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SOME REMARKS ON RELATIVE TOR AND REGULAR HOMOMORPHISMS

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Let $A \to B$ be a flat homomorphism of (always commutative in this paper) Noetherian rings, M a $B \otimes_A B$-module, and let $HH(B, M)$ denote its Hochschild homology [**6**], [**2**]. In [**12**], it is proved that if the Hochschild dimension of B is finite (i.e., if $HH_n(B, -) = 0$ for all $n \gg 0$), then the homomorphism $A \to B$ has geometrically regular fibers and, therefore, since it is flat, it is a regular homomorphism. In this paper we show that the flatness hypothesis can be dropped in the sense that the finiteness of Hochschild dimension suffices to imply that the fibers are geometrically regular.

More precisely, if $A \to B$ is not flat, we have several different definitions of Hochschild homology which are all equivalent in the flat case (see, e.g., [**14**]). We choose the one in the paper by Hochschild [**6**] which can be thought of as a relative Tor [**7**]: $\mathrm{Tor}^{B \otimes_A B|A}(B, M)$, where B is considered as a $B \otimes_A B$-module via the multiplication map. We shall show that if $\mathrm{Tor}_n^{B \otimes_A B|A}(B, -) = 0$ for all $n \gg 0$, then the fibers of $A \to B$ are geometrically regular.

We deduce this result from an elementary (nonflat) base change lemma for relative Tor, and some classical results also used in [**12**]. We also give some examples of how this base change lemma allows us to obtain relative Tor characterizations of smooth and complete intersection homomorphisms in some cases.

First we recall the definitions of relative Tor [**7**]. Let $R \to S$ be a ring homomorphism. An S-module P is $S|R$-projective if for every epimorphism of S-modules $f : M \to N$ which is R-split, and every homomorphism of S-modules $g : P \to N$, there is an S-module homomorphism $h : P \to M$ such that $fh = g$. An exact sequence of S-module homomorphisms $\dots \to P_1 \to P_0 \to M \to 0$ is called R-split if it has a contracting homotopy as an exact sequence of R-modules.

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An $S|R$-projective resolution of an S-module M is an exact sequence $\ldots \to P_1 \to P_0 \to M \to 0$ of S-modules which is R-split and where P_i is $S|R$-projective for all $i \geq 0$. We denote $\mathbf{P} = \ldots \to P_1 \to P_0$. $S|R$-projective resolutions exist and they are unique up to S-homotopy [**7**, Section 2], so we can define the relative Tor functor as follows.

Let M, N be S-modules, $\mathbf{P} \to M \to 0$ an $S|R$-projective resolution of M. We define $\operatorname{Tor}_n^{S|R}(M,N) := H_n(\mathbf{P} \otimes_S N)$. Note that if R is a field, then $\operatorname{Tor}_n^{S|R}(M,N) = \operatorname{Tor}_n^S(M,N)$.

Lemma 1. *Let $R \to S$, $R \to T$ be ring homomorphisms, M an S-module, N an $S \otimes_R T$-module. Then*

$$\operatorname{Tor}_n^{S|R}(M,N) = \operatorname{Tor}_n^{S\otimes_R T|T}(M \otimes_R T, N)$$

for all $n \geq 0$. In particular, if $R \to T$ is surjective and M is also an $S \otimes_R T$-module, then

$$\operatorname{Tor}_n^{S|R}(M,N) = \operatorname{Tor}_n^{S\otimes_R T|T}(M,N).$$

Proof. Let $\mathbf{P}$ be an $S|R$-projective resolution of M, and let $\mathbf{Q} = \mathbf{P} \otimes_R T$. From the isomorphisms of $S \otimes_R T$-module functors $\operatorname{Hom}_{S\otimes_R T}(P_i \otimes_R T, -) = \operatorname{Hom}_S(P_i, -)$ we deduce easily that each $Q_i = P_i \otimes_R T$ is $S \otimes_R T|T$-projective. Moreover, since $\ldots \to P_1 \to P_0 \to M \to 0$ is a homotopically trivial (exact) sequence of R-modules, after applying the additive functor $- \otimes_R T$, we obtain a homotopically trivial sequence of T-modules $\ldots \to Q_1 \to Q_0 \to M \otimes_R T \to 0$, which is also a sequence of $S \otimes_R T$-modules. So $\mathbf{Q}$ is an $S \otimes_R T|T$-projective resolution of $M \otimes_R T$. Therefore $\operatorname{Tor}_n^{S|R}(M,N) = H_n(\mathbf{P} \otimes_S N) = H_n(\mathbf{P} \otimes_R T \otimes_{S\otimes_R T} N) = H_n(\mathbf{Q} \otimes_{S\otimes_R T} N) = \operatorname{Tor}_n^{S\otimes_R T|T}(M,N)$.

Theorem 2. *Let $f : A \to B$ be a homomorphism of Noetherian rings. Assume that there exists a positive integer m such that*

$$\operatorname{Tor}_n^{B\otimes_A B|A}(B, -) = 0$$

for all $n \geq m$. Then the fibers of f are geometrically regular of dimension $\leq m$. In particular, $\dim B \leq \dim A + m$.

Proof. Let $\mathfrak{p}$ be a prime ideal of A and $k(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ the residue field at $\mathfrak{p}$. Let $F|k(\mathfrak{p})$ be a finite field extension. By Lemma 1, for any $(B \otimes_A F) \otimes_F (B \otimes_A F)$-module M, we have $\mathrm{Tor}_n^{B\otimes_A B|A}(B, M) = \mathrm{Tor}_n^{B\otimes_A B\otimes_A F|F}(B \otimes_A F, M) = \mathrm{Tor}_n^{(B\otimes_A F)\otimes_F (B\otimes_A F)|F}(B \otimes_A F, M) = \mathrm{Tor}_n^{(B\otimes_A F)\otimes_F (B\otimes_A F)}(B \otimes_A F, M) = 0$ for all $n \gg 0$. Then, by [**2**, Proposition IX.7.6], the global dimension of $B \otimes_A F$ is $\leq m$ and so $B \otimes_A F$ is a regular ring of dimension $\leq m$ by Serre's theorem [**13**]. Therefore $k(\mathfrak{p}) \to B \otimes_A k(\mathfrak{p})$ is geometrically regular.

If f is flat, note that the result about the dimension of B in Theorem 2 was obtained in [**9**, Corollary 2.8].

The same ideas allow us to obtain, in some cases, characterizations of smooth, regular and complete intersection properties of a homomorphism $A \to B$ in terms of relative $\mathrm{Tor}^{B|A}$. We shall give two examples. Note that, in these examples, the flatness hypothesis could also be treated separately, though we shall assume it in order to simplify the statements.

Proposition 3. *Let $(A, \mathfrak{m}, K) \to (B, \mathfrak{n}, L)$ be a flat local homomorphism of Noetherian local rings such that K is of characteristic zero (or more generally $L|K$ is separable). The following are equivalent*:

(i) *B is a formally smooth A-algebra for the $\mathfrak{n}$-adic topology* [**3**, 0_{IV}, 19.3.1].

(ii) $\mathrm{Tor}_n^{B|A}(L, L) = 0$ *for all* $n \gg 0$.

Proof. Under the flatness hypothesis B is a formally smooth A-algebra if and only if $B \otimes_A K$ is a formally smooth K-algebra [**3**, 0_{IV}, 19.7.1]. Since $L|K$ is separable, by [**3**, 0_{IV}, 19.6.4], this is equivalent to the regularity of the ring $B \otimes_A K$.

Now, having in mind that K is a field, we deduce from Lemma 1

$$\mathrm{Tor}_n^{B|A}(L, L) = \mathrm{Tor}_n^{B\otimes_A K|K}(L, L) = \mathrm{Tor}_n^{B\otimes_A K}(L, L)$$

and so $\mathrm{Tor}_n^{B|A}(L, L) = 0$ for all $n \gg 0$ if and only if the global dimension of $B \otimes_A K$ is finite. This is equivalent to the regularity of $B \otimes_A K$.

Note that the hypothesis on the characteristic is not superfluous: if $A \to B$ is a (possibly nonseparable) field extension, $\mathrm{Tor}_n^{B|A}(-, -) = 0$ for all $n > 0$.

If $A \to B$ is a ring homomorphism and C a B-algebra, the (unnormalized) bar resolution $\beta(B,C)$ give us a $B|A$-projective resolution of C which is a simplicial B-algebra [**8**, X, Section 2]. Therefore, $\mathrm{Tor}^{B|A}(C,C) = H(\beta(B,C) \otimes_B C)$ is a graded C-algebra with divided powers. We say that a flat homomorphism of Noetherian rings is a complete intersection if its fibers are complete intersection rings [**4**, IV, 19.3.6] (we do not assume finite type hypothesis).

Proposition 4. *Let $f : (A, \mathfrak{m}, K) \to (B, \mathfrak{n}, L)$ be a flat local homomorphism of Noetherian local rings. Assume that A is a homomorphic image of a complete intersection ring,* (*e.g., if A is complete*). *The following are equivalent*:

(i) *f is a complete intersection homomorphism.*

(ii) *$\mathrm{Tor}^{B|A}(L,L)$ is an L-algebra with divided powers generated by the elements of degree* 1 *and* 2.

Proof. Since A is of the form R/I with R a complete intersection ring, then the formal fibers of A are complete intersections (we can mimic the proof of [**1**, Suppl. 33] with a shift of one dimension; details are provided in [**11**, Lemma 1.6]). Then by [**10**], [**15**], f is a complete intersection if and only if $B \otimes_A K$ is a complete intersection ring.

Since the isomorphism provided by Lemma 1, $\mathrm{Tor}^{B|A}(L,L) = \mathrm{Tor}^{B\otimes_A K|K}(L,L) = \mathrm{Tor}^{B\otimes_A K}(L,L)$ preserves this divided powers algebra structure, the result follows from the absolute case [**16**, Theorem 6], [**5**, Theorem 2.3.5, Theorem 3.5.1].

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