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THE ASYMPTOTIC OF SOLUTIONS FOR A CLASS OF DELAY DIFFERENTIAL EQUATIONS

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ABSTRACT. We study the asymptotic properties of solutions of the differential equation

$$\dot{x}(t) = -c(t)[x(t) - Lx(\tau(t))]$$

with a positive continuous function $c(t)$, a nonzero real constant L and unbounded lag. We establish conditions under which each solution of this equation approaches a solution of the auxiliary functional equation

$$\psi(t) = L\psi(\tau(t)).$$

Moreover, we investigate some modifications of the studied equation and give comparisons with the known results.

1. Introduction. We investigate the asymptotic behavior of solutions of the delay differential equation

$$\dot{x}(t) = -c(t)[x(t) - Lx(\tau(t))], \quad t \in I = [t_0, \infty), \tag{1.1}$$

where $c(t)$ is a positive continuous function on I, L is a nonzero real scalar and $\tau(t)$ is a continuously differentiable function such that $\tau(t) \to \infty$ as $t \to \infty$, $\tau(t) < t$ and $0 < \dot{\tau}(t) \le \lambda < 1$ for every $t \in I$.

Our assumptions imply that the lag $t - \tau(t)$ must be necessarily unbounded on I. Equations with this type of a delay have diverse applications in areas ranging from the number theory to industrial problems. The objective of many authors have been especially equations with the proportional argument (see [**6**], [**8**], [**10**], [**14**] and others) and equations with the linearly transformed argument (see [**3**] and references

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therein). Among papers devoted to the study of differential equations with the general form of unbounded delay, we can mention those of Heard [**5**], Jaroš [**7**], Makay and Terjéki [**11**] and [**1**], [**2**]. We note that these authors investigated equation (1.1) and its modifications under assumptions close to ours.

In this paper we wish to generalize and improve asymptotic results discussed in the above papers. Moreover, we relate asymptotic behavior of all solutions of (1.1) to the behavior of a solution of an auxiliary functional equation.

2. Preliminaries. Throughout this paper we assume that $\tau(t) \in C^1(I)$, $\tau(t) \to \infty$ as $t \to \infty$, $\tau(t) < t$ and $0 < \dot{\tau}(t) \le \lambda < 1$ for every $t \in I$. We note that without significant complication we can consider also $\tau(t)$ with the property $\tau(t_0) = t_0$.

By a solution of (1.1) we understand a real or complex valued function $x(t) \in C^0([\tau(t_0), t_0] \cup I)$, $x(t) \in C^1(I)$ such that $x(t)$ satisfies (1.1) for every $t \in I$. Similarly we introduce the notion of a solution for other delay equations occurring in this paper.

The key tool in establishing our results is a transformation converting an equation with unbounded delay to an equation with constant delay. The substitutions of this type have been already employed in the above cited papers [**5**], [**8**], [**11**], [**14**] and [**1**]. It should be noted that the systematic transformation theory of functional differential equations has been started and developed especially by [**5**], [**12**], [**13**], [**15**] and [**16**]. In this section, we state some simple facts that we use in our investigations.

We start with the study of the functional equation

$$\varphi(\tau(t)) = \lambda\varphi(t), \quad t \in I, \tag{2.1}$$

where the real parameter λ is introduced in accordance with our previous notation as

$$\lambda = \sup\{\dot{\tau}(t), t \in I\}.$$

In the sequel we mean by the symbol $\tau^k(t)$, $k \in \mathbf{Z}$, the k-th iterate of $\tau(t)$ (for $k > 0$) or the $-k$-th iterate of the inverse function $\tau^{-1}(t)$ (for $k < 0$) and put $\tau^0(t) \equiv t$.

Proposition 2.1. *Let* $\varphi_0(t) \in C^1(I_0)$*, where* $I_0 = [\tau(t_0), t_0]$ *be a function such that* $\varphi_0(t) > 0$*,* $\dot{\varphi}_0(t) > 0$ *for every* $t \in I_0$ *and*

$$(\varphi_0 \circ \tau)^{(n)}(t_0) = \lambda \varphi_0^{(n)}(t_0), \quad n = 0, 1.$$

Then the formula

$$\varphi(t) = \lambda^{-k} \varphi_0(\tau^k(t)), \quad \tau^{-k+1}(t_0) \le t \le \tau^{-k}(t_0), \quad k = 0, 1, 2, \dots , \tag{2.2}$$

defines a unique solution $\varphi(t) \in C^1(I_0 \cup I)$ *of* (2.1) *such that* $\varphi(t) \to \infty$ *as* $t \to \infty$*,* $\varphi(t) > 0$*,* $0 < \dot{\varphi}(t) \le M$ *for every* $t \in I_0 \cup I$ *and a suitable constant* $M > 0$ *and* $\varphi(t) \equiv \varphi_0(t)$ *on* I_0*.*

Proof. The validity of formula (2.2) can be easily proved by the step method. We show the above stated properties of $\varphi(t)$. The solution $\varphi(t)$ must be unbounded as $t \to \infty$ because $\tau(t)$ is unbounded as $t \to \infty$. The fact that $\varphi(t) > 0$ on $I_0 \cup I$ is obvious. Differentiating (2.1) we get that $\dot{\varphi}(t)$ is a solution of

$$\dot{\varphi}(\tau(t)) = \frac{\lambda}{\dot{\tau}(t)} \, \dot{\varphi}(t).$$

Hence, $\dot{\varphi}(t)$ is positive on $I_0 \cup I$ and due to the inequality $\lambda / \dot{\tau}(t) \ge 1$, is also bounded as $t \to \infty$. □

Remark 2.2. If we admit functions $\tau(t)$ intersecting the identity as t_0, then the assumptions of Proposition 2.1 have to be slightly modified. Nevertheless, all the conclusions remain valid (see also [**9**, Chapter 2]).

Remark 2.3. The required solution of equation (2.1) can be given in several important cases explicitly. These cases are discussed in Section 3.

Remark 2.4. Notice that the substitution $s = \log \varphi(t)$ enables us to converge equation (1.1) to an equation with constant delay. This fact is used in the proofs of our asymptotic results, where we introduce also the change of the dependent variable to obtain the transformed equation in a more convenient form.

3. Asymptotic behavior of solutions. In this section, we first derive the asymptotic estimate of solutions of the more general equation than (1.1).

Theorem 3.1. *Let $\varphi(t)$ be a solution of* (2.1) *given by* (2.2). *Let $x(t)$ be a solution of the equation*

$$\dot{x}(t) = -a(t)x(t) + b(t)x(\tau(t)), \quad t \in I, \tag{3.1}$$

where $a(t), b(t) \in C^0(I)$, $a(t) \geq K/(\varphi(t))^\kappa$, $0 < |b(t)| \leq Qa(t)$ for every $t \in I$ and suitable reals $\kappa < 1$, $K > 0$, $Q > 0$. Then

$$x(t) = O((\varphi(t))^\gamma) \quad \text{as } t \to \infty, \quad \gamma = \frac{\log Q}{\log \lambda^{-1}}. \tag{3.2}$$

Proof. We introduce a change of variables

$$s = \log \varphi(t), \quad w(s) = (\varphi(t))^{-\gamma} x(t)$$

in (3.1) to obtain

$$w'(s) = -(a(h(s))h'(s) + \gamma)w(s) + b(h(s))\lambda^\gamma h'(s)w(s-c), \\ s \in J = [s_0, \infty),$$

where "$'$" means d/ds, $h(s) \equiv \varphi^{-1}(e^s)$ on J, $c = \log \lambda^{-1}$ and $s_0 = \log \varphi(t_0)$. From here we get

$$\frac{d}{ds}\Big[\exp\Big\{\gamma s + \int_{s_0}^{h(s)} a(u)\, du\Big\} w(s)\Big] \\ = b(h(s))\lambda^\gamma h'(s) \exp\Big\{\gamma s + \int_{s_0}^{h(s)} a(u)\, du\Big\} w(s-c). \tag{3.3}$$

Due to Proposition 2.1,

$$\frac{1}{h'(s)} = \frac{\dot{\varphi}(h(s))}{\varphi(h(s))} = O(e^{-s}) \quad \text{as } s \to \infty.$$

Consequently,

$$a(h(s))h'(s) \geq K_1 e^{(1-\kappa)s} \tag{3.4}$$

for a suitable real $K_1 > 0$ and every $s \geq s_0$. Thus we can choose $\xi_0 \geq s_0$ such that $\gamma + a(h(s))h'(s) > 0$ for every $s \geq \xi_0$. Put $\xi_i := \xi_0 + ic$, $J_i := [\xi_{i-1}, \xi_i]$ and $M_i := \max\{|w(s)|, s \in J_i\}$, $i = 1, 2, \ldots$. If we choose any $s^* \in J_{i+1}$, then the integration (3.3) over $[\xi_i, s^*]$ yields

$$\exp\Big\{\gamma s + \int_{s_0}^{h(s)} a(u)\,du\Big\} w(s)\Big|_{\xi_i}^{s^*} = \int_{\xi_i}^{s^*} b(h(s))\lambda^\gamma h'(s) \exp\Big\{\gamma s + \int_{s_0}^{h(s)} a(u)\,du\Big\} w(s-c)\,ds.$$

Then

$$\begin{aligned} w(s^*) = {} & \exp\Big\{\gamma(\xi_i - s^*) - \int_{h(\xi_i)}^{h(s^*)} a(u)\,du\Big\} w(\xi_i) \\ & + \exp\Big\{-\int_{s_0}^{h(s^*)} a(u)\,du - \gamma s^*\Big\} \int_{\xi_i}^{s^*} b(h(s))\lambda^\gamma h'(s) \\ & \cdot \exp\Big\{\gamma s + \int_{s_0}^{h(s)} a(u)\,du\Big\} w(s-c)\,ds. \end{aligned}$$

Consequently,

$$\begin{aligned} |w(s^*)| \leq {} & M_i \exp\Big\{\gamma(\xi_i - s^*) - \int_{h(\xi_i)}^{h(s^*)} a(u)\,du\Big\} \\ & + M_i \exp\Big\{-\int_{s_0}^{h(s^*)} a(u)\,du - \gamma s^*\Big\} \\ & \cdot \int_{\xi_i}^{s^*} |b(h(s))|\lambda^\gamma h'(s) \exp\Big\{\gamma s + \int_{s_0}^{h(s)} a(u)\,du\Big\}\,ds \\ \leq {} & M_i \exp\Big\{\gamma(\xi_i - s^*) - \int_{h(\xi_i)}^{h(s^*)} a(u)\,du\Big\} \qquad (3.5) \\ & + M_i \exp\Big\{-\int_{s_0}^{h(s^*)} a(u)\,du - \gamma s^*\Big\} \\ & \cdot \int_{\xi_i}^{s^*} a(h(s))h'(s) \exp\Big\{\gamma s + \int_{s_0}^{h(s)} a(u)\,du\Big\}\,ds. \end{aligned}$$

Now we estimate the integral

$$I_1 := \int_{\xi_i}^{s^*} a(h(s))h'(s)\exp\Big\{\gamma s + \int_{s_0}^{h(s)} a(u)\,du\Big\}\,ds$$

occurring in (3.5). Obviously

$$I_1 \le \exp\Big\{\gamma s + \int_{s_0}^{h(s)} a(u)\,du\Big\}\Big|_{\xi_i}^{s^*} + |\gamma|\int_{\xi_i}^{s^*}\exp\Big\{\gamma s + \int_{s_0}^{h(s)} a(u)\,du\Big\}\,ds.$$

Let

$$I_2 := |\gamma|\int_{\xi_i}^{s^*}\exp\Big\{\gamma s + \int_{s_0}^{h(s)} a(u)\,du\Big\}\,ds.$$

Then

$$I_2 = \int_{\xi_i}^{s^*}\frac{|\gamma|}{\gamma + a(h(s))h'(s)}\,\frac{d}{ds}\Big[\exp\Big\{\gamma s + \int_{s_0}^{h(s)} a(u)\,du\Big\}\Big]\,ds.$$

Notice that

$$\frac{|\gamma|}{\gamma + a(h(s))h'(s)} = O(\exp\{(\kappa - 1)s\}) \quad \text{as } s \to \infty$$

by use of (3.4). Put $\delta = 1 - \kappa > 0$. Then

$$\begin{aligned} I_2 &\le K_2\int_{\xi_i}^{s^*} e^{-\delta s}\frac{d}{ds}\Big[\exp\Big\{\gamma s + \int_{s_0}^{h(s)} a(u)\,du\Big\}\Big]\,ds \\ &\le K_2 e^{-\delta\xi_i}\exp\Big\{\gamma s + \int_{s_0}^{h(s)} a(u)\,du\Big\}\Big|_{\xi_i}^{s^*} \end{aligned}$$

for a suitable $K_2 > 0$. Summarizing these estimates, we get

$$I_1 \le \exp\Big\{\gamma s + \int_{s_0}^{h(s)} a(u)\,du\Big\}\Big|_{\xi_i}^{s^*}(1 + K_2 e^{-\delta\xi_i}).$$

Substituting this into (3.5), we obtain

$$\begin{aligned} |w(s^*)| \le{}& M_i\exp\Big\{\gamma(\xi_i - s^*) - \int_{h(\xi_i)}^{h(s^*)} a(u)\,du\Big\} \\ &+ M_i\exp\Big\{-\int_{s_0}^{h(s^*)} a(u)\,du - \gamma s^*\Big\} \\ &\qquad\cdot\exp\Big\{\gamma s + \int_{s_0}^{h(s)} a(u)\,du\Big\}\Big|_{\xi_i}^{s^*}(1 + K_2 e^{-\delta\xi_i}) \\ \le{}& M_i(1 + K_2 e^{-\delta\xi_i}). \end{aligned}$$

This implies that

$$M_{i+1} \le M_i(1+K_2e^{-\delta\xi_i}) \le M_1 \prod_{k=1}^{i}(1+K_2e^{-\delta\xi_k}), \quad i=1,2,\dots .$$

Thus, the convergence of the infinite product

$$\prod_{k=1}^{\infty}(1+K_2e^{-\delta\xi_k})$$

implies that (M_i) is bounded as $i \to \infty$. This proves that $w(s)$ is bounded as $s \to \infty$, hence (3.2) holds. □

Now we use the conclusion of Theorem 3.1 to obtain a stronger asymptotic result for solutions of equation (1.1).

Theorem 3.2. *Let $\varphi(t)$ be a solution of (2.1) given by (2.2). Let $x(t)$ be a solution of (1.1), where $c(t) \in C^1(I)$, $c(t) \ge K/(\varphi(t))^\kappa$, $\dot c(t) \le M(c(t))^2$ for every $t \in I$ and suitable reals $K > 0$, $0 \le M < 1-\lambda$, $\kappa < \theta = 1 - \log(1-M)/\log\lambda$. Then there exists a continuous periodic function $g(s)$ of period $\log\lambda^{-1}$ such that*

$$x(t) = (\varphi(t))^\alpha g(\log\varphi(t)) + O((\varphi(t))^\beta) \quad \text{as } t \to \infty, \tag{3.6}$$

where α is a (possibly complex) constant such that $\lambda^\alpha = 1/L$ and $\beta = \operatorname{Re}\alpha - \theta + \kappa$.

Proof. We set

$$s = \log\varphi(t), \quad w(s) = (\varphi(t))^{-\alpha}x(t)$$

in (1.1) to obtain the equation

$$\begin{gathered} w'(s) = -c(h(s))h'(s)(w(s)-w(s-c)) - \alpha w(s), \\ s \in J = [s_0, \infty), \end{gathered} \tag{3.7}$$

where we use the same notation as in the proof of Theorem 3.1. By Theorem 3.1,

$$x(t) = O((\varphi(t))^\gamma) \quad \text{as } t \to \infty, \quad \gamma = \operatorname{Re}\alpha,$$

hence $w(s)$ is bounded on J. We derive the asymptotic estimate of $w'(s)$.

Differentiating (1.1) we have

$$\ddot{x}(t) = -\Big(c(t) - \frac{\dot{c}(t)}{c(t)}\Big)\dot{x}(t) + L\dot{\tau}(t)c(t)\dot{x}(\tau(t)). \tag{3.8}$$

We verify the assumptions of Theorem 3.1 for equation (3.8). The inequality

$$\dot{c}(t) \le M(c(t))^2$$

is equivalent to

$$\lambda c(t) \le \lambda_1\Big(c(t) - \frac{\dot{c}(t)}{c(t)}\Big), \quad \text{where } \lambda_1 = \frac{\lambda}{1-M}.$$

Consequently,

$$|L|\dot{\tau}(t)c(t) \le |L|\lambda c(t) \le |L|\lambda_1\Big(c(t) - \frac{\dot{c}(t)}{c(t)}\Big).$$

Further,

$$c(t) - \frac{\dot{c}(t)}{c(t)} \ge (1-M)c(t) \ge \frac{(1-M)K}{(\varphi(t))^\kappa}.$$

Then the repeated application of Theorem 3.1 (with $a(t) = c(t) - \dot{c}(t)/c(t)$, $b(t) = L\dot{\tau}(t)c(t)$, $Q = |L|\lambda_1$) yields

$$\dot{x}(t) = O((\varphi(t))^{\gamma-\theta}) \quad \text{as } t \to \infty,$$
$$\theta = 1 - \frac{\log(1-M)}{\log \lambda} > 0.$$

Thus we can estimate $w'(s)$ as

$$w'(s) = \alpha e^{-\alpha s}x(h(s)) + e^{-\alpha s}x'(h(s))h'(s) = O(e^{-\theta s}h'(s)) \quad \text{as } s \to \infty.$$

Now from (3.7)

$$w(s) - w(s-c) = O(e^{-\delta s}) \quad \text{as } s \to \infty, \quad \delta = \theta - \kappa > 0.$$

Hence, the sequence $(w(s+nc))_{n=1}^{\infty}$ is Cauchy and converges to a continuous periodic function $g(s)$ of period $c = \log \lambda^{-1}$ such that

$$w(s) = g(s) + O(e^{-\delta s}) \quad \text{as } s \to \infty.$$

This proves the asymptotic relation (3.6) and completes the proof. □

Remark 3.3. It is easy to verify that the function $\psi(t) = (\varphi(t))^{\alpha}$ occurring in (3.6) defines a solution of the functional equation

$$\psi(t) - L\psi(\tau(t)) = 0, \quad t \in I. \tag{3.9}$$

Thus asymptotic relation (3.6) essentially says that each solution of (1.1) approaches a solution of functional equation (3.9).

Corollary 3.4. *Let $x(t)$ be a solution of the equation*

$$\dot{x}(t) = -c(t)[x(t) - Lx(\lambda t)], \quad t \in I = [0, \infty), \tag{3.10}$$

where $0 < \lambda < 1$, $c(t) \in C^1(I)$, $c(t) \geq K/t^{\kappa}$, $\dot{c}(t) \leq M(c(t))^2$ for every $t \in I$ and suitable reals $K > 0$, $0 \leq M < 1 - \lambda$, $\kappa < \theta = 1 - \log(1-M)/\log \lambda$. Then there exists a continuous periodic function $g(s)$ of period log λ^{-1} such that

$$x(t) = t^{\alpha} g(\log t) + O(t^{\beta}), \quad \text{as } t \to \infty,$$

where α, β are the same as in Theorem 3.2.

Proof. It is enough to verify that $\varphi(t) = t$ is a required solution of (2.1). Then our conclusion follows immediately from (3.6). □

Remark 3.5. The asymptotic behavior of solutions of equation (3.10) with $L = 1$ has been the object of a paper by Makay and Terjéki [**11**]. If we put $L = 1$ in Corollary 3.4, we just obtain Theorem 5 of [**11**]. Hence, formula (3.6) generalizes some parts of this paper.

Corollary 3.6. *Let $x(t)$ be a solution of the equation*

$$\dot{x}(t) = -c(t)[x(t) - Lx(t^{\omega})], \quad t \in I = [1, \infty),$$

where $0 < \omega < 1$, $c(t) \in C^1(I)$, $c(t) \geq K/(\log t)^\kappa$, $\dot{c}(t) \leq M(c(t))^2$ *for every* $t \in I$ *and suitable reals* $K > 0$, $0 \leq M < 1 - \omega$, $\kappa < \theta = 1 - \log(1 - M)/\log\omega$. *Then there exists a continuous periodic function* $g(s)$ *of period* $\log\omega^{-1}$ *such that*

$$x(t) = (\log t)^\alpha g(\log\log t) + O((\log t)^\beta) \quad \text{as } t \to \infty,$$

where α *is a complex constant such that* $\omega^\alpha = 1/L$ *and* $\beta = Re\,\alpha - \theta + \kappa$.

Proof. Apply again Theorem 3.2 with $\lambda = \omega$ and with $\varphi(t) = \log t$ as a solution of (2.1) having the required properties. □

In the following assertions, we again assume that $\tau(t)$ fulfills the assumptions introduced in Section 2. Since the proofs of these corollaries are obvious, we can omit them.

Corollary 3.7. *Let* $\varphi(t)$ *be a solution of* (2.1) *given by* (2.2). *Let* $x(t)$ *be a solution of the equation*

$$\dot{x}(t) = -ax(t) + bx(\tau(t)), \quad t \in I, \tag{3.11}$$

where $a > 0$, $b \neq 0$ *are real scalars. Then there exists a continuous periodic function* $g(s)$ *of period log* λ^{-1} *such that* (3.6) *holds, where* α *is a complex constant such that* $b\lambda^\alpha = a$ *and* $\beta = \operatorname{Re}\alpha - 1$.

Remark 3.8. This asymptotic expansion of solutions of (3.11) has been formulated also in Theorem 3.1 of [**5**]. Consequently, Theorem 3.2 generalizes this assertion. Moreover we do not require $\tau(t) \in C^2(I)$ and $\dot{\tau}(t)$ decreasing on I as it has been assumed in [**5**].

Corollary 3.9. *Let* $\varphi(t)$ *be a solution of* (2.1) *given by* (2.2). *Let* $x(t)$ *be a solution of the equation*

$$\dot{x}(t) = -c(t)[x(t) - x(\tau(t))], \quad t \in I, \tag{3.12}$$

where $c(t) \in C^1(I)$, $c(t) \geq K/(\varphi(t))^\kappa$, $\dot{c}(t) \leq M(c(t))^2$ *for every* $t \in I$ *and suitable reals* $K > 0$, $0 \leq M < 1 - \lambda$, $\kappa < 1 - \log(1 - M)/\log\lambda$.

Then there exists a continuous periodic function $g(s)$ of period $\log\lambda^{-1}$ *such that*

$$|x(t) - g(\log\varphi(t))| \to 0 \quad \text{as } t \to \infty.$$

Remark 3.10. Equation (3.12) has been investigated in many papers (for results and references, see Diblík [**4**]). Corollary 3.9 extends the validity of some of these results.

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