \tau_2(x) & \cdots & \tau_N(x) \end{vmatrix}
\end{aligned}
$$

by Theorem 1. $\square$

Note. Strictly speaking, in the above definition we have to assume that $N > 0$, but here and in what follows, we adopt the convention that $\mathcal{S}_0(x) = 1$.

3. Some facts about the little q-Jacobi polynomials.

A fairly full account of these polynomials is given in [**3**, pages 166–168]. Their moment generating function is given in [**2**, pages 32–33]. A nice summary of their properties and their relationships to other orthogonal polynomials of hypergeometric type is contained in the reference [**4**].

The original polynomials were not monic, but we prefer to deal with the monic polynomials, so we normalize them accordingly:

$$\bar{p}(x;a;b;q) = \sum_{j}^{n} \begin{bmatrix} n \\ j \end{bmatrix} \frac{(aq^{j+1};q)_{n-j}}{(bq^{n+j};q)_{n-j}} x^j q^{\binom{n-j}{2}} (-1)^{n-j}.$$

These polynomials are orthogonal with respect to a discrete distribution, consisting of weights ω_i at the points q^i, $i = 0, 1, 2, \ldots$, $0 < q < 1$, with

$$\omega_i = \frac{a^i q^i (q^{i+1};q)_\infty}{(bq^i/a;q)_\infty}.$$

The nth moment of this distribution is

$$\frac{(q;q)_\infty (bq;q)_\infty}{(b/a;q)_\infty (aq;q)_\infty} \cdot \frac{(aq;q)_n}{(bq;q)_n}.$$

We define a slightly modified moment by

$$\mu_n(a,b) = \frac{(aq;q)_n}{(bq;q)_n},$$

and

$$M_n(a,b) = \begin{vmatrix} \mu_0(a,b) & \mu_1(a,b) & \cdots & \mu_n(a,b) \\ \mu_1(a,b) & \mu_2(a,b) & \cdots & \mu_{n+1}(a,b) \\ \vdots & \vdots & \ddots & \vdots \\ \mu_n(a,b) & \mu_{n+1}(a,b) & \cdots & \mu_{2n}(a,b) \end{vmatrix}.$$

We may then represent the $\bar{p}$ in terms of these moments as follows:

$$\bar{p}_n(x;a;b;q) = \frac{1}{M_{n-1}(a,b)} \begin{vmatrix} \mu_0(a,b) & \mu_1(a,b) & \cdots & \mu_n(a,b) \\ \mu_1(a,b) & \mu_2(a,b) & \cdots & \mu_{n+1}(a,b) \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1}(a,b) & \mu_n(a,b) & \cdots & \mu_{2n-1}(a,b) \\ 1 & x & \cdots & x^n \end{vmatrix}.$$

We shall now apply our knowledge of the Little q-Jacobi polynomials, plus Theorems 1 and 2, to the special case

$$\frac{t}{\sum_{n=0}^\infty \frac{t^n (a;q)_n}{(b;q)_n} - 1} = \frac{1}{\sum_{n=1}^\infty \frac{t^n (a;q)_{n+1}}{(b;q)_{n+1}}} := \sum_{n=0}^\infty s_n t^n.$$

Thus

$$s_0 = \frac{1-b}{1-a},$$

and from the definition

$$\sum_{j=0}^n \mu_j(a,b) s_{N-j} = \begin{cases} s_0 & N = 0; \\ 0 & N > 0. \end{cases}$$

Thus, Theorems 1 and 2 are applicable with s_n, as above, and $t_n = \mu_n(a,b)$.

We apply Theorem 2 to obtain a useful representation of $\mathcal{S}_N(x)$:

$$\mathcal{S}_N(x) = \frac{1}{T_{N-1}} \begin{vmatrix} \mu_2(a,b) & \mu_3(a,b) & \cdots & \mu_{N+1}(a,b) \\ \mu_3(a,b) & \mu_4(a,b) & \cdots & \mu_{N+2}(a,b) \\ \vdots & \vdots & \ddots & \vdots \\ \mu_N(a,b) & \mu_{N+1}(a,b) & \cdots & \mu_{2N-1}(a,b) \\ \tau_1(x) & \tau_2(x) & \cdots & \tau_N(x) \end{vmatrix}$$

$$= \frac{(1-aq)^{N-1}(1-aq^2)^{N-1}}{T_{N-1}(1-bq)^{N-1}(1-bq^2)^{N-1}}$$

$$\times \begin{vmatrix} \mu_0(aq^2,bq^2) & \mu_1(aq^2,bq^2) & \cdots & \mu_{N-1}(aq^2,bq^2) \\ \mu_1(aq^2,bq^2) & \mu_2(aq^2,bq^2) & \cdots & \mu_N(aq^2,bq^2) \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{N-2}(aq^2,bq^2) & \mu_{N-1}(aq^2,bq^2) & \cdots & \mu_{2N-3}(aq^2,bq^2) \\ \tau_1(x) & \tau_2(x) & \cdots & \tau_N(x) \end{vmatrix}$$

$$= \frac{(1-aq)^{N-1}(1-aq^2)^{N-1}}{T_{N-1}(1-bq)^{N-1}(1-bq^2)^{N-1}} M_{N-2}(aq^2,bq^2)$$
$$\times \sum_{j=0}^{N-1} \begin{bmatrix} N-1 \\ j \end{bmatrix} \frac{(aq^{j+3};q)_{N-1-j}}{(bq^{N+1+j};q)_{N-1-j}} \tau_{j+1}(x)$$

$$\times\, q^{\binom{N-1-j}{2}} (-1)^{N-1-j}.$$

Finally, we note that

$$T_{N-1} = \begin{vmatrix} t_2 & t_3 & \cdots & t_N \\ t_3 & t_4 & \cdots & t_{N+1} \\ \vdots & \vdots & \ddots & \vdots \\ t_N & t_{N+1} & \cdots & t_{2N-2} \end{vmatrix}$$

$$= \begin{vmatrix} \mu_2(a,b) & \mu_3(a,b) & \cdots & \mu_N(a,b) \\ \mu_3(a,b) & \mu_4(a,b) & \cdots & \mu_{N+1}(a,b) \\ \vdots & \vdots & \ddots & \vdots \\ \mu_N(a,b) & \mu_{N+1}(a,b) & \cdots & \mu_{2N-2}(a,b) \end{vmatrix}$$
$$= \frac{(1-aq)^{N-1}(1-aq^2)^{N-1}}{(1-bq)^{N-1}(1-bq^2)^{N-1}} M_{N-2}(aq^2, bq^2).$$

We have proved

Theorem 3.

$$\mathcal{S}_N(x) = \sum_{j=0}^{N-1} \begin{bmatrix} N-1 \\ j \end{bmatrix} \frac{(aq^{j+3};q)_{N-1-j}}{(bq^{N+1+j};q)_{N-1-j}} \tau_{j+1}(x)$$
$$\times q^{\binom{N-1-j}{2}} (-1)^{N-1-j}, \quad N > 0,$$

where

$$\tau_n(x) = \sum_{s=0}^{n} \frac{(aq;q)_{n-s}}{(bq;q)_{n-s}} x^s.$$

Note. Recall our convention $\mathcal{S}_0(x) = 1$.

It follows from the work in [**4**] and the definition in Theorem 2 that the polynomials $\mathcal{S}_N(x)$ are orthogonal with respect to a distribution, namely, the one giving the moments s_n. This distribution will be complex. In fact, the polynomials are orthogonal with respect to the distribution defined by

$$L(h(z)) = \frac{1}{2\pi i} \oint h(z) \left(\frac{f(1/z)}{z} \right) dz,$$

where

$$f(t) = \frac{1}{\sum_{n=0}^{\infty} \frac{t^n (a;q)_{n+1}}{(b;q)_{n+1}}},$$

and the path of integration is a simple closed curve encircling the origin which lies outside all singularities of the integrand. Thus the polynomials satisfy a three-term recurrence relation,

$$\mathcal{S}_{N+1}(x) = (x + B_N)\mathcal{S}_N(x) - C_N S_{N-1}(x).$$

Our goal is to compute B_N and C_N. In order to do this, we must compute α_N and β_N in

$$\mathcal{S}_N(x) = x^N + \alpha_N x^{N-1} + \beta_N x^{N-2} + \cdots .$$

Comparing coefficients of x^N gives

$$B_N = \alpha_{N+1} - \alpha_N,$$

and comparing coefficients of x^{N-1} in the recurrence gives

$$\beta_{N+1} = \beta_N + \beta_N \alpha_N - C_N,$$

so

$$C_N = \beta_N - \beta_{N+1} + \beta_N \alpha_N.$$

We have

$$\begin{aligned}
\mathcal{S}_N(x) &= \tau_N(x) - \begin{bmatrix} N-1 \\ N-2 \end{bmatrix} \frac{(1-aq^{N+1})}{(1-bq^{2N-1})} \tau_{N-1}(x) \\
&\quad + \begin{bmatrix} N-1 \\ N-3 \end{bmatrix} \frac{(1-aq^N)(1-aq^{N+1})}{(1-bq^{2N-2})(1-bq^{2N-1})} \tau_{N-2}(x) q + \cdots \\
&= \left\{ x^N + \frac{(1-aq)}{(1-bq)} x^{N-1} + \frac{(1-aq)}{(1-bq)} \frac{(1-aq^2)}{(1-bq^2)} x^{N-2} + \cdots \right\} \\
&\quad - \begin{bmatrix} N-1 \\ 1 \end{bmatrix} \frac{(1-aq^{N+1})}{(1-bq^{2N-1})} \left\{ x^{N-1} + \frac{(1-aq)}{(1-bq)} x^{N-2} + \cdots \right\} \\
&\quad + \begin{bmatrix} N-1 \\ 2 \end{bmatrix} \frac{(1-aq^N)}{(1-bq^{2N-2})} \frac{(1-aq^{N+1})q}{(1-bq^{2N-1})} x^{N-2} + \cdots ,
\end{aligned}$$

thus

$$\alpha_N = \frac{(1-aq)}{(1-bq)} - \begin{bmatrix} N-1 \\ 1 \end{bmatrix} \frac{(1-aq^{N+1})}{(1-bq^{2N-1})}.$$

Returning to our expansion immediately above for $\mathcal{S}_N(x)$, we see that

$$\begin{aligned}
\beta_N &= \frac{(1-aq)(1-aq^2)}{(1-bq)(1-bq^2)} - \begin{bmatrix} N-1 \\ 1 \end{bmatrix} \frac{(1-aq)(1-aq^{N+1})}{(1-bq)(1-bq^{2N-1})} \\
&\quad + \begin{bmatrix} N-1 \\ 2 \end{bmatrix} \frac{(1-aq^N)(1-aq^{N+1})q}{(1-bq^{2N-2})(1-bq^{2N-1})}.
\end{aligned}$$

Using these expressions in the formulas for B_N and C_N and doing some fearsome algebraic rearranging and simplification give

Theorem 4. *In the three-term recurrence*

$$\mathcal{S}_{N+1}(x) = (x + B_N)\mathcal{S}_N(x) - C_N\mathcal{S}_{N-1}(x),$$

we have

$$\begin{aligned}
B_N &= \frac{-q^{N-1}(1-aq^{N+1})(1-bq^{N+1}) - q^2(a-bq^{N-2})(1-q^N)}{(1-bq^{2N-1})(1-bq^{2N+1})};\\
C_N &= \frac{q^{2N-1}(1-q^{N-1})(1-aq^{N+1})(1-bq^N)(a-bq^{N-2})}{(1-bq^{2N-2})(1-bq^{2N-1})^2(1-bq^{2N})}, \quad N > 1;\\
C_1 &= -\frac{(1-aq)(1-aq^2)}{(1-bq)(1-bq^2)}.
\end{aligned}$$

Note. $\mathcal{S}_N(x)$ gives an explicit evaluation of these polynomials. The formula for C_1 is obtained by a direct computation.

Corollary 1. *Let*

$$f(t) = \frac{t}{\sum_{n=0}^{\infty} \frac{t^n (a)_n}{(b)_n} - 1} = \frac{t}{{}_2F_1(a,1;b;t) - 1} = \sum_{n=0}^{\infty} s_n t^n,$$

where we have used the standard notation for the symbol $(A)_n$ and the Gaussian hypergeometric function ${}_2F_1$. Then the coefficients in the recurrence relation satisfied by the polynomials in Theorem 4 are given by

$$\begin{aligned}
B_N &= -\frac{(a+1)(b+1) + 2N(N+b)}{(b+2N-1)(b+2N+1)};\\
C_N &= \frac{(N-1)(a+N+1)(b+N)(N+b-a-2)}{(b+2N-2)(b+2N-1)^2(b+2N)}, \quad N > 1;\\
C_1 &= -\frac{(a+1)(a+2)}{(b+1)(b+2)}.
\end{aligned}$$

Proof. In Theorem 4 we let $a \to q^a$, $b \to q^b$, and take the limit as $q \to 1$.

Corollary 2. *Let*

$$f(t) = \frac{t}{{}_1F_1(1, \nu+1; t) - 1} = \sum_{n=0}^{\infty} s_n t^n.$$

Then the polynomials $\mathcal{S}_N(x)$ satisfy a three term recurrence with

$$\begin{aligned} B_N &= -\frac{(\nu+2)}{2(\nu+2N)(\nu+2N+2)}; \\ C_N &= -\frac{(N-1)(\nu+N+1)}{(\nu+2N-1)(\nu+2N)^2(\nu+2N+1)}, \quad N > 1; \\ C_1 &= -\frac{1}{(\nu+2)(\nu+3)}. \end{aligned}$$

Proof. In the previous corollary, we replace t by t/a and let $a \to \infty$. To obtain the formula for S_N, we use the fact that

$$|a^{-i-j} s_{i+j}|_{i,j=0\dots n} = a^{-n(n+1)} |s_{i+j}|_{i,j=0\dots n}.$$

Finally, we let $b = \nu + 1$. $\square$

Corollary 3. *Let*

$$f(t) = \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} s_n t^n,$$

so that

$$s_n = \frac{B_n}{n!},$$

the B_n denoting the Bernoulli numbers in the standard notation. Then the polynomials

$$\mathcal{S}_N(x) = \frac{1}{S_{N-1}} \begin{vmatrix} B_0 & B_1/1! & \cdots & B_N/N! \\ B_1/1! & B_2/2! & \cdots & B_{N+1}/(N+1)! \\ \vdots & \vdots & \ddots & \vdots \\ B_{N-1}/(N-1)! & B_N/N! & \cdots & B_{2N-1}/(2N-1)! \\ 1 & x & \cdots & x^N \end{vmatrix}$$

are orthogonal with respect to the distribution defined by

$$L(h(z)) = \oint h(z)\left(\frac{z^{-2}}{e^{1/z}-1}\right) dz,$$

the path of integration being a circle with center 0, radius $> (1/2\pi)$, *and the coefficients in the recurrence are given by*

$$B_N = -\frac{1}{2N(N+1)}; \quad C_N = -\frac{N^2-1}{4N^2(4N^2-1)}, \quad N > 1; \quad C_1 = -\frac{1}{6}.$$

Furthermore,

$$\begin{aligned} S_N &= \begin{vmatrix} B_0 & B_1/1! & \cdots & B_N/N! \\ B_1/1! & B_2/2! & \cdots & B_{N+1}/(N+1)! \\ \vdots & \vdots & \ddots & \vdots \\ B_N/N! & B_{N+1}/(N+1)! & \cdots & B_{2N}/(2N)! \end{vmatrix} \\ &= \left(\frac{1}{6}\right)^N (-1)^{N(N+1)/2} \sum_{j=2}^{N} \left[\frac{j^2-1}{4j^2(4j^2-1)}\right]^{N+1-j}. \end{aligned}$$

Proof. We put $\nu = 0$ in the previous corollary and use the formula for S_N in Section 1. □

We make no claims that the above expression for the Hankel determinant of Bernoulli numbers is new, since it seems that almost any fact about the Bernoulli numbers can be found somewhere in the literature.

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