ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 33, Number 3, Fall 2003

ON THE COMMUTANT OF MULTIPLICATION OPERATORS WITH ANALYTIC SYMBOLS

B. KHANI ROBATI AND S.M. VAEZPOUR

ABSTRACT. Let $\mathcal{B}$ be a certain Banach space consisting of analytic functions defined on a bounded domain G in the complex plane. Let $\phi \in \mathcal{B}$ be a function which is analytic on G and continuous on $\overline{G}$. Assume that M_ϕ denotes the operator of multiplication by ϕ. We characterize the commutant of M_ϕ that is the set of all bounded operators T such that $M_\phi T = TM_\phi$. Under certain conditions on ϕ, we show that $T = M_\varphi$ for some function φ in $\mathcal{B}$.

1. Introduction. Let $\mathcal{B}$ be a Banach space consisting of analytic functions defined on a bounded domain G in the complex plane such that $\mathcal{B}$ satisfies conditions a, b, c, d as follows:

(a) $1 \in \mathcal{B}$, $z\mathcal{B} \subset B$.

(b) For every $\lambda \in G$ the evaluation functional at λ, $e_\lambda : \mathcal{B} \to \mathbf{C}$, given by $f \mapsto f(\lambda)$, is bounded.

(c) $\operatorname{ran}(M_z - \lambda) = \ker e_\lambda$ for every $\lambda \in G$.

(d) If $f \in \mathcal{B}$ and $|f(\lambda)| > c > 0$ for every $\lambda \in G$, then $1/f$ is a multiplier of $\mathcal{B}$.

Throughout this article by a Banach space of analytic functions $\mathcal{B}$ on G we mean one satisfying the above conditions.

Some examples of such spaces are as follows:

1) The algebra $A(G)$ which is the algebra of all continuous functions on the closure of G that are analytic on G.

2) The Bergman space of analytic functions defined on G, $L^p_a(G)$ for $1 \leq p \leq \infty$.

AMS *Mathematics Subject Classification.* Primary 47B35, Secondary 47B38.
Key words and phrases. Commutant, multiplication operators, Banach space of analytic functions, univalent function, bounded point evaluation.
This research was in part supported by a grant from IPM.
Received by the editors on February 13, 2000, and in revised form on June 4, 2001.

3) The spaces D_α of all functions $f(z) = \sum \hat{f}(n)z^n$, holomorphic in $\mathcal{D}$, for which

$$\|f\|_\alpha^2 = \sum (n+1)^\alpha |\hat{f}(n)|^2 < \infty$$

for every $\alpha \geq 1$ or $\alpha \leq 0$.

4) The analytic Lipschitz spaces $\mathrm{Lip}\,(\alpha, \overline{G})$ for $0 < \alpha < 1$, i.e., the space of all analytic functions defined on G that satisfy a Lipschitz condition of order α.

5) The subspce $\mathrm{lip}\,(\alpha, \overline{G})$ of $\mathrm{Lip}\,(\alpha, \overline{G})$ consisting of functions f in $\mathrm{Lip}\,(\alpha, \overline{G})$ for which

$$\lim_{z \to w} \frac{|f(z) - f(w)|}{|z - w|^\alpha} = 0.$$

6) The classical Hardy spaces H^p for $1 \leq p \leq \infty$.

A complex-valued function ϕ defined on G is called a multiplier of $\mathcal{B}$ if $\phi\mathcal{B} \subset \mathcal{B}$, i.e., ϕf is in $\mathcal{B}$ for every $f \in \mathcal{B}$, and the set of all multipliers of $\mathcal{B}$ is denoted by $\mathcal{M}(\mathcal{B})$. As is shown in [**6**], each multiplier ϕ is bounded on G. Given a multiplier ϕ, let M_ϕ be defined by $M_\phi(f) = \phi f$ denotes the operator of multiplication by ϕ. By the closed graph theorem M_ϕ is bounded. The algebra of all bounded operators on $\mathcal{B}$ is denoted by $L(\mathcal{B})$. Let $X \in L(\mathcal{B})$ and $XM_z = M_zX$, it is easy to see that $X = M_\phi$ for some function $\phi \in \mathcal{M}(\mathcal{B})$. A good source on this topic is [**6**]. We denote by $\{M_\phi\}'$ the set of operators $X \in L(\mathcal{B})$ such that $M_\phi X = XM_\phi$, i.e., the commutant of M_ϕ. Let $f \in A(G)$ and $z_0 \in G$. If $f(z)$ has a zero of order one at z_0 and $f(z) \neq 0$ for all $z \neq z_0$ in $\overline{G}$ we say that f has only a simple zero in $\overline{G}$.

Shields and Wallen [**8**] studied the commutant of the operator of multiplication by z on the Hilbert spaces of analytic functions and introduced interesting function theoretic methods. The commutant of Toeplitz operator on certain Hilbert spaces of functions was studied by many mathematicians. See, for example, [**1, 9, 10**], Cuckovic in [**3**] investigates the commutant of M_{z^n} on the Bergman space $L_a^2(\mathcal{D})$. Seddighi and Vaezpour [**7**] studied the commutants of certain multiplication operators on Hilbert space of analytic functions with special reproducing kernels. Also the commutant of M_{z^2} on Banach space of analytic functions and the commutant of M_{z^n} on certain Hilbert spaces of functions were studied in [**4**]. In [**2**] Axler, Cuckovic and Rao have shown that if two Toeplitz operators on Bergman space commute, and the symbol of one of them is analytic and nonconstant, then the other one is analytic. Also in [**5**] Khani and Vaezpour characterize the commutant of M_ϕ for a univalent function $\phi \in \mathcal{M}(\mathcal{B}) \cap A(\mathcal{D})$ on a Banach space of continuous functions and investigate the commutant of M_{ϕ^2} under certain conditions. In Section 2 of this article we investigate the commutant of the operator M_ϕ for some function $\phi \in \mathcal{M}(\mathcal{B}) \cap A(G)$ which is not necessarily univalent but we show that $\{M_\phi\}' = \{M_\Psi : \Psi \in \mathcal{M}(\mathcal{B})\}$. In particular we investigate the commutant of M_ϕ when ϕ is a certain polynomial.

2. The main results. First we state a theorem which will be used in the proof of other theorems that we state in this section.

Theorem 2.1. *Let $\mathcal{B}$ be a Banach space of analytic functions and let $\phi \in \mathcal{M}(\mathcal{B}) \cap A(G)$. If for some $\lambda \in G$, $\phi - \phi(\lambda)$ has only a simple zero in $\overline{G}$, then $T(f)(\lambda) = T(1)(\lambda) f(\lambda)$ for each $f \in \mathcal{B}$ and every $T \in \{M_\phi\}'$.*

Proof. First we will show that $\operatorname{ran}(M_\phi - \phi(\lambda)) = \ker e_\lambda$. It is clear that $\operatorname{ran}(M_\phi - \phi(\lambda)) \subset \ker e_\lambda$.

To show the converse, since $\operatorname{ran}(M_z - \lambda) = \ker e_\lambda$ we have $\phi(z) - \phi(\lambda) = (z - \lambda) g(z)$ for some $g \in \mathcal{B}$. By assumption, $g(z) \neq 0$ on $\overline{G}$. Therefore $1/g$ is in $\mathcal{M}(\mathcal{B})$ and we have $z - \lambda = \dfrac{\phi(z) - \phi(\lambda)}{g(z)}$. Now assume that $h \in \ker e_\lambda$ so $h = (z - \lambda) u$ for some function $u \in \mathcal{B}$. Hence

$$h = \frac{\phi - \phi(\lambda)}{g} u = (\phi - \phi(\lambda)) \frac{u}{g}.$$

Since $u/g \in \mathcal{B}$, we conclude that $\ker e_\lambda \subset \operatorname{ran}(M_\phi - \phi(\lambda))$. Now let $T \in \{M_\phi\}'$, an easy calculation shows that $M_\phi^* T^*(e_\lambda) = \phi(\lambda) T^*(e_\lambda)$. Hence $(M_\phi - \phi(\lambda))^*(e_\lambda) = (M_\phi - \phi(\lambda))^* T^*(e_\lambda) = 0.$ Since $\dim \ker (M_\phi - \phi(\lambda))^* = 1$, we conclude that $T^*(e_\lambda) = \psi(\lambda) e_\lambda$ for some constant $\psi(\lambda)$. Therefore we have

$$\begin{aligned} T(f)(\lambda) &= \langle T(f), e_\lambda \rangle = \langle f, T^*(e_\lambda) \rangle \\ &= \psi(\lambda) \langle f, e_\lambda \rangle = \psi(\lambda) f(\lambda) \end{aligned}$$

for every $f \in \mathcal{B}$, in particular if we set $f = 1$ in the above relation we have $\psi(\lambda) = T(1)(\lambda)$. $\square$

Let U be a subset of the complex plane and let a be a constant. We define $U_{-} - a = \{-z - a : z \in U\}$.

Theorem 2.2. *Let $\mathcal{B}$ be a Banach space of analytic functions on G and let, for $a, b \in \mathbf{C}$, $\phi(z) = z^2 + az + b$. If $G - \{G_{-} - a\} \neq \varnothing$, then $\{M_\phi\}' = \{M_\Psi : \Psi \in \mathcal{M}(\mathcal{B})\}$.*

Proof. It is easy to see that $W = (G - \{\overline{G_{-} - a}\}) - \{-a/2\}$ is a nonempty open subset of G. Assume $\lambda \in W$ we have $\phi(z) - \phi(\lambda) = (z - \lambda)(z + a + \lambda)$, since $\overline{G_{-} - a} = \overline{G_{-}} - a$, $\phi - \phi(\lambda)$ has only a simple zero in $\overline{G}$. Now let $T \in \{M_\phi\}'$ and $f \in \mathcal{B}$. By Theorem 2.1, $T(f)(\lambda) = T(1)(\lambda)f(\lambda)$ for every $\lambda \in W$. Since $T(f)$ is analytic on G and G is connected, we conclude that $T(f) = T(1)f$ and the proof is complete. $\square$

Remark. Let ϕ and G be as in Theorem 2.2. By this theorem it is easy to see that for each G there is at most one a such that $\{M_\phi\}' \neq \{M_\Psi : \Psi \in \mathcal{M}(\mathcal{B})\}$. In fact by Theorem 2.6 of [**4**], we can see that for some Banach spaces of analytic functions defined on $\mathcal{D}$ we have $\{M_{z^2}\}' \neq \{M_\Psi : \Psi \in \mathcal{M}(\mathcal{B})\}$ and $G - \{G_{-} - a\} = \varnothing$. Also for $M_{(z+1)^2}$ on $\mathcal{H}^2(\mathcal{D} - 1)$ we have $G - (G_{-} - a) = \varnothing$ and it is known that $\{M_{(z+1)^2}\}' \neq \{M_\Psi : \Psi \in \mathcal{M}(\mathcal{B})\}$.

Theorem 2.3. *Let $\mathcal{B}$ be a Banach space of analytic functions on $\mathcal{D}$. Let $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree n for some positive integer $n \geq 2$, and let $|a_n| + |a_{n-1}| + \cdots + |a_2| < |a_1|$. Then $\{M_p\}' = \{M_\Psi : \Psi \in \mathcal{M}(\mathcal{B})\}$.*

Proof. Let $\lambda \in \mathcal{D}$ be such that

$$|\lambda| < \frac{|a_1| - |a_n| - |a_{n-1}| - \cdots - |a_2|}{|a_1| + |a_n| + |a_{n-1}| + \cdots + |a_2|}.$$

It is easy to see that $|p(z) - p(\lambda) - a_1 z| < |a_1 z|$ on the circle $|z| = 1$. Hence by Rouche's theorem $p - p(\lambda)$ has only a simple zero on $\overline{\mathcal{D}}$. Now by Theorem 2.1 and a similar argument as in the proof of Theorem 2.2, we have $\{M_p\}' = \{M_\Psi : \Psi \in \mathcal{M}(\mathcal{B})\}$. $\square$

Let $\mathcal{B}$ be a Banach space of analytic functions on $\mathcal{D}$ and let n be a positive integer. If a is a constant with $|a| > 1$, then by Theorem 2.3 we have $\{M_{z^n+az}\}' = \{M_\Psi : \Psi \in \mathcal{M}(\mathcal{B})\}$. In the next theorem we investigate the commutant of M_{z^3+az}.

Theorem 2.4. *Let $\mathcal{B}$ be a Banach space of analytic functions on $\mathcal{D}$. If a is a constant such that $|Re\,(a)| > 1/8$ or $|Im\,(a)| > 1/8$, then $\{M_{z^3+az}\}' = \{M_\Psi : \Psi \in \mathcal{M}(\ B)\}$.*

Proof. Let $\phi(z) = z^3+az$, then $\phi(z)-\phi(\lambda) = (z-\lambda)(z^2+\lambda z+\lambda^2+a)$. Now suppose that $\mathrm{Re}\,(-a) < -1/8$ and λ is a positive number. Let $z = x + iy$. We have

$$\begin{aligned}\mathrm{Re}\,(z^2+\lambda z+\lambda^2) &= x^2 - y^2 + \lambda x + \lambda^2\\ &\geq 2x^2 + \lambda x + \lambda^2 - 1\\ &= 2(x+\lambda/4)^2 + (7/8)\lambda^2 - 1\\ &\geq (7/8)\lambda^2 - 1.\end{aligned}$$

Since $(7/8)\lambda^2 - 1 \to -1/8$ whenever $\lambda \to 1$ and $\mathrm{Re}\,(-a) < -1/8$. We can choose a sequence of distinct real numbers $\{\lambda_n\}$ which converges to a real number λ_0 in $(0, 1)$ such that $\mathrm{Re}\,(z^2 + \lambda_n z + \lambda_n^2) > c$ for some constant c, with $\mathrm{Re}\,(-a) < c < -1/8$ and for all positive integers n. Now let $T \in \{M_\phi\}'$ and $f \in \mathcal{B}$. By assumption $\phi(z) - \phi(\lambda_n)$ has only a simple zero in $\overline{\mathcal{D}}$, therefore by Theorem 2.1 $T(f)(\lambda_n) = T(1)f(\lambda_n)$ for each positive integer n. Since $T(f) - T(1)f$ is an analytic function on $\mathcal{D}$ whose zero set has a limit point in $\mathcal{D}$, it is the zero function on $\mathcal{D}$ and we conclude that $T(f) = T(1)f$.

If $\mathrm{Re}\,(-a) > 1/8$, by a similar argument as before and substituting λ with $i\lambda$ we obtain the result.

Now assume that $\mathrm{Im}\,(-a) < -1/8$ if we set $\lambda = \alpha + i\alpha$, then

$$\mathrm{Im}\,(z^2 + \lambda z + \lambda^2) = 2xy + \alpha(x+y) + 2\alpha^2.$$

By a rotation with measure $-\pi/4$ and substitution $x = \dfrac{X+Y}{\sqrt{2}}$ and

$y = \dfrac{Y - X}{\sqrt{2}}$, we have

$$\begin{aligned} 2xy + \alpha(x + y) + 2\alpha^2 &= Y^2 - X^2 + \sqrt{2}\alpha Y + 2\alpha^2 \\ &\geq 2Y^2 - 1 + \sqrt{2}\alpha Y + 2\alpha^2 \\ &= 2\left(Y + \frac{\alpha}{2(\sqrt{2})}\right)^2 - \frac{\alpha^2}{4} + 2\alpha^2 - 1 \\ &\geq -1 + (7/4)\alpha^2. \end{aligned}$$

We see that $(7/4)\alpha^2 - 1 \to -1/8$ whenever $\alpha \to \sqrt{2}/2$. Now by a similar argument as we used in the first stage of the theorem we can prove the theorem in this case.

Finally assume that $\mathrm{Im}\,(-a) > 1/8$ if we set $\lambda = \alpha - i\alpha$ for some real number α in the former stage and, using a similar argument, we obtain the result. $\square$

In the next theorem we improve the result obtained in Theorem 2.3.

Theorem 2.5. *Let $\mathcal{B}$ be a Banach space of functions defined on G, and let G be the interior of $\overline{G}$. Suppose h and $g \in \mathcal{M}(\mathcal{B}) \cap A(G)$ and assume that g has only a simple zero in $\overline{G}$ at a point z_0 in G and $h(z_0) = 0$. If $|h(z)| < |g(z)|$ at each point of $\overline{G} - G$, then*

$$\{M_{h+g}\}' = \{M_\Psi : \Psi \in \mathcal{M}(\mathcal{B})\}.$$

Proof. Since $|g(z)| - |h(z)| > 0$ at each point of $\overline{G} - G$ there is a constant $a > 0$ such that $|g(z)| - |h(z)| > a$ for each $z \in \overline{G} - G$. Since $g(z_0) = h(z_0) = 0$ there is a $\delta > 0$ such that $|g(\lambda)| < a/2$ and $|h(\lambda)| < a/2$ whenever $\lambda \in B(z_0; \delta)$ where $B(z_0; \delta) = \{z \in G : |z - z_0| < \delta\}$. Now we set $\phi = h + g$ and we have

$$|\phi(z) - \phi(\lambda) - g(z)| = |h(z) - h(\lambda) - g(\lambda)| < |h(z)| + a < |g(z)|$$

for each $\lambda \in B(z_0; \delta)$ and $z \in \overline{G} - G$. Hence by the general form of Rouche's theorem $\phi(z) - \phi(\lambda)$ has only a simple zero for each $\lambda \in \mathcal{B}(z_0; \delta)$; therefore, by Theorem 2.1, for each $T \in \{M_\phi\}'$, $f \in \mathcal{B}$, and every $\lambda \in \mathcal{B}(z_0;\delta)$ we have $T(f)(\lambda) = T(1)(\lambda)f(\lambda)$. Since $T(f)$ is analytic, the proof is complete. $\square$

Theorem 2.6. *Let $\mathcal{B}$ be a Banach space of functions defined on G and let G be the interior of $\overline{G}$. Suppose h and $g \in \mathcal{M}(\mathcal{B}) \cap A(G)$ and assume that g has only a simple zero in $\overline{G}$ at a point z_0 in G. If there is a constant c such that $|h(z)| < c < (1/2)|g(z)|$ for each $z \in \overline{G} - G$, then*

$$\{M_{h+g}\}' = \{M_\Psi : \Psi \in \mathcal{M}(\mathcal{B})\}.$$

Proof. Since $|g(z)| - 2c > 0$ at each point of $\overline{G} - G$ there is a constant $a > 0$ such that $|g(z)| - 2c > a$ for each $z \in \overline{G} - G$. Since $g(z_0) = 0$ there is a $\delta > 0$ such that $|g(\lambda)| < a$ whenever $\lambda \in B(z_0;\delta)$. Now we set $\phi = h + g$ and we have

$$|\phi(z) - \phi(\lambda) - g(z)| = |h(z) - h(\lambda) - g(\lambda)| < 2c + a < |g(z)|$$

for each $\lambda \in B(z_0;\delta)$ and $z \in \overline{G} - G$. Hence by the general form of Rouche's theorem $\phi(z) - \phi(\lambda)$ has only a simple zero for each $\lambda \in B(z_0;\delta)$; therefore, by Theorem 2.1, for each $T \in \{M_\phi\}'$, $f \in \mathcal{B}$, and every $\lambda \in B(z_0;\delta)$ we have $T(f)(\lambda) = T(1)(\lambda)f(\lambda)$. Since $T(f)$ is analytic the proof is complete. $\square$

Example 2.7. Let $\mathcal{B} = L^p_a(\mathcal{D})$ be the Bergman space of analytic functions, and let $h(z) = z^n e^z$ for some nonnegative integer n and $g(z) = az$ where a is a constant. We set $\phi(z) = h(z) + g(z)$. If $n > 1$ and $|a| > e$ then, by Theorem 2.5, $\{M_\phi\}' = \{M_\Psi : \Psi \in \mathcal{M}(\mathcal{B})\}$. Also if $n = 0$ and $|a| > 2e$, by Theorem 2.6 we have $\{M_\phi\}' = \{M_\Psi : \Psi \in \mathcal{M}(\mathcal{B})\}$.

Example 2.8. Let $\mathcal{B}$ be a Banach space of analytic function on $\mathcal{D}$. Let $\phi = \sum_{n=1}^\infty a_n z^n$ belong to $\mathcal{M}(\mathcal{B}) \cap A(G)$.

a) If $a_1 = 1$ and $\sum_{n=2}^\infty n|a_n| < 1$, then ϕ is a univalent function. Hence by Theorem 2.1, $\{M_\phi\}' = \{M_\Psi : \Psi \in \mathcal{M}(\mathcal{B})\}$.

b) If $a_1 = 1$ and $\sum_{n=2}^\infty |a_n| < 1$ then, by Theorem 2.5, we have $\{M_\phi\}' = \{M_\Psi : \Psi \in \mathcal{M}(\mathcal{B})\}$.

REFERENCES

1. S. Axler and Z. Cuckovic, *Commuting Toeplitz operators with harmonic symbols*, Integral Equations Operator Theory, vol. 14, 1991, pp. 1–12.

2. S. Axler, Z. Cuckovic and N.V. Rao, *Commutant of analytic toeplitz operators on the Bergman space*, Proc. Amer. Math. Soc., **128** (2000), 1951–1953.

3. Z. Cuckovic, *Commutant of Toeplitz operators on the Bergman spaces*, Pacific J. Math. **162** (1994), 277–285.

4. B. Khani Robati, *On the commutant of certain multiplication operators on spaces of analytic functions*, Rend. Cerc. Mat. Palermo **49** (2000), 601–608.

5. B. Khani Robati and S.M. Vaezpour, *On the commutant of operators of multiplication by univalent functions*, Proc. Amer. Math. Soc. **129** (2001), 2379–2383.

6. S. Richter, *Invariant subspaces in Banach spaces of analytic functions*, Trans. Amer. Math. Soc. **304** (1987), 585–616.

7. K. Seddighi and S.M. Vaezpour, *Commutant of certain multiplication operator on Hilbert spaces of analytic functions*, Studia Math. **133** (1999), 121–130.

8. A.L. Shields and L.J. Wallen, *The commutants of certain Hilbert space operators*, Indiana Univ. Math. J. **20** (1971), 777–788.

9. J.E. Thomson, *Intersections of commutants of analytic Toeplitz operators*, Proc. Amer. Math. Soc. **52** (1975), 305–310.

10. ———, *The commutant of certain analytic Toeplitz operators*, Proc. Amer. Math. Soc. **54** (1976), 165–169.

Department of Mathematics, College of Sciences, Shiraz University, Shiraz 71454, Iran
Institute for Studies in Theoretical Physics and Mathematics
E-mail address: khani@math.susc.ac.ir

Department of Mathematics, College of Sciences, Yazd University, Yazd, Iran
Institute for Studies in Theoretical Physics and Mathematics