ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 33, Number 2, Summer 2003

NEVANLINNA MATRICES FOR THE STRONG HAMBURGER MOMENT PROBLEM

O. NJÅSTAD

ABSTRACT. Let $\{c_n\}_{n=-\infty}^{\infty}$ be a doubly infinite sequence of real numbers. The strong Hamburger moment problem consists of finding positive measures σ on $\mathbf{R}$ such that $c_n = \int_{-\infty}^{\infty} t^n \, d\sigma(t)$ for $n = 0, \pm 1, \pm 2, \dots$. The problem is indeterminate if there is more than one solution. For an indeterminate problem there is a one-to-one correspondence between all Pick functions φ and all solutions σ of the moment problem, expressed by $\int_{-\infty}^{\infty} (z-t)^{-1} \, d\sigma(t) = [\alpha(z)\varphi(z) - \gamma(z)][\beta(z)\varphi(z) - \delta(z)]^{-1}$. The functions $\alpha, \beta, \gamma, \delta$ are holomorphic in the complex plane outside the origin. The purpose of this paper is to study growth properties of these functions $\alpha, \beta, \gamma, \delta$, analogous to properties of corresponding entire functions connected with the classical Hamburger moment problem.

1. Introduction. A solution of the classical *Hamburger moment problem* for a given sequence $\{c_n\}_{n=0}^{\infty}$ of real numbers is a positive measure σ on the real line $\mathbf{R}$ such that $c_n = \int_{-\infty}^{\infty} t^n \, d\sigma(t)$ for $n = 0, 1, 2, \dots$. A solvable moment problem is called *determinate* when there is a unique solution, *indeterminate* otherwise.

The *Stieltjes transform* of a finite measure σ is defined as the function

$$F(z, \sigma) = \int_{-\infty}^{\infty} \frac{d\sigma(t)}{z-t}. \tag{1.1}$$

The correspondence between measures and their Stieltjes transforms is one-to-one.

A function φ is called a *Pick function* (or *Nevanlinna function*) if it is holomorphic in the open upper half-plane $\mathbf{U}$ and maps $\mathbf{U}$ into $\mathbf{U} \cup \hat{\mathbf{R}}$ (where $\hat{\mathbf{R}}$ denotes the extended real line $\mathbf{R} \cup \{\infty\}$).

AMS *Mathematics Subject Classification.* 30D15, 30E05, 42C05, 44A60.
Key words and phrases. Strong moment problems, Nevanlinna parametrization, Nevanlinna matrices.
Received by the editors on September 20, 2001, and in revised form on April 2, 2002.

A Pick function is either a constant in $\hat{\mathbf{R}}$ (the function with the constant value ∞ is included) or a function mapping $\mathbf{U}$ into $\mathbf{U}$.

There is a one-to-one correspondence between the Pick functions φ and the solutions σ of the moment problem, given by

$$F(z,\sigma) = \frac{A(z)\varphi(z) - C(z)}{B(z)\varphi(z) - D(z)} \tag{1.2}$$

(*Nevanlinna parametrization* of the solutions of the Hamburger moment problem). Here A, B, C, D are entire transcendent functions of at most minimal type of order 1. That a function f has this property means that for every positive number ε there exists a constant M_ε such that

$$|f(z)| \leq M_\varepsilon \exp(\varepsilon|z|) \quad \text{for all } z \in \mathbf{C} \tag{1.3}$$

(where exp denotes the exponential function).

The matrix $\begin{bmatrix} A(z) & B(z) \\ C(z) & D(z) \end{bmatrix}$ is called a *Nevanlinna matrix*.

For detailed treatments of important aspects of the Hamburger moment problem, see, e.g., [**1**–**8, 10, 15**–**19, 25**–**28**].

The *strong Hamburger moment problem* is defined in the same way as the classical problem, except that a doubly infinite sequence $\{c_n\}_{n=-\infty}^{\infty}$ is given for which the equality $c_n = \int_{-\infty}^{\infty} t^n \, d\sigma(t)$ is required to hold. A theory in the main analogous to the classical theory has been developed for this moment problem up to the existence of a Nevanlinna parametrization for the solutions of an indeterminate problem, with functions $\alpha, \beta, \gamma, \delta$ holomorphic in $\mathbf{C} \setminus \{0\}$ replacing the functions A, B, C, D.

The aim of this paper is to study growth properties of the functions $\alpha, \beta, \gamma, \delta$ at the origin and at infinity, partly analogous to those at infinity of the functions A, B, C, D, expressed by (1.3).

In Section 2 we sketch the theory of orthogonal Laurent polynomials and their use in the study of strong moment problems, including the Nevanlinna parametrization of the solutions of an indeterminate strong Hamburger problem. Section 3 is devoted to the study of growth properties of $\alpha, \beta, \gamma, \delta$, inspired by (1.3). A crucial role is here played by a Riesz criterion for indeterminacy, proved in [**23**].

The organization and presentation of our material is strongly influenced by Akhiezer's [**1**] work on the classical moment problem. Other very instructive treatments of the classical moment problem can be found in the treatises by Riesz [**24**] and by Shohat and Tamarkin [**27**]. At some crucial points this classical approach has to be modified for the strong moment problem, and our final results are not quite as conclusive as in the classical case.

2. Orthogonal Laurent polynomials and the strong moment problem. For detailed treatments of the topics discussed in this section, see [**9, 11–14, 19–21, 23, 24**].

The linear space spanned by all monomials z^n, $n = 0, \pm 1, \pm 2, \dots$ is denoted by Λ, and the elements of Λ are called *Laurent polynomials.*

Let $\{c_n\}_{n=-\infty}^{\infty}$ be a doubly infinite sequence of real numbers, and let S be the linear functional on Λ defined by

$$S[z^n] = c_n \quad \text{for } n = 0, \pm 1, \pm 2, \dots . \tag{2.1}$$

A solution of the *strong Hamburger moment problem* is a positive measure σ on $\mathbf{R}$ such that

$$c_n = \int_{-\infty}^{\infty} t^n \, d\sigma(t) \quad \text{for } n = 0, \pm 1, \pm 2, \dots , \tag{2.2}$$

or equivalently such that $S[L] = \int_{-\infty}^{\infty} L(t)\, d\sigma(t)$ for all $L \in \Lambda$. A necessary and sufficient condition for the existence of at least one solution is that S is positive. *We shall in this paper assume that this condition is satisfied and thus that the moment problem is solvable.*

An inner product $\langle \cdot, \cdot \rangle$ is defined on Λ by

$$\langle f, g \rangle = S[f(z) \cdot \overline{g(z)}]. \tag{2.3}$$

By applying the Gram-Schmidt process to the basis $\{1, z^{-1}, z, z^{-2}, z^2, \dots\}$ an orthonormal sequence $\{\varphi_n\}_{n=0}^{\infty}$ is obtained. These orthogonal Laurent polynomials have the form

$$\varphi_{2m}(z) = \frac{u_{2m}}{z^m} + \cdots + v_{2m} z^m, \quad v_{2m} > 0 \tag{2.4}$$

$$\varphi_{2m+1}(z) = \frac{v_{2m+1}}{z^{m+1}} + \cdots + u_{2m+1} z^m, \quad v_{2m+1} > 0 \tag{2.5}$$

for $m = 0, 1, 2, \dots$. All the coefficients in φ_n are real.

The orthonormal Laurent polynomial φ_n is called *regular* if $u_n \neq 0$. Either φ_n or φ_{n+1} is always regular, hence there is always an infinite subsequence of $\{\varphi_n\}$ consisting of regular elements. *For simplicity we shall in the following assume that all the φ_n are regular.*

The *associated orthogonal Laurent polynomials* $\{\psi_n\}$ are defined by

$$\psi_n(z) = S\left[\frac{\varphi_n(t) - \varphi_n(z)}{t - z}\right] \quad \text{for } n = 0, 1, 2, \dots . \tag{2.6}$$

(The functional is applied to its argument as a function of t.) The coefficients in the Laurent polynomial ψ_n are all real.

Let x_0 be an arbitrary fixed point in $\mathbf{R} \setminus \{0\}$. We define functions $\alpha_n, \beta_n, \gamma_n, \delta_n$ (depending on x_0) by

$$\alpha_n(z) = (z - x_0) \sum_{k=0}^{n-1} \psi_k(x_0)\psi_k(z) \tag{2.7}$$

$$\beta_n(z) = -1 + (z - x_0) \sum_{k=0}^{n-1} \psi_k(x_0)\varphi_k(z) \tag{2.8}$$

$$\gamma_n(z) = 1 + (z - x_0) \sum_{k=0}^{n-1} \varphi_k(x_0)\psi_k(z) \tag{2.9}$$

$$\delta_n(z) = (z - x_0) \sum_{k=0}^{n-1} \varphi_k(x_0)\varphi_k(z). \tag{2.10}$$

These functions are Laurent polynomials with real coefficients. By utilizing Christoffel-Darboux type formulas for orthogonal Laurent polynomials (see e.g. [**13, 16, 25, 31**]) we find that

$$\alpha_n(z)\delta_n(z) - \beta_n(z)\gamma_n(z) = 1 \tag{2.11}$$

for all $z \in \mathbf{C} \setminus \{0\}$.

Remark. In the definition of analogous functions A_n, B_n, C_n, D_n connected with the classical moment problem, the value of x_0 is usually taken to be 0, but any other value in $\mathbf{R}$ may also be used. In the definition of $\alpha_n, \beta_n, \gamma_n, \delta_n$, the value $x_0 = 0$ cannot be used since the functions φ_n and ψ_n are singular at the origin.

For $z \notin \mathbf{R}$ we define the linear fractional transformation $t \to T_n(z,t)$ by

$$T_n(z,t) = \frac{\alpha_n(z)t - \gamma_n(z)}{\beta_n(z)t - \delta_n(z)}. \tag{2.12}$$

This transformation maps the closed upper half-plane $\mathbf{U} \cup \hat{\mathbf{R}}$ onto a closed disk $\Delta_n(z)$ and the extended real line onto the boundary $\partial\Delta_n(z)$. The sequence $\{\Delta_n(z)\}$ is *nested*, i.e., $\Delta_{n+1}(z) \subset \Delta_n(z)$ for all n. The radius $r_n(z)$ of $\Delta_n(z)$ is given by

$$r_n(z) = \frac{1}{|z-\bar{z}|\omega_n(z)}, \quad \text{where } \omega_n(z) = \sum_{k=0}^{n-1} |\varphi_k(z)|^2. \tag{2.13}$$

For $t \in \hat{\mathbf{R}}$ all the zeros of the numerator and the denominator of $T_n(z,t)$ are real and simple, and $T_n(z,t)$ has a partial fraction decomposition of the form

$$T_n(z,t) = \sum_{k=1}^{n} \frac{\lambda_{n,k}(t)}{z - \xi_{n,k}(t)} \tag{2.14}$$

with $\xi_{n,k}(t)$ real, $\lambda_{n,k}(t) > 0$ for $k = 1, \dots, n$ and $\lambda_{n,1}(t) + \cdots + \lambda_{n,n}(t) = c_0$.

We define $\Delta_\infty(z) = \cap_{n=1}^\infty \Delta_n(z)$. Then $\Delta_\infty(z)$ is either a single point for every $z \notin \mathbf{R}$ (the *limit point case*) or a proper closed disk for every $z \notin \mathbf{R}$ (the *limit circle case*). Furthermore, $\Delta_\infty(z)$ consists of exactly all values $F(z,\sigma)$, where σ is a solution of the moment problem. The moment problem is *determinate*, i.e., has exactly one solution, in the limit point case, *indeterminate*, i.e., has more than one solution, in the limit circle case. The radius $r(z)$ of $\Delta_\infty(z)$ is given by

$$r(z) = \frac{1}{|z-\bar{z}|\omega(z)}, \quad \text{where } \omega(z) = \sum_{n=0}^{\infty} |\varphi_n(z)|^2. \tag{2.15}$$

We formulate as a theorem some basic results for indeterminate problems.

Theorem 2.1. *For an indeterminate moment problem the following hold*:

A. *The series* $\sum_{n=0}^{\infty} |\varphi_n(z)|^2$ *and* $\sum_{n=0}^{\infty} |\psi_n(z)|^2$ *converge locally uniformly in* $\mathbf{C} \setminus \{0\}$.

B. *The sequence* $\{\alpha_n(z)\}$, $\{\beta_n(z)\}$, $\{\gamma_n(z)\}$, $\{\delta_n(z)\}$ *converge locally uniformly in* $\mathbf{C} \setminus \{0\}$ *to functions* $\alpha(z), \beta(z), \gamma(z), \delta(z)$ *which are holomorphic in* $\mathbf{C} \setminus \{0\}$ *and satisfy*

$$\alpha(z)\delta(z) - \beta(z)\gamma(z) = 1. \tag{2.16}$$

Clearly the disk $\Delta_\infty(z)$ can be represented as

$$\Delta_\infty(z) = \left\{ w = \frac{\alpha(z)t - \gamma(z)}{\beta(z)t - \delta(z)} : t \in \mathbf{U} \cup \hat{\mathbf{R}} \right\}. \tag{2.17}$$

Of crucial importance in the following discussion is a Riesz criterion for indeterminate problems (see [**23**]).

Theorem 2.2. *For an indeterminate moment problem, the following inequality holds*:

$$\int_{-\infty}^{\infty} \frac{\ln \omega(t)}{1+t^2}\, dt < \infty. \tag{2.18}$$

The *Nevanlinna parametrization of the strong Hamburger moment problem* can be stated as follows:

Theorem 2.3. *For an indeterminate moment problem, there exists, for a given* x_0, *a one-to-one correspondence between all Pick functions* φ *and all solutions* σ *of the moment problem. The correspondence is given by*

$$F(z, \sigma) = \frac{\alpha(z)\varphi(z) - \gamma(z)}{\beta(z)\varphi(z) - \delta(z)}. \tag{2.19}$$

In analogy with the classical case, we may call $\left[\begin{smallmatrix} \alpha(z) & \beta(z) \\ \gamma(z) & \delta(z) \end{smallmatrix}\right]$ a *Nevanlinna matrix for the strong moment problem.*

3. Growth estimates. *We assume in the whole of this section that the moment problem under discussion is indeterminate.* Our main purpose is to study growth properties of the functions $\alpha, \beta, \gamma, \delta$ and ω. We shall first discuss relationships between the functions ω_n, ω (see (2.12) and (2.14)) on the one hand and the functions $\alpha_n, \beta_n, \gamma_n, \delta_n, \alpha, \beta, \gamma, \delta$ (see (2.6)–(2.9)) on the other hand. Note that the functions $\alpha, \beta, \gamma, \delta$ can be written as

$$\alpha(z) = (z - x_0) \sum_{n=0}^{\infty} \psi_n(x_0)\psi_n(z) \tag{3.1}$$

$$\beta(z) = -1 + (z - x_0) \sum_{n=0}^{\infty} \psi_n(x_0)\varphi_n(z) \tag{3.2}$$

$$\gamma(z) = 1 + (z - x_0) \sum_{n=0}^{\infty} \varphi_n(x_0)\psi_n(z) \tag{3.3}$$

$$\delta(z) = (z - x_0) \sum_{n=0}^{\infty} \varphi_n(x_0)\varphi_n(z). \tag{3.4}$$

As usual, let $x = \operatorname{Re} z$, $y = \operatorname{Im} z$.

Lemma 3.1. *For every $z \in \mathbf{C} \setminus \mathbf{R}$, we have*

$$\omega_n(z) \le \frac{|\beta_n(z)\delta_n(z)|}{|y|}. \tag{3.5}$$

Proof. According to (2.12), both of the points $\alpha_n(z)/\beta_n(z)$ and $\gamma_n(z)/\delta_n(z)$ belong to the disk $\Delta_n(z)$. The diameter of the disk is $1/(|y|\omega_n(z))$ by (2.13) and so $|\alpha_n(z)/\beta_n(z) - \gamma_n(z)/\delta_n(z)| \le 1/|y|\omega_n(z)$. Taking into account (2.11) we then get (3.5). $\square$

Lemma 3.2. *For every $z \in \mathbf{C} \setminus \mathbf{R}$, we have*

$$\left|\frac{\alpha_n(z)}{\beta_n(z)}\right| \le \frac{c_0}{|y|}, \quad \left|\frac{\gamma_n(z)}{\delta_n(z)}\right| \le \frac{c_0}{|y|}. \tag{3.6}$$

Proof. The quotient $\gamma_n(z)/\delta_n(z)$ is obtained from (2.12) for $t = 0$. It follows from (2.14) that

$$\left|\frac{\gamma_n(z)}{\delta_n(z)}\right| \leq \sum_{k=1}^{n} \frac{\lambda_{n,k}(0)}{\sqrt{(x-\xi_{n,k}(0))^2+y^2}}\,,$$

hence

$$\left|\frac{\gamma_n(z)}{\delta_n(z)}\right| \leq \sum_{k=1}^{n} \frac{\lambda_{n,k}(0)}{|y|} = \frac{c_0}{|y|},$$

from which the second inequality of (3.6) follow. By a similar argument, we obtain the first inequality for $t = \infty$. □

Proposition 3.3. *There exists a constant M independent of n such that*

(3.7)
$$|h_n(z)| \leq 1 + M|z-x_0|\sqrt{\omega(z)} \quad \textit{for } z \in \mathbf{C} \setminus \{0\}$$

(3.8)
$$|g_n(z)| \leq \frac{c_0}{|y|}[1 + M|z-x_0|\sqrt{\omega(z)}\,] \quad \textit{for } z \in \mathbf{C} \setminus \{0\}$$

where g_n is any of the functions α_n, γ_n and h_n is any of the functions β_n, δ_n.

Proof. It follows from the definition (2.8), (2.10), together with Schwartz's inequality that

(3.9)
$$|\beta_n(z)| \leq 1 + |z-x_0|\left[\sum_{k=0}^{\infty}|\psi_k(x_0)|^2\right]^{1/2}\left[\sum_{k=0}^{\infty}|\varphi_k(z)|^2\right]^{1/2},$$

(3.10)
$$|\delta_n(z)| \leq 1 + |z-x_0|\left[\sum_{k=0}^{\infty}|\varphi_k(x_0)|^2\right]^{1/2}\left[\sum_{k=0}^{\infty}|\varphi_k(z)|^2\right]^{1/2}.$$

We then conclude from Theorem 2.1 and the definition in (2.15) that (3.7) is satisfied. The inequality (3.8) follows from (3.7) and Lemma 3.2. □

We define the function φ by

$$\varphi(\xi) = \ln[1 + M|\xi - x_0|\sqrt{\omega(\xi)}\,] \quad \text{for } \xi \in \mathbf{R} \setminus \{0\} \tag{3.11}$$

where M is a constant as given in Proposition 3.3. We note that $\varphi(\xi) \geq 0$ for all ξ.

Lemma 3.4. *The following inequality holds*:

$$\int_{-\infty}^{\infty} \frac{\varphi(\xi)\, d\xi}{(\xi - x)^2 + y^2} < \infty \quad \text{for } x \in \mathbf{R},\ y \in \mathbf{R} \setminus \{0\}. \tag{3.12}$$

Proof. We have $\varphi(\xi) \leq \ln^+ \big[2 \cdot \max\{1, M|\xi - x_0|\sqrt{\omega(\xi)}\,\}\big]$. Since

$$\int_{-\infty}^{\infty} \frac{\ln[M]|\xi - x_0|\sqrt{\omega(\xi)}\,]}{1 + \xi^2}\, d\xi < \infty$$

by Riesz's criterion (Theorem 2.2), it follows that $\int_{-\infty}^{\infty} \varphi(\xi)/(1 + \xi^2)\, d\xi < \infty$. Furthermore, $(\xi - x)^2 + y^2 = y^2\big[1 + ((\xi - x)/y)^2\big]$, from which we see that, for fixed x and y, $y \neq 0$, also the inequality in (3.12) holds. $\square$

Proposition 3.5. *The following inequality holds for $x \in \mathbf{R}$, $y \in \mathbf{R} \setminus \{0\}$*:

$$\ln|h_n(x + iy)| \leq \frac{|y|}{\pi} \int_{-\infty}^{\infty} \frac{\varphi(\xi)\, d\xi}{(\xi - x)^2 + y^2} \tag{3.13}$$

where h_n is any of the functions β_n, δ_n.

Proof. The function h_n is a Laurent polynomial with only real zeros (cf. the remark following the formula (2.13)). We may therefore write $h_n(z) = H_n(z)/z^p$, where H_n is a polynomial with real zeros and p is a natural number. Poisson's formula gives

$$\ln|H_n(x + iy)| = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{\ln|H_n(\xi)|\, d\xi}{(\xi - x)^2 + y^2} \tag{3.14}$$

and

$$\ln(|x+iy|^p) = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{\ln(|\xi|^p)\,d\xi}{(\xi-x)^2+y^2} \tag{3.15}$$

from which it follows

$$\ln|h_n(x+iy)| = \frac{|y|}{\pi}\int_{-\infty}^{\infty}\frac{\ln|h_n(\xi)|\,d\xi}{(\xi-x)^2+y^2}. \tag{3.16}$$

The inequality (3.13) now follows from Proposition 3.3, the definition (3.11) and formula (3.16). □

We introduce the following notation

$$\Omega_\eta(r) = \{z \in \mathbf{C} : \eta \le |\arg z| \le \pi-\eta, |z| \ge r\} \tag{3.17}$$

where η and r are arbitrary positive numbers, $\eta < \pi/2$, and

$$\Omega_\eta = \{z \in \mathbf{C} : \eta \le |\arg z| \le \pi-\eta,\ |z| > 0\} \tag{3.18}$$

where η is an arbitrary positive number $\eta < \pi/2$. We observe that

$$(\xi-x)^2+y^2 = |z-\xi|^2 \ge \xi^2\sin^2\eta \ge \frac{1+\xi^2}{2}\sin^2\eta \tag{3.19}$$

for $|\xi| \ge 1$, ξ real and $z \in \Omega_\eta$.

The result of the following Proposition 3.6 is crucial. We give a detailed argument along the same lines as the argument given by Akhiezer for the classical case. It could also be obtained by essentially arguing from Proposition 3.5 and general results on estimations of a positive harmonic function (the Poisson integral of φ) in sectors in the upper half plane.

Proposition 3.6. *For fixed numbers ε and η, with $\varepsilon > 0$ and $0 < \eta < \pi/2$, there exists a constant $B(\eta,\varepsilon)$, independent of n such that*

$$|h_n(z)| \le B(\eta,\varepsilon)\exp\left[\varepsilon(|z|+|z|^{-1})\right] \quad \textit{for } z \in \Omega_\eta, \tag{3.20}$$

where h_n is any of the functions β_n, δ_n.

Proof. Let η and ε be fixed numbers as specified. It follows from (3.19) that for $T > 1$ we have

$$\frac{1}{\pi}\int_{|\xi|\geq T}\frac{\varphi(\xi)\,d\xi}{(\xi-x)^2+y^2}\leq\frac{2}{\pi\sin^2\eta}\int_{|\xi|\geq T}\frac{\varphi(\xi)\,d\xi}{1+\xi^2}.$$

Since the integral $\int_{-\infty}^{\infty}\varphi(\xi)\,d\xi/(1+\xi^2)$ converges by Lemma 3.4, we can make the righthand side of this inequality arbitrarily small by choosing the value of T sufficiently large. For a fixed T, we can again by Lemma 3.4 make the value of $\int_{-T}^{T}\varphi(\xi)\,d\xi/((\xi-x)^2+y^2)$ arbitrarily small by choosing $|y|$ sufficiently large. We conclude from Proposition 3.5 that $\ln|h_n(x+iy)|\leq\varepsilon|y|$ for $z\in\Omega_\eta$ and $|y|$ sufficiently large. It follows that, for every positive r there exists a constant $k(\eta,\varepsilon,r)$ such that

$$\ln|h_n(x+iy)|\leq\varepsilon|y|+k(\eta,\varepsilon,r)\quad\text{for } z\in\Omega_\eta(r). \tag{3.21}$$

Clearly (by Lemma 3.4) $\int_{-T}^{T}\varphi(\xi)\,d\xi<\infty$ for every finite T. Since $y^2/((\xi-x)^2+y^2)\leq 1$ we can then make the integral $\int_{-T}^{T}(y^2\varphi(\xi)\,d\xi)/((\xi-x)^2+y^2)$ arbitrarily small by choosing the value of T sufficiently small. Furthermore, by (3.19), we have

$$\frac{1}{\pi}\int_{|\xi|\geq 1}\frac{y^2\varphi(\xi)\,d\xi}{(\xi-x)^2+y^2}\leq\frac{2}{\pi}\int_{|\xi|\geq 1}\frac{y^2\varphi(\xi)\,d\xi}{(1+\xi^2)\sin^2\eta}$$

for $z\in\Omega_\eta$. By Lemma 3.4 we can make $\int_{|\xi|\geq 1}(y^2\varphi(\xi)\,d\xi)/((1+\xi^2)\sin^2\eta)$ arbitrarily small by choosing $|y|$ sufficiently small. By Lebesgue's dominated convergence theorem, we can for arbitrary $T\in(0,1)$ make the value of $\int_{T\leq|\xi|\leq 1}(y^2\varphi(\xi)\,d\xi)/((\xi-x)^2+y^2)$ arbitrarily small by choosing $|y|$ sufficiently small. We conclude from Proposition 3.5 that $\ln|h_n(x+iy)|\leq\varepsilon/|y|\sin\eta$ for $z\in\Omega_\eta$ and $|y|$ sufficiently small. It follows that, for every positive r, there exists a constant $K(\eta,\varepsilon,r)$ such that

$$\ln|h_n(x+iy)|\leq\frac{\varepsilon\sin\eta}{|y|}+K(\eta,\varepsilon,r)\quad\text{for } z\in\Omega_\eta\setminus\Omega_\eta(r). \tag{3.22}$$

We conclude from (3.21)–(3.22) that there exists a constant $C(\eta,\varepsilon)$ such that $\ln|h_n(x+iy)| \le \varepsilon\big[|y| + (\sin\eta)/|y|\big] + C(\eta,\varepsilon)$ for $z \in \Omega_\eta$ and hence a constant $B(\eta,\eta)$ such that

$$|h_n(x+iy)| \le B(\eta,\varepsilon)\exp\left[\left(|y| + \frac{\sin\eta}{|y|}\right)\right] \quad \text{for } z \in \Omega_\eta. \tag{3.23}$$

Since $|y| \le |z|$ and $|y| \ge |z|\sin\eta$ for $z \in \Omega_\eta$, the inequality (3.20) follows. □

Our main results may now be formulated as follows.

Theorem 3.7. *For fixed numbers ε and η, with $\varepsilon > 0$ and $0 < \eta < \pi/2$, there exists a constant $M(\eta,\varepsilon)$ such that*

$$|F(z)| \le M(\eta,\varepsilon)\exp\big[\varepsilon(|z| + |z|^{-1})\big] \quad \textit{for } z \in \Omega_\eta, \tag{3.24}$$

where F is any of the functions $\alpha, \beta, \gamma, \delta$ and ω.

Proof. We conclude from Lemma 3.2 and Proposition 3.6 that

$$|g_n(z)| \le c_0 B\left(\eta, \frac{\varepsilon}{2}\right)\frac{1}{|z|\sin\eta}\exp\left[\frac{\varepsilon}{2}(|z| + |z|^{-1})\right] \quad \text{for } z \in \Omega_\eta, \tag{3.25}$$

where g_n is any of the functions α_n, γ_n. For each positive number w there exists a constant $E(w)$ such that $t \le E(w)e^{wt}$ for all $t \ge 0$. With this notation we get from (3.25) that

$$|g_n(z)| \le \frac{c_0}{\sin\eta}E\left(\frac{\varepsilon}{2}\right)B\left(\eta, \frac{\varepsilon}{2}\right)\exp\big[\varepsilon(|z| + |z|^{-1})\big] \quad \text{for } z \in \Omega_\eta. \tag{3.26}$$

From Lemma 3.1 and Proposition 3.6 we conclude that

$$\omega_n(z) \le \frac{1}{\sin\eta}E\left(\frac{\varepsilon}{2}\right)B\left(\eta, \frac{\varepsilon}{2}\right)^2\exp\big[\varepsilon(|z| + |z|^{-1})\big] \quad \text{for } z \in \Omega_\eta. \tag{3.27}$$

Since the constants $B(\eta,\varepsilon)$ and $E(\varepsilon)$ are independent of n, the desired result now follows from (3.20), (3.26) and (3.27). □

Remark. We easily see that an inequality $|F(z)| \leq M \exp\left[\varepsilon(|z| + |z|^{-1})\right]$ is equivalent to two inequalities $|F(z)| \leq M_1 \exp[\varepsilon|z|]$ and $|F(z)| \leq M_2 \exp[\varepsilon|z|^{-1}]$. It may therefore be natural to express the statements of Theorem 3.7 concerning $\alpha, \beta, \gamma, \delta$ by saying that these functions (which are holomorphic in $\mathbf{C} \setminus \{0\}$) are at most minimal type of order 1 at infinity and at the origin, though only in all angular regions Ω_η.

If we knew that $|F_n(x+iy)|$ were increasing functions of $|y|$ for F_n equal to $\alpha_n, \beta_n, \gamma_n, \delta_n, \omega_n$, as in the classical situation, we could infer the validity of (3.24) in the whole of $\mathbf{C} \setminus \{0\}$ for F equal to $\alpha, \beta, \gamma, \delta$ and ω. Building on results presented in this paper, it is shown in [**22**] that if the *strong Stieltjes moment problem* is solvable, i.e., if the strong Hamburger moment problem has a solution with support on the nonnegative real axis, then (3.24) is valid for every region given by $|\arg z| \leq \pi - \eta$, $\eta > 0$.

REFERENCES

1. N.I. Akhiezer, *The classical moment problem and some related questions in analysis*, Hafner, New York, 1965.

2. C. Berg, *Indeterminate moment problems and the theory of entire functions*, J. Comp. Appl. Math. **65** (1995), 27–55.

3. C. Berg and H.L. Pedersen, *On the order and type of the entire functions associated with an indeterminate Hamburger moment problem*, Ark. Mat. **32** (1994), 1–11.

4. ———, *Nevanlinna matrices of entire functions*, Math. Nachr. **171** (1995), 29–52.

5. H. Buchwalter and G. Cassier, *La paramétrisation de Nevanlinna dans le problème des moments de Hamburger*, Exposition. Math. **2** (1984), 155–178.

6. T.S. Chihara, *On indeterminate Hamburger moment problems*, Pacific J. Math. **27** (1968), 475–484.

7. ———, *An introduction to orthogonal polynomials*, Gordon and Breach, New York, 1978.

8. T.S. Chihara and M.E.H. Ismail, *Extremal measures for a system of orthogonal polynomials*, Constructive Approx. **9** (1993), 111–119.

9. L. Cochran and S. Clement Cooper, *Orthogonal Laurent polynomials on the real line*, in *Continued fractions and orthogonal functions* (S. Clement Cooper and W.J. Thron, eds.), Marcel Dekker, New York, 1994, pp. 47–100.

10. H. Hamburger, *Über eine Erweiterung des Stieltjesschen Momentproblems*, Parts I, II, III, Math. Ann. **81** (1920), 235–319; **82** (1921), 120–164; **82** (1921), 168–187.

11. E. Hendriksen and H. van Rossum, *Orthogonal Laurent polynomials*, Indag. Math. **89** (1986), 17–36.

12. W.B. Jones and O. Njåstad, *Orthogonal Laurent polynomials and strong moment theory*: *A survey*, J. Comp. Appl. Math. **105** (1999), 51–91.

13. W.B. Jones, O. Njåstad and W.J. Thron, *Orthogonal Laurent polynomials and the strong Hamburger moment problem*, J. Math. Anal. Appl. **98** (1984), 528–554.

14. W.B. Jones and G. Shen, *Regular strong Hamburger moment problems*, in *Orthogonal functions, moment theory and continued fractions*: *Theory and applications* (W.B. Jones and A. Sri Ranga, eds.), Marcel Dekker, New York, 1998, pp. 209–256.

15. T.H. Kjeldsen, *The early history of the moment problem*, Historia Math. **20** (1993), 19–44.

16. H.J. Landau, *The classical moment problem*: *Hilbertian proofs*, J. Funct. Anal. **38** (1980), 255–272.

17. ———, ed., *Moments in mathematics*, Proc. Symposium Appl. Math., vol. 37, Amer. Math. Society, Providence, RI, 1987.

18. R. Nevanlinna, *Über beschränkte analytische Funktionen*, Ann. Acad. Sci. Fenn. A **32** (1929), 1–75.

19. O. Njåstad, *Classical and strong moment problems*, Comm. Analytic Theory Continued Fractions **4** (1995), 4–38.

20. ———, *Solutions of the strong Hamburger moment problem*, J. Math. Anal. Appl. **197** (1996), 227–248.

21. ———, *Remarks on canonical solutions of strong moment problems*, in *Orthogonal functions, moment theory and continued fractions*: *Theory and applications* (W.B. Jones and A. Sri Ranga, eds.), Marcel Dekker, New York, (1998), pp. 359–374.

22. ———, *Nevanlinna matrices for strong Stieltjes moment problems*, J. Comp. Appl. Math. **99** (1998), 311–318.

23. ———, *Riesz criterion for the strong Hamburger moment problem*, Comm. Analytic Theory Continued Fractions **7** (1999), 34–50.

24. O. Njåstad and W.J. Thron, *Unique solvability of the strong Hamburger moment problem*, J. Austral. Math. Soc. Ser. A **40** (1986), 5–19.

25. H.L. Pedersen, *The Nevanlinna matrix of entire functions associated with a shifted indeterminate Hamburger moment problem*, Math. Scand. **74** (1994), 152–160.

26. M. Riesz, *Sur le problème des moments*, I, II, III, Arkiv för Matematik, Astronomi och Fysik **16** (12) (1922); **16** (19) (1922); **17** (16) (1923).

27. J.A. Shohat and J.D. Tamarkin, *The problem of moments*, Math. Surveys **1**, Amer. Math. Soc., Providence, RI, 1943.

28. M.H. Stone, *Linear transformations in Hilbert space and their applications to analysis*, Amer. Math. Soc. Colloq. Publ. **15**, New York, 1932.

Department of Mathematical Sciences, Norwegian University of Science and Technology, N-7491, Trondheim, Norway
E-mail address: njastad@math.ntnu.no