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CONJUGATE INEQUALITIES

A.M. FINK

ABSTRACT. We propose a general theory of conjugate inequalities which relate the norm of a function and Lf, where L is some linear differential operator. The best constants in these inequalities are shown to be the same as for an inequality involving the adjoint operator $L+$. We concentrate on the properties of the best constants in the hope that they can be computed exactly when sufficient numbers of properties are derived. Good estimates are provided for most of the constants.

1. Introduction. Brink [**1**] in his thesis considered the problem of finding best constants in the inequalities

$$\|f\|_p \leq M(n,p,q)\|f^{(n)}\|_q \tag{1}$$

on the interval $[0,1]$ with f having n zeros on the interval. He showed that the best constant M is determined by having all of the zeros at the ends of the interval. He was led to the problem of finding the best constants in the inequality

$$\|f\|_p \leq K(n,\alpha,p,q)\|f^{(n)}\|_q \tag{2}$$

with the boundary conditions

(3) f has α zeros at 0 and $n-\alpha$ zeros at 1,

i.e.,

(4) $f^{(i)}(0)=0,\ i=0,1,\dots,\alpha-1$ and $f^{(i)}(1)=0,\ i=0,\dots,n-\alpha-1.$

If one of α or $n-\alpha-1$ is negative, then that condition is deleted. It is to be mentioned that p and q are not related in this formulation. In the course of Brink's study, using a formal application of the Pontryagin

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maximal principle (it does not apply because of absolute values) he found that

$$K(n, \alpha, p, q) = K(n, \alpha, q', p') \tag{5}$$

where $1/p' + 1/p = 1$ with the usual provisions when $p = 1$. Brink was able to prove some of the special cases. He also proved some results on these kinds of problems which are now extended in this paper to fairly general differential operators.

Later I found a proof of this result which does not use any calculus of variations and also gave equations for the extremals. Here I propose a general theory of such inequalities.

2. The general theory. We begin with an nth order linear differential operator L defined for functions in $C^{(n)}[a, b]$. This can be weakened by considering L in disconjugate form. But for our purposes here we do not consider ourselves with this generalization. Together with the operator L, we consider a set of boundary conditions

$$Uy = 0 \tag{6}$$

which are linear conditions on y and its derivatives at a and b. They may be separated or not.

We will assume

(H_1) Zero is not an eigenvalue of the operator L together with the boundary conditions (6).

Because of the linearity of the problem we immediately have Theorem 1.

Theorem 1. *If H_1 holds, then for every continuous g, the problem*

$$Ly = g, \quad Uy = 0 \tag{7}$$

has a unique solution.

Together with such an operator and boundary conditions, there is the formal adjoint L^+ and the adjoint boundary conditions $U^+ y = 0$. We have the Lagrange identity

$$\langle Lf, g \rangle = \langle f, L^+ g \rangle \tag{8}$$

where $\langle , \rangle$ is the usual inner product. See Coddington and Levinson [**2**, Chapter 11].

For the fixed interval $[a,b]$ we consider the problem: Find the best possible constant $C\{L,[a,b],p,q\}$ in the inequality

$$\|f\|_p \le C\{L,[a,b],p,q\}(b-a)^{1/p-1/q}\|Lf\|_q \quad \text{for } Uf = 0. \tag{9}$$

We limit ourselves to the case when U has rank n so that the adjoint boundary condition also has rank n and Theorem 1 also applies to the adjoint problem, i.e., the problem

$$L^+y = g, \quad U^+y = 0, \tag{10}$$

has a unique solution for any continuous g.

We can now state and prove the main result.

Theorem 2. *If H_1 holds and U has rank n then the problems* (9) *for L and L^+ both have best constants and for $1 < p$, $q < \infty$,*

$$C\{L,[a,b],p,q\} = C\{L^+,[a,b],q',p'\}. \tag{11}$$

Furthermore, extremals exist and satisfy the equations:

$$L^+g = |f|^{p-1}\operatorname{sgn}(f), \tag{12}$$

$$Lf = c|g|^{q'-1}\operatorname{sgn}(g) \quad \textit{for some constant } c. \tag{13}$$

Here f is the extremal for the L inequality and g is the extremal for the L^+ inequality.

Note 1. The inequalities are homogeneous and the boundary conditions are also. If f is an extremal so is any multiple of f. It may be shown therefore that the constant in (13) could be chosen to be 1 if the correct multiples of f and g are chosen. This actually is insignificant.

Proof. From the hypotheses, there is a Green's function G for each of the problems for L and L^+ so if f satisfies the boundary conditions $Uf = 0$, then

$$f(t) = \int_a^b G(t,s)Lf(s)\,ds \tag{14}$$

and G is a continuous function. Then it is elementary to show that there is some constant which satisfies the inequality (9). One applies Hölder's to the righthand side to get $|f(t)| \leq |G(t,\cdot)|_{q'}$. Now take pth powers and integrate. In general the constant obtained in this way is far from being optimal.

That there are extremals follows from the weak compactness of the unit ball in L_q, once we know that the mapping is continuous.

So let f satisfy the boundary conditions and look at

$$\int_a^b |f(t)|^p \, dt = \int_a^p |f(t)|^{p-1} \operatorname{sgn}(f(t))(f(t)) \, dt.$$

Let g be defined by $L^+ g = |f(t)|^{p-1} \operatorname{sgn}(f(t)), U^+ g = 0$. Then $\int_a^b |f(t)|^p \, dt = \langle f, L^+ g\rangle = \langle Lf, g\rangle \leq \|g\|_{q'} \|Lf\|_q \leq C\{L^+, [a,b], q', p'\}$ $(b-a)^{1/q'-1/p'} \|Lf\|_q \|L^+ g\|_{p'}$ where we have used Hölder's inequality and the inequality for g since $U^+ g = 0$. Write out what $\|L^+ g\|_{p'}$ is in terms of f. It is $(\int_a^b |f(t)|^p \, dt)^{1/p'}$. Divide by this quantity and we have

$$\|f\|_p \leq C\{L^+, [a,b], q', p'\}(b-a)^{1/p-1/q} \|Lf\|_q. \tag{15}$$

But $C\{L, [a,b], p, q\}$ is the smallest constant for such an inequality to hold. Therefore,

$$C\{L, [a,b], p, q\} \leq C\{L^+, [a,b], q', p'\}. \tag{16}$$

We can get the reverse by interchanging the roles of g and f in the above string of inequalities. This proves (11). If f is an extremal for the L problem, then equality must hold in (15) and therefore in all of the intermediate inequalities. This means that g must be an extremal also and equality holds in the Hölder's inequality place. This gives (13) while (12) holds by definition.

Corollary 1. *The equality of the constants in* (11) *extends to* $1 \leq p$, $q \leq \infty$.

Proof. The proof of the theorem works for p and q in this extended interval if Hölder's inequality is replaced by the obvious inequality in Case 1 or ∞ is one of the indices of the norm.

We can extend the range over which extremals exist.

Corollary 2. *Extremals exist except for the four cases* $(p, q) \in \{(1,1), (1,\infty), (\infty,1), (\infty,\infty)\}$.

Proof. Again we look at the proof, observing that if an extremal exists for one of the problems it also exists for the conjugate problem. The equality of the constants is crucial here. The weak compactness argument gives extremals for $1 \le p \le \infty$ and $1 < q < \infty$. But applying this to the conjugate problem we get extremals existing for $1 \le q' \le \infty$ and $1 < p' < \infty$. This is $1 < p < \infty$ and $1 \le q \le \infty$. So only the above four remain. □

Corollary 3. *If the Green's function for* L *is of constant sign on* $[a,b] \otimes [a,b]$, *then extremals exist also for* $p = 1$ *and* $p = \infty$ *and* $q = \infty$.

Proof. We have from (14) that $f(t) = \int_a^b G(t,s) Lf(s)\, ds$. Then

$$|f(t)| \le |Lf|_\infty \int_a^b |G(t,s)|\, ds = |Lf|_\infty \int_a^b G(t,s) \operatorname{sgn}(G(t,s))\, ds. \tag{17}$$

Consequently we have

$$|f|_\infty \le |Lf|_\infty \sup_t \int_a^b G(t,s) \operatorname{sgn}(G(t,s))\, ds. \tag{18}$$

If $Lf = \operatorname{sgn} G(t,s)$, then $f \ge 0$ and equality holds throughout the argument, in particular in (18). So there is an extremal when $p = \infty$ and $q = \infty$. For $p = 1$ we again begin with (17) and arrive at

$$|f|_1 \le |Lf|_\infty \int_a^b \int_a^b |G(t,s)|\, ds\, dt. \tag{19}$$

Again if $Lf = \operatorname{sgn} G(t,s)$, we have equality in all of the inequalities. So there is an extremal when $p = 1$ and $q = \infty$.

There are other consequences of the Green's function being of constant sign.

Theorem 3. *If the Green's function for L with $Uy = 0$ is of constant sign, then we may restrict ourselves to nonnegative functions. In particular, the extremals have f and Lf each of constant sign.*

Proof. Let f be an admissible function so that Lf is well defined and $Uf = 0$. Then $f(t) = \int_a^b G(t,s)Lf(s)\,ds$ and let $g(t)$ be defined by

$$g(t) = \int_a^b G(t,s)|Lf(s)|\operatorname{sgn} G(t,s)\,ds, \text{ i.e.,} \tag{21}$$

$Lg = |Lf(s)|\operatorname{sgn} G(t,s)$ and $Ug = 0$. Then

$$\begin{aligned} |f(t)| &\le \int_a^b |G(t,s)||Lf(s)|\,ds \\ &= \int_a^b G(t,s)|Lf(s)|\operatorname{sgn} G(t,s)\,ds = g(t). \end{aligned} \tag{22}$$

Now $\|f\|_p \le \|g\|_p$, with $g \ge 0$ and $\|Lf\|_q = \|Lg\|_q$ and Lg is of constant sign. Unless f is nonnegative to begin with, the first inequality is strict. Therefore, the extremal will satisfy the requirements of the theorem.

Theorem 4 (Boyd [**3**]). *If the Green's function is of one sign, then the extremals are unique if $p \le q$.*

We will not prove this theorem here. Boyd considers inequalities of a slightly different type and proves some theorems on relationships between various constants as we will do.

The constants derived in the above manner satisfy other properties which we now address.

Theorem 5. *The constant $C\{L[a,b],p,q\}$*

(i) *is increasing in p,*

(ii) *is decreasing in q,*

(iii) *the pth power is log convex in p, and*

(iv) *the qth power is log convex in q.*

Proof. We begin by noting that for any continuous f

$$\|f\|_p/(b-a)^{1/p} \quad \text{is increasing in } p. \tag{23}$$

For i) let $p_1 < p_2$. Then we have

$$\begin{aligned}\frac{\|f\|_{p_1}}{(b-a)^{\frac{1}{p_1}}} &\leq \frac{\|f\|_{p_2}}{(b-a)^{\frac{1}{p_2}}} \\ &\leq C\{L,[a,\beta],p_2,q\}(b-a)^{-1/q}\|Lf\|_q.\end{aligned} \tag{24}$$

Since $C\{L,[a,b],p_1,q\}$ is the smallest constant for which the extremes of (24) hold, we have $C\{L,[a,b],p_1,q\} \leq C\{L,[a,b],p_2,q\}$ which is i).

For ii) we apply the same argument except with $\|Lf\|_{q_1}/(b-a)^{1/q_1} \leq \|Lf\|_{q_2}/(b-a)^{1/q_2}$.

For iii) we write for $p = \lambda p_1 + (1-\lambda)p_2$, $0 < \lambda < 1$, and use the log convexity of $(\|f\|_p)^p$. We have

$$\begin{aligned} p\log\|f\|_p &\leq \lambda p_1 \log\|f\|_{p_1} + p_2(1-\lambda)\log\|f\|_{p_2} \\ &\leq \lambda p_1 \log\{C\{L,[a,b],p_1,q\}(b-a)^{1/p_1-1/q}\|Lf\|_q\} \\ &\quad + p_2(1-\lambda)\log\{C\{L,[a,b],p_2,q\}(b-a)^{1/p_2-1/q}\|Lf\|_q\} \\ &= \{\lambda p_1 \log\{C\{L,[a,b],p_1,q\}(b-a)^{1/p_1-1/q}\} \\ &\quad + p_2(1-\lambda)\log\{C\{L,[a,b],p_2,q\}\}(b-a)^{1/p_2-1/q}\} \\ &\quad + p\log\|Lf\|_q. \end{aligned}$$

If we now exponentiate and use that $C\{L,[a,b],p,q\}(b-a)^{1/p-1/q}$ is the smallest number for which the inequality is to hold, we get the result.

The proof of iv) is a little more computational. Let $1 < q_1 < q < q_2$. We will use the identity

$$C(p,q) = C\left(\frac{q}{q-1}, p'\right) \tag{25}$$

where we have suppressed the reference to the operator and interval. Now

$$\frac{q_2}{q_2-1} < \frac{q}{q-1} < \frac{q_1}{q_1-1}. \tag{26}$$

In particular, for appropriate $0 < \alpha < 1$, let

$$\frac{q}{q-1} = \alpha \frac{q_1}{q_1-1} + (1-\alpha) \frac{q_2}{q_2-1}. \tag{27}$$

Since $C(\cdot, p')^{(\cdot)}$ is log convex, we have

$$C\left(\frac{q}{q-1}, p'\right)^{\frac{q}{q-1}} \leq C\left(\frac{q_1}{q_1-1}, p'\right)^{\alpha \frac{q_1}{q_1-1}} C\left(\frac{q_2}{q_2-1}, p'\right)^{\frac{(1-\alpha)q_2}{q_2-1}}.$$

On the other hand, we have

$$q = \lambda q_2 + (1-\lambda) q_1. \tag{29}$$

We can solve (29) for λ and (27) for α and we find that $\alpha q_1(q-1)/(q_1 - 1) = (1-\lambda)q_1$ and $(1-\alpha)q_2(q-1)/(q_2-1) = \lambda q_2$.

Inserting these into (28) we get the log convexity as required.

3. Examples. We now look at some examples. For this section we let $Lf = f^{(n)}$ so that $L^+ f = (-1)^n f^{(n)}$.

We first look at Brink's result for this case. The boundary conditions for the K constants in (2) are given by (4). We call these the $(\alpha, n-\alpha)$ boundary conditions. A set of adjoint conditions are given by the boundary conditions $(n-\alpha, \alpha)$, so that the Lagrange identity is merely integration by parts and all the boundary terms which are of the form $f^{(i)}(a)g^{(n-i)}(a)$, or the same at b, are zero. We make a slight change in the notation here. To do things on the interval $[a, b]$ and not make the constants K depend on the interval, we write Brink's inequalities as

$$\|f\|_p \leq K(n, \alpha, p, q)(b-a)^{n+1/p-1/q} \|f^{(n)}\|_q. \tag{31}$$

Then the constants are independent of the interval and are the ones for the standard interval $[0, 1]$. This is easy to prove by a linear change of variables.

We note that by graphing the functions backwards that $K(n, \alpha, p, q) = K(n, n-\alpha, p, q)$ so that the conjugate relation can be written as

$$K(n, \alpha, p, q) = K(n, \alpha, q', p'). \tag{32}$$

Now we give some of the results here from Brink's paper.

Theorem 6 (Brink). *For any p, $1 \leq p < \infty$,*

$$K(n,\alpha,p,\infty) = \|g\|_p, \tag{33}$$

where $g(t) = t^\alpha(1-t)^{n-\alpha}/n!$. Hence

$$\begin{aligned} K(n,\alpha,p,\infty) &= K(n,\alpha,1,p') \\ &= (1/n!)\left\{\frac{\Gamma(p\alpha+1)\Gamma(pn-p\alpha+1)}{\Gamma(pn+2)}\right\}^{1/p} \end{aligned} \tag{34}$$

and for $1 \leq p < \infty$,

$$K(n,\alpha,\infty,\infty) = K(n,\alpha,1,1) = \frac{\alpha^\alpha(1-\alpha)^{n-\alpha}}{n!n^n}. \tag{35}$$

Proof. Let g be given as in the theorem. We know that we may assume $f \geq 0$. So

$$\begin{aligned} \|f\|_1 &= \int_0^1 f(t)1\,dt = \int_0^1 f(t)g^{(n)}\,dt \\ &= (-1)^n \int_0^1 g(t)f^{(n)}(t)\,dt \leq \|f^{(n)}\|_q\|g\|_{q'} \end{aligned} \tag{36}$$

with equality when $|f^{(n)}|^q$ and $|g|^{q'}$ are proportional. This gives $K(n,\alpha,1,q)$ and the result (and also the extremals).

We will quote one of Brink's other theorems.

Theorem 7. *Let $G(t,s)$ be the Green's function for the $(\alpha, n-\alpha)$ problem, then*

$$K(n,\alpha,\infty,q) = K(n,\alpha,q',1) = \max\{\|G(t,\cdot)\|_{q'} : 0 \leq t \leq 1\}. \tag{37}$$

Proof. We use the fact that we may assume $f \geq 0$ and the formula

$$f(t) = \int_0^1 G(t,s)f^{(n)}(s)\,ds. \tag{38}$$

Then $f(t) \leq \|G(t,\cdot)\|_{q'}\|f^{(n)}\|_q \leq \|G(t_0,\cdot)\|_{q'}\|f^{(n)}\|_q$, where t_0 is where the norm of G is maximized. If we take $f^{(n)}(s) = G(t_0, s)$, then equality holds throughout the above inequalities.

We also note that

$$K(2,1,2,2) = \frac{1}{\pi^2} \tag{39}$$

is a well-known result and is called Wirtinger's inequality. For other scattered results, see Brink's paper and Fink [**4**]. On the other hand, Hardy, Littlewood and Polya [**5**] show that

$$K(1,1,2k,2k) = (2k-1)^{-1/2k}(2k/\pi)\sin(\pi/2k). \tag{40}$$

The equations of the extremals are given as in (12) and (13) with the proviso that, according to our theorems, f and $f^{(n)}$ are of constant sign

$$f^{(n)} = (-1)^n \lambda |g|^{q'-1}\mathrm{sgn}\,(g) \tag{41}$$

and

$$g^{(n)} = |f|^{p-1}\mathrm{sgn}\,(f) \tag{42}$$

with f having α zeros at 0 and $n-\alpha$ zeros at 1 and g switching these two numbers. The signs of the functions can be predicted. The sign of the Green's function for the $(\alpha, n-\alpha)$ is $(-1)^{n-\alpha}$, so if we take $f \geq 0$, then $\mathrm{sgn}\, g = (-1)^\alpha$ and $\mathrm{sgn}\, f^{(n)} = (-1)^{n-\alpha}$ so $\lambda \geq 0$. For $p = q = 2$, these equations are $f^{(n)} = (-1)^n\lambda g$ and $g^{(n)} = f$ so that these combine into the single equation

$$f^{(2n)} = (-1)^n \lambda f. \tag{43}$$

The computation of $K(n,\alpha,2,2)$ amounts to solving the eigenvalue problem of the differential equation (43) with the $(\alpha, n-\alpha)$ boundary conditions for f and the $(n-\alpha,\alpha)$ boundary conditions for $f^{(n)}$.

To take the simplest case, take $n = \alpha = 1$. Then $K(1,1,2,2) = \max\{\|f\|_2/\|f\|_2 : f$ is an eigenfunction of (43)$\}$. This is $K(1,1,2,2) = \max\{(\pi/2 + k\pi)^{-1}, k = 0, 1, \dots\} = 2/\pi$.

For $K(2,1,2,2)$ the differential equation is $f^{iv} - \alpha^4 f = 0$, with $f(0) = f''(0) = f(1) = f''(1) = 0$. The eigenfunctions are $\sin(n\pi x)$

and $K(2,1,2,2) = \max\{f\|_2/\|f''\|_2\} = 1/\pi^2$, which is Wirtinger's inequality.

Our second example is to consider inequalities, see Fink [**6**], of the form

$$\|f\|_p \le H(n,p,q)(b-a)^{n+1/p-(1/q)}\|f^{(n)}\|_q \tag{44}$$

with the boundary conditions: a non-decreasing sequence $a_i \in [a,b]$ exists such that $f^{(i)}(a_i) = 0$, $i = 0,1,\dots,n-1$.

The integral representation

$$f(x) = \int_{a_0}^{x} dx_1 \int_{a_1}^{x_1} dx_2 \int_{a_2}^{x_2} \cdots \int_{a_{n-1}}^{x_{n-1}} f^{(n)}(x_n)\, dx_n \tag{45}$$

shows that the constants exist. Hartman [**7**] and Levin [**8**] both considered such inequalities for $p = \infty$ and $q = 1$ or ∞. In particular, Levin showed that if some $a_i \in (0,1)$, then f is the midpoint of two other functions f_1 and f_2, which are admissible for (44) with the same nth derivative (but different a_i). Since the p norm is convex in f, one of the functions f_1 or f_2 has a larger p norm. As a result $H(n,p,q)$ is the maximum over α of the numbers $H(n,\alpha,p,q)$ for the problem of finding the best constants in

$$\|f\|_p \le H(n,\alpha,p,q)(b-a)^{n+1/p-1/q}\|f^{(n)}\|_q \tag{46}$$

where f has α zeros at a and $f^{(\alpha)}$ has $n-\alpha$ zeros at b. Call these boundary conditions $Z(\alpha, n-\alpha)$. That is, all the a_i are at the ends of the interval. Again the factor of $(b-a)$ is selected to make H independent of the interval so for the remaining discussion we restrict ourselves to the interval $[0,1]$.

Our general theory applies with the adjoint operator being the operator $(-1)^n f^{(n)}$ with the boundary conditions $Z(n-\alpha,\alpha)$. So that $H(n,\alpha,p,q) = H(n,\alpha,q',p')$, extremals existing etc. To get some specific results, introduce the polynomials $g(t,\alpha,\beta)$ for integral α and β by

$$\begin{gathered} g(t,\alpha,\beta) = (1/\alpha!\beta!)\int_0^t (1-s)^\alpha (t-s)^\beta\, ds, \quad \text{for } \alpha \ge 0, \beta \ge 0; \\ g(t,\alpha-1) = (1-t)^\alpha/\alpha!, \quad \alpha \ge 0; \\ g(t,-1,\beta) = t^\beta/\beta!, \quad \beta \ge 0. \end{gathered} \tag{47}$$

Theorem 8. *For any* q

(48)
$$H(n,\alpha,\infty,q)=\|g(\cdot,\alpha-1,n-\alpha-1)\|_{q'}\quad \textit{if } 0\le\alpha<n;$$

and

$$H(n,\alpha,1,q)=\|g(\cdot,n-\alpha,\alpha-1)\|_{q'}\quad \textit{if } 0\le\alpha\le n.$$

In particular

(49)
$$\begin{aligned}H(n,\alpha,\infty,\infty)=H(n,\alpha,1,1)&={}_{n-1}C_\alpha/n!,\quad 0\le\alpha<n;\\ H(n,\alpha,1,\infty)&={}_nC_\alpha/(n+1)!,\quad 0\le\alpha<n;\\ H(n,\alpha,\infty,1)&={}_{n-2}C_{\alpha-1}/(n-1)!,\quad 0<\alpha<n;\\ H(n,0,\infty,1)&=1/(n-1)!;\\ H(n,n,p,1)=H(n,0,\infty,p')&=[(n-1)p+1]^{-1/p}/(n-1)!;\end{aligned}$$

and

$$H(n,n,p,\infty)=H(n,0,1,p')=(np+1)^{-1/p}/n!.$$

It follows from these computations, the monotonicity of the H constants and the definitions that

$$\frac{C_n}{(n+1)!}\le H(n,p,q)\le\frac{C_{n-1}}{(n-1)!},\tag{50}$$

where C_n is the central binomial coefficient ${}_nC_{[n/2]}$. Some better bounds can be obtained than this.

Theorem 9. *Let* $0<\alpha<n$. *For any* p,q *and* r

$$H(n,\alpha,p,q)\le H(\alpha,\alpha,p,r)H(n-\alpha,n-\alpha,r,q).\tag{51}$$

Proof. We note that if f satisfies the boundary conditions for the (n,α) problem, then f satisfies the boundary conditions for the (α,α)

problem, and $f^{(\alpha)}$ satisfies the boundary conditions for the $(0, n-\alpha)$ problem, so

$$\|f\|_p \leq H(\alpha,\alpha,p,r)\|f^{(\alpha)}\|_r \quad \text{and} \quad \|f^{(\alpha)}\|_r \leq H(n-\alpha,0,r,q)\|f^{(n)}\|_q.$$

Combining these two we have

$$\|f\|_p \leq H(\alpha,\alpha,p,r)H(n-\alpha,0,r,q)\|f^{(n)}\|_q.$$

But $H(n,\alpha,p,q)$ is the smallest number for which this is to hold. Now by graphing the functions backwards, $H(n-\alpha,0,p,q) = H(n-\alpha, n-\alpha, p, q)$.

If we replace r by 1 and ∞, we get the following estimates.

Corollary. *For $0 < \alpha < n$ and any p and q*

(52)
$$H(n,\alpha,p,q) \leq \{(\alpha-1)!(n-\alpha)![(\alpha-1)p+1]^{1/p}[(n-\alpha)q'+1]^{1/q'}\}^{-1}$$

and

(53)
$$H(n,\alpha,p,q) \leq \{\alpha!(n-\alpha-1)![\alpha p+1]^{1/p}[(n-\alpha-1)q'+1]^{1/q'}\}^{-1}.$$

Proof. We apply the results of the theorem and (48).

We now look at a third example from Fink [**9**]. Consider the problem of finding the best constants in the inequality

$$\|f\|_p \leq D(n,p,q)(b-a)^{n+1/p-1/q}\|f^{(n)}\|_q \tag{54}$$

with the boundary condition that f has n zeros at a and n zeros at b which we write as $f \in Z(n,n)$.

The related inequality is

$$\|f\|_p \leq F(n,p,q)(b-a)^{n+1/p-1/q}\|f^{(n)}\|_q \tag{55}$$

with the boundary condition that $|f|^{p-1}\mathrm{sgn}\,(f(t)) \perp \pi_{n-1}$, i.e., $\int_a^b |f|^{p-1}\mathrm{sgn}\,(f(t))p(t)\,dt = 0$ for all polynomials of degree $\leq n-1$. Here $p < \infty$ and π_{n-1} is the set of polynomials of degree $\leq n-1$.

The powers of $(b-a)$ are chosen so that the constants are independent of the interval so we will do the statements on $[0,1]$.

We plan to show that these are adjoint problems, but we need a couple of preliminary results.

Lemma 1. *If f has the required differentiability, then $f \in Z(n,n)$ if and only if $f^{(n)} \perp \pi_{n-1}$. The relationship is given by*

$$f(t) = \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1} f^{(n)}(s)\,ds. \tag{56}$$

Proof. If $f \in Z(n,n)$, then f has the representation (56). It is easy to show that this representation having n zeros at 1 implies that $f^{(n)} \perp$ the powers of $(1-t)^j$ for $j = 0,\dots,n-1$. These are a basis for π_{n-1}. Conversely, if $f^{(n)} \perp \pi_{n-1}$, then define f by (56) and $f \in Z(n,n)$.

Lemma 2. *If g is any integrable function there is a unique $f \perp \pi_{n-1}$ such that $f^{(n)} = g$.*

Proof. Let f_0 be any function so that $f_0^{(n)} = g$ and p be a polynomial on degree $\le n-1$. Then $f = f_0 + p$ also solves the differential equation. Consider the problem $\min_{p\in\pi_{n-1}} \|f_0 + p\|_2$. This finite dimensional problem has a solution $f_0 + p_0 \equiv f_1$. The inequality $\|f_1\|_2 \le \|f_1 + ap\|_2$ for arbitrary real a, and $p \in \pi_{n-1}$ leads to $\int_{-1}^1 f_1 p = 0$ as required. Uniqueness now follows from $\|f_1\|_2 = \|f_1 + p\|_2$, $p \in \pi_{n-1} \Rightarrow \|p^2\|_2 = 0$.

These two lemmas show that the boundary conditions are adjoint with respect to the operator $Ly = y^{(n)}$ so that we immediately have the next theorem and all of the results of the general theory.

Theorem 10. *We have $D(n,p,q) = F(n,q',p')$ provided $1 < q$.*

There is an interesting connection between this theorem and best approximation by polynomials. Let $p > 1$ and f be $n+1$ times differentiable and consider the number

$$E(f,n,p) = \min\{\|f-P\|_p : P \in \pi_n\}. \tag{57}$$

This minimum exists and it is well known that P solves this problem if and only if $\|f-P|^{p-1}\mathrm{sgn}\,(f-P)\perp\pi_n$. We seek the best possible constant in the inequality

$$E(f,n,p)\le B(n,p,q)(b-a)^{n+1+1/p-1/q}\|f^{(n+1)}\|_q. \tag{58}$$

We have

Theorem 11. *For any $p>1$, $B(n,p,q)=F(n+1,p,q)$.*

Proof. If f is given and P is a polynomial in π_n that gives $E(f,n,p)$, then $E(f,n,p)=\|f-P\|_p$ and $\|f-P\|^{p-1}\mathrm{sgn}\,(f-P)\perp\pi_n$. Consequently, we have $E(f,n,p)\le F(n+1,p,q)(b-a)^{(n+1+1/p-1/q}\|f^{(n+1)}\|_q$ where we have used $P^{(n+1)}=0$. Since $B(n,p,q)$ is the smallest constant, we have that $B(n,p,q)\le F(n+1,p,q)$. Conversely, if f satisfies $|f|^{p-1}\mathrm{sgn}\,(f)\perp\pi_n$, the boundary conditions for the $F(n+1,p,q)$ problem, then the best approximation to f by polynomials is the zero polynomial and we have $\|f\|_p=E(f,p,q)=\|f-P\|_p\le B(n,p,q)(b-a)^{n+1+1/p-1/q}\|f^{(n+1)}\|_q$ so that $F(n+1,p,q)\le B(n,p,q)$.

Phillips [**10**] showed that $B(n,p,\infty)=\delta(n,p)/(n+1)!2^{n+1+1/p}$, where $\delta(n,p)=\inf\{\|x^n-q\|_p : q\in\pi_{n-1}\}$. He shows that

$$\delta(n,1)=\delta(n,\infty)=\frac{1}{2^n}$$

and

$$\delta(n,2)=\sqrt{\frac{2}{2n+3}}\left(\frac{2^{n+1}}{2n+2}\right)C_{n+1}.$$

We mention in passing one version of the above inequalities for discrete functions, Fink [**11**]. Specifically there are inequalities of the form

$$\left(\sum_0^m |x_k|^p\right)^{1/p} \le C(m,n,\alpha,p,q)(m-n+1)^{n+1/p-1/q}\left(\sum_0^{m-n}|\Delta^n x_k|^q\right)^{1/q} \tag{59}$$

with the boundary conditions $x_0 = x_1 = \cdots = x_{\alpha-1} = 0 = x_{\alpha+1} = \cdots = x_{m-1} = x_m$ and Δ is the forward difference operator $\Delta x = x_{k-1} - x_k$, $\alpha + \beta = n$. These constants satisfy the same sort of monotonicity conditions as in the general continuous theory and are self-conjugate in the same way as the K constants in our first example.

4. Continuity properties. There are undoubtedly other interesting examples of the above inequalities, especially for operators which are not simply $f^{(n)}$. In addition, there are some other properties of the constants.

By an argument in Fink [**9**], one can show that extremals are unique in the above examples. We do not reproduce that proof here.

Since the constants themselves are log convex, it follows that as functions of p and q they are continuous and bounded on the open intervals $(1, \infty)$ and therefore have derivatives almost everywhere and one-sided derivatives everywhere. The continuity in p for fixed q can be extended to the interval $(1, \infty]$. For let f be given. Because as $p \to \infty$, both $\|f\|_p$ and the constants $C\{L, [a,b], p, q\}$ are increasing and bounded, they have limits. The limit of $\|f\|_p$ is $\|f\|_\infty$ so that we have

$$\|f\|_\infty \le \lim_{p\to\infty} C\{L, [a,b], p, q\}(b-a)^{-1/q}\|Lf\|_q$$

so that $C\{L, [a,b], \infty, q\} \le \lim_{p\to\infty} C\{L, [a,b], p, q\}$. But since $C\{L, [a,b], p, q\} \le C\{L, [a,b], \infty, q\}$ the reverse inequality also holds. For $q < \infty$, we can also prove that the constant is continuous at $p = 1$. For let $p_j \downarrow 1$ and f_j be selected so that f_j satisfies $\|Lf_j\|_q = 1$ and $\|f_j\|_p > C\{L, [a,b], p_j, q\} - 1/j$. By weak compactness of the unit ball in L_q we can assume that everything converges, f_j converges to g, Lf_j converges to Lg, so that

$$\begin{aligned} C\{L, [a,b], p_j, q\}(b-a)^{1-1/q}\|Lg\| &\ge \|g\|_1 \\ &\ge \lim_{j\to\infty} C\{L, [a,b], p_j, q\}(b-a)^{(1/p_j)-1/q}\|Lg\|. \end{aligned}$$

Since the constants are increasing in $p, C\{L, [a,b], 1, q\} \le \lim C\{L, [a,b], p_j, q\}$. But the above argument gives the reverse. By the conjugacy property, this gives the same continuity properties in the variable q. These continuity properties allow us to extend the log convexity also.

Proposition. *For any q and $p_1 < p_2$,*

$$C(p_2, q)^{p_2} \le C(p_1, q)^{p_1} C(p_1, q)^{p_2 - p_1}. \tag{60}$$

Proof. Let $p_1 < p_2 < p_3$ and write $p_2 = (1-\lambda)p_1 + \lambda p_3$. Then $C(p_2, q)^{p_2} \le C(p_1, q)^{(\lambda-1)p_1} C(p_1, q)^{\lambda p_3}$. Solving the p_2 equation for λ we find that $\lambda \to 0$ as $p_3 \to \infty$, but $\lambda p_3 \to p_2 - p_1$. Using the continuity of the C functions in the first variable, we find the result.

The case when $p_1 = 1$ is particularly interesting, i.e., the inequality

$$C(p, q)^p \le C(1, q) C(q', 1)^{p-1}. \tag{61}$$

5. Further work. Of course one would also like to be able to compute more constants explicitly. It is worth trying to discover all the properties of the constants themselves as p and q vary. The continuity and differentiability almost everywhere leads one to wonder if one could derive a differential equation or differential inequality that the constants satisfy. With enough properties of the constants one would hope to be able to prove that there is a unique solution to the list of properties. This would provide a way to get the constants by guessing a function that satisfies all the requirements. I leave this as a challenge to the interested reader.

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Mathematics Department, Iowa State University, Ames, Iowa 50011
E-mail address: `fink@math.iastate.edu`