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COHEN-MACAULAYNESS OF TENSOR PRODUCTS

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ABSTRACT. Let (R, m) be a commutative noetherian local ring. Suppose that M and N are finitely generated modules over R such that M has finite projective dimension and such that $\operatorname{Tor}_i^R(M, N) = 0$ for all $i > 0$. The main result of this note gives a condition on M which is necessary and sufficient for the tensor product of M and N to be a Cohen-Macaulay module over R, provided N is itself a Cohen-Macaulay module.

1. Introduction. Throughout this note $(R, \mathfrak{m})$ is a commutative noetherian local ring with nonzero identity and the maximal ideal $\mathfrak{m}$. By M and N we always mean nonzero finitely generated R-modules. The projective dimension of an R-module M is denoted by $\operatorname{proj.dim} M$.

The well-known notion "grade of M", grade M, has been introduced by Rees, see [**8**], as the least integer $t \geq 0$ such that $\operatorname{Ext}_R^t(M, R) \neq 0$. In [**10**], we have defined the "grade of M and N", grade (M, N), as the least integer $t \geq 0$ such that $\operatorname{Ext}_R^t(M, N) \neq 0$.

One of the main results of this note is Theorem 1.8, and it states:

Let N be a Cohen-Macaulay R-module, and let M be an R-module with finite projective dimension. If $\operatorname{Tor}_i^R(M, N) = 0$ for all $i > 0$, then $M \otimes_R N$ is Cohen-Macaulay if and only if $\operatorname{grade}(M, N) = \operatorname{proj.dim} M$.

This theorem can be considered as a generalization of the following well-known statement, cf. [**4**, Theorem 2.1.5]:

(T1) Let R be a Cohen-Macaulay local ring, and let M be a finite R-module with finite projective dimension. Then M is a Cohen-Macaulay if and only if $\operatorname{grade} M = \operatorname{proj.dim} M$.

On the other hand the following statement from Yoshida can be concluded from our result:

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Yoshida [**11**, Proposition 2.4]. "Suppose that grade $M = \text{proj.dim}\, M (< \infty)$ and that N is a maximal Cohen-Macaulay R-module (that is depth $N = \dim N = \dim R$). Then $M \otimes_R N$ is Cohen-Macaulay and $\dim M \otimes_R N = \dim M$."

In another theorem of the first section we improve a theorem due to Kawasaki:

Kawasaki [**6**, Theorem 3.3(i)]. "Let R be a Cohen-Macaulay local ring, and let K be a canonical module of R. Let M be a finite R-module of finite projective dimension. Then $M \otimes_R K$ is Cohen-Macaulay if and only if M is Cohen-Macaulay."

The following statement, which is our Theorem 1.11, generalizes Kawasaki's theorem:

Let R be a Cohen-Macaulay local ring, and let K be a canonical module of R. If M is an R-module with finite Gorenstein dimension, then $M \otimes_R K$ is Cohen-Macaulay if and only if M is Cohen-Macaulay.

Recall that the Gorenstein dimension is an invariant for finite modules which was introduced by Auslander in [**2**]. It is a finer invariant than projective dimension in the sense that for every finite nonzero R-module M, $G\text{-dim}\, M \leq \text{proj.dim}\, M$ and equality holds when $\text{proj.dim}\, M < \infty$. There exist modules with finite Gorenstein dimension which have infinite projective dimension.

In the second section we consider Serre's conditions. We say M satisfies Serre's condition (S_n) for a nonnegative integer n when, for every $\mathfrak{p} \in \text{Supp}\, M$ the following inequality holds:

$$\text{depth}\, M_{\mathfrak{p}} \geq \min(n, \dim M_{\mathfrak{p}}).$$

Obviously, every Cohen-Macaulay module satisfies (S_n) for all nonnegative integers n.

The main result of Section 2 is Theorem 2.4 which states:

Let M and N be R-modules such that $\text{Tor}_i^R(M, N) = 0$ for all $i > 0$. If projective dimension of M is finite and $M \otimes_R N$ satisfies (S_n), then so does N.

This result generalizes [**11**, Proposition 4.1].

1. Cohen-Macaulayness.

Definition 1.1. We define

$$\operatorname{grade}(M,N) = \inf\{i \mid \operatorname{Ext}^i_R(M,N) \neq 0\}.$$

Since M is finite, using [**4**, 1.2.10] we have that

$$\begin{aligned}\operatorname{grade}(M,N) &= \inf\{\operatorname{depth} N_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Supp} M\} \\ &= \inf\{\operatorname{depth} N_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Supp} M \cap \operatorname{Supp} N\}.\end{aligned}$$

The second equality holds because the depth of the zero module is defined to be infinite.

Proposition 1.2 [**10**, Theorem 2.1]. *The following inequalities hold*:

(a) $\operatorname{depth} N - \dim M \leq \operatorname{grade}(M,N)$;

(b) *If* $\operatorname{Supp} M \subseteq \operatorname{Supp} N$, *then* $\operatorname{grade}(M,N) \leq \dim N - \dim M$.

For a finite R-module M of finite projective dimension, the invariant $\operatorname{imp} M$, imperfection of M, is defined to be $\operatorname{proj.dim} M - \operatorname{grade} M$. This is, using the Auslander-Buchsbaum equality, equal to $\operatorname{depth} R - \operatorname{depth} M - \operatorname{grade} M$.

Definition 1.3. For finite R-modules M and N, which may have infinite projective dimensions, we define $\operatorname{imp}(M,N) = \operatorname{depth} N - \operatorname{depth} M - \operatorname{grade}(M,N)$ (this may be negative).

It is clear that if $\operatorname{proj.dim} M < \infty$, then $\operatorname{imp} M = \operatorname{imp}(M,R)$.

By $\operatorname{cmd} M$ we mean the difference $\dim M - \operatorname{depth} M$.

Proposition 1.4. *The following inequalities hold*:

(a) $\operatorname{imp}(M,N) \leq \operatorname{cmd} M$;

(b) *If* $\operatorname{Supp} M \subseteq \operatorname{Supp} N$, *then* $\operatorname{cmd} M \leq \operatorname{imp}(M,N) + \operatorname{cmd} N$.

Proof. This is clear from Proposition 1.2 and the definition.

Corollary 1.5. *Let N be a Cohen-Macaulay R-module and* $\operatorname{Supp} M \subseteq \operatorname{Supp} N$. *Then* $\operatorname{cmd} M = \mathit{imp}\,(M,N)$; *in particular, M is a Cohen-Macaulay module if and only if* $\operatorname{imp}(M,N) = 0$.

(T1) says that over a Cohen-Macaulay local ring R, the R-module M with finite projective dimension is Cohen-Macaulay if $\operatorname{Ext}^i_R(M,R) = 0$ for $i \neq \operatorname{proj.dim} M$. The following corollary is a generalization of (T1).

Corollary 1.6. *Let N be a Cohen-Macaulay R-module with* depth $N = \operatorname{depth} R$. *Let M have finite projective dimension and* $\operatorname{Supp} M \subseteq \operatorname{Supp} N$. *Then M is Cohen-Macaulay if and only if* $\operatorname{Ext}^i_R(M,N) = 0$ *for* $i \neq \operatorname{proj.dim} M$.

Proof. Note that $\operatorname{proj.dim} M = \sup\{i \mid \operatorname{Ext}^i_R(M,N) \neq 0 \text{ for any } N\}$, cf. [**7**] and so it is always greater than or equal to $\operatorname{grade}(M,N)$.

$$\begin{aligned} \operatorname{imp}(M,N) &= \operatorname{depth} N - \operatorname{depth} M - \operatorname{grade}(M,N) \\ &= \operatorname{depth} R - \operatorname{depth} M - \operatorname{grade}(M,N) \\ &== \operatorname{proj.dim} M - \operatorname{grade}(M,N). \end{aligned}$$

Now the claim is clear from Corollary 1.5. $\square$

Recall that a finite R-module M with finite projective dimension is called perfect if $\operatorname{proj.dim} M = \operatorname{grade} M$.

Definition 1.7. Let M and N be R-modules with $\operatorname{proj.dim} M < \infty$. We say that M is N-perfect if $\operatorname{proj.dim} M = \operatorname{grade}(M,N)$.

In the proof of the following statements we use the well-known result, cf. [**1**, Theorem 1.2].

(T2) Let M and N be finite R-modules with $\operatorname{proj.dim} M < \infty$. If $\operatorname{Tor}^R_i(M,N) = 0$ for all $i > 0$, then we have the equality $\operatorname{depth} M \otimes_R N = \operatorname{depth} N - \operatorname{proj.dim} M$.

Theorem 1.8. *Let N be a Cohen-Macaulay R-module, and let M be an R-module with finite projective dimension. If* $\operatorname{Tor}^R_i(M,N) = 0$

for all $i > 0$, then $M \otimes_R N$ is Cohen-Macaulay if and only if M is N-perfect.

Proof. We claim that M is N-perfect if and only if $\operatorname{imp}(M \otimes_R N, N) = 0$, and then the assertion will be clear from Corollary 1.5. We know that $\operatorname{depth} M \otimes_R N = \operatorname{depth} N - \operatorname{proj.dim} M$. On the other hand,

$$\begin{aligned}\operatorname{grade}(M \otimes_R N, N) &= \inf\{\operatorname{depth} N_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Supp} M \otimes_R N\}\\ &= \inf\{\operatorname{depth} N_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Supp} M \cap \operatorname{Supp} N\}\\ &= \operatorname{grade}(M, N).\end{aligned}$$

Then we have the equality $\operatorname{imp}(M \otimes_R N, N) = \operatorname{proj.dim} M - \operatorname{grade}(M, N)$, which proves our claim. □

Now [**11**, 2.4] can be deduced from the above theorem, for when N is a maximal Cohen-Macaulay and $\operatorname{proj.dim} M < \infty$ by [**11**, 2.2] we have that $\operatorname{Tor}_i^R(M, N) = 0$ for all $i > 0$. For every $\mathfrak{p} \in \operatorname{Supp} N$, the $R_{\mathfrak{p}}$-module $N_{\mathfrak{p}}$ is maximal Cohen-Macaulay module and, then $\operatorname{depth} N_{\mathfrak{p}} = \dim R_{\mathfrak{p}} \geq \operatorname{depth} R_{\mathfrak{p}}$ and hence we have inequalities

$$\operatorname{grade} M \leq \operatorname{grade}(M, N) \leq \operatorname{proj.dim} M.$$

This means that every perfect module is N-perfect.

Definition 1.9. A finite R-module N is said to be of Gorenstein dimension zero and we write $G\text{-}\dim N = 0$, if and only if

(a) $\operatorname{Ext}_R^i(N, R) = 0$ for $i > 0$.

(b) $\operatorname{Ext}_R^i(\operatorname{Hom}_R(N, R), R) = 0$ for $i > 0$.

(c) The canonical map $N \to \operatorname{Hom}_R(\operatorname{Hom}_R(N, R), R)$ is an isomorphism.

For a nonnegative integer n, the R-module N is said to be of Gorenstein dimension at most n, if and only if there exists an exact sequence

$$0 \longrightarrow G_n \longrightarrow G_{n-1} \longrightarrow \cdots \longrightarrow G_0 \longrightarrow N \longrightarrow 0$$

where $G\text{-}\dim G_i = 0$ for $0 \leq i \leq n$. If such a sequence does not exist, then $G\text{-}\dim N = \infty$.

Lemma 1.10 [**3**, 3.7, 3.14, 4.12]. *If G-$\dim M < \infty$, then the following hold*:

(a) G-$\dim M + \operatorname{depth} M = \operatorname{depth} R$.

(b) G-$\dim M = \sup\{t \mid \operatorname{Ext}_R^t(M,R) \neq 0\}$.

(c) $\operatorname{Tor}_i^R(M,P) = 0$ *for all* $i > G$-$\dim M$ *and all modules* P *with finite projective dimension.*

The following theorem improves Kawasaki's result [**6**, 3.3(i)].

Theorem 1.11. *Let R be a Cohen-Macaulay local ring, and let K be a canonical module of R. If M is an R-module with finite Gorenstein dimension, then $M \otimes_R N$ is Cohen-Macaulay if and only if M is Cohen-Macaulay.*

Proof. Proposition [**5**, 2.5] says that $\operatorname{Tor}_i^R(M,K) = 0$ for $i > 0$ and then, since injective dimension of K is finite, we have that $\operatorname{depth} M \otimes_R K = \operatorname{depth} K - G$-$\dim M$, cf., [**9**, 2.13]. Then $\operatorname{imp}(M \otimes_R K, K) = G$-$\dim K - \operatorname{grade}(M \otimes_R K, K)$. Since $\operatorname{Supp} K = \operatorname{Spec} R$, we have that

$$\begin{aligned}\operatorname{grade}(M \otimes_R K, K) &= \inf\{\operatorname{depth} K_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Supp} M \otimes K\}\\ &= \inf\{\operatorname{depth} K_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Supp} M\}.\end{aligned}$$

But, since $\operatorname{depth} K_{\mathfrak{p}} = \operatorname{depth} R_{\mathfrak{p}}$ for all prime ideals $\mathfrak{p} \in \operatorname{Supp} K = \operatorname{Spec} R$ we have that $\operatorname{grade}(M \otimes_R K, K) = \operatorname{grade} M$. The claim of the theorem is now clear from Corollary 1.5 and the fact that over a Cohen-Macaulay local ring R, the R-module M with finite Gorenstein dimension is Cohen-Macaulay if and only if $\operatorname{grade} M = G$-$\dim M$, cf. [**10**]. $\square$

2. Serre conditions.

First recall that, for a nonnegative integer n, we say that a finite R-module M satisfies Serre's condition (S_n) if $\operatorname{depth} M_{\mathfrak{p}} \geq \min(n, \dim M_{\mathfrak{p}})$ for every $\mathfrak{p} \in \operatorname{Supp} M$ or equivalently if $M_{\mathfrak{p}}$ is a Cohen-Macaulay $R_{\mathfrak{p}}$-module for every $\mathfrak{p} \in \operatorname{Supp} M$ such that $\operatorname{depth} M_{\mathfrak{p}} < n$.

We also recall a consequence of the new intersection theorem, cf., [**4**, Corollary 9.4.6].

(T3) Let M and N be finite R-modules with $\operatorname{proj.dim} M < \infty$. We have the inequality $\dim N \leq \operatorname{proj.dim} M + \dim (M \otimes_R N)$.

Theorem 2.1. *Let N be a finite R-module which satisfies (S_n). Let M be an N-perfect R-module with $t = \operatorname{proj.dim} M \leq n$, such that $\operatorname{Tor}_i^R(M, N) = 0$ for $i > 0$. Then $M \otimes_R N$ satisfies (S_{n-t}).*

Proof. For every $\mathfrak{p} \in \operatorname{Supp}(M \otimes_R N)$ it is clear that

$$\operatorname{grade}(M, N) \leq \operatorname{grade}(M_\mathfrak{p}, N_\mathfrak{p}) \leq \operatorname{proj.dim} M_\mathfrak{p} \leq \operatorname{proj.dim} M = t.$$

Since M is N-perfect, $M_\mathfrak{p}$ is $N_\mathfrak{p}$-perfect with $\operatorname{proj.dim} M_\mathfrak{p} = t$. From Proposition 1.2 we have that $t = \operatorname{grade}(M_\mathfrak{p}, N_\mathfrak{p}) = \operatorname{grade}((M \otimes_R N)_\mathfrak{p}, N_\mathfrak{p}) \leq \dim N_\mathfrak{p} - \dim (M \otimes_R N)_\mathfrak{p}$.

On the other hand, from the fact that N satisfies (S_n), we have the following claim

$$\begin{aligned}\operatorname{depth}(M \otimes_R N)_\mathfrak{p} &= \operatorname{depth}(M_\mathfrak{p} \otimes_{R_\mathfrak{p}} N_\mathfrak{p})\\ &= \operatorname{depth} N_\mathfrak{p} - \operatorname{proj.dim} M_\mathfrak{p}\\ &= \operatorname{depth} N_\mathfrak{p} - t\\ &\geq \min(n, \dim N_\mathfrak{p}) - t\\ &= \min(n - t, \dim N_\mathfrak{p} - t).\end{aligned}$$

Now the assertion holds. $\square$

Corollary 2.2. *If R satisfies (S_n), then every perfect R-module with projective dimension t, less than or equal to n, satisfies (S_{n-t}).*

It is well known that if a local ring admits a finite Cohen-Macaulay module with finite projective dimension, then the ring itself is Cohen-Macaulay.

In [**11**, 4.1] Yoshida has proved a more general statement, by replacing "being Cohen-Macaulay" with "satisfying Serre's condition (S_n)."

Our next two theorems improve those results by similar proofs. Theorem 2.3 is a special case of Theorem 2.4, and the proof of 2.3 is only included because it is so simple.

Theorem 2.3. *Let M and N be R-modules such that* $\operatorname{Tor}_i^R(M,N) = 0$ *for all $i > 0$. If* $\operatorname{proj.dim} M < \infty$ *and $M \otimes_R N$ is Cohen-Macaulay, then so is N.*

Proof. The intersection theorem (T3) gives the inequality

$$\dim N \leq \dim M \otimes_R N + \operatorname{proj.dim} M.$$

On the other hand (T2) gives the equality

$$\operatorname{depth} N = \operatorname{depth} M \otimes_R N + \operatorname{proj.dim} M.$$

Since $\dim N \geq \operatorname{depth} N$, the assertion is clear. $\square$

Theorem 2.4. *Let M and N be R-modules such that* $\operatorname{Tor}_i^R(M,N) = 0$ *for all $i > 0$. If* $\operatorname{proj.dim} M < \infty$ *and $M \otimes_R N$ satisfies (S_n), then so does N.*

Proof. Choose $\mathfrak{p} \in \operatorname{Supp} N$. There are two cases.

The first case is when $\mathfrak{p} \in \operatorname{Supp} M$ and then $\mathfrak{p} \in \operatorname{Supp} M \otimes_R N$.

If $\operatorname{depth} (M \otimes_R N)_\mathfrak{p} < n$, then $(M \otimes_R N)_\mathfrak{p}$ is Cohen-Macaulay and, by the Theorem 2.3 so is $N_\mathfrak{p}$.

If $\operatorname{depth} (M \otimes_R N)_\mathfrak{p} \geq n$, then $\operatorname{depth} N_\mathfrak{p} \geq n$ because by (T2) we have the equality $\operatorname{depth} N_\mathfrak{p} = \operatorname{depth} (M \otimes_R N)_\mathfrak{p} + \operatorname{proj.dim} M_\mathfrak{p}$. The second case is when $\mathfrak{p} \notin \operatorname{Supp} M$. Let $\mathfrak{q}$ be a minimal prime over the ideal $(\operatorname{Ann} M + \mathfrak{p})$. From (T3) we have the inequality

$$\dim R_\mathfrak{q}/\mathfrak{p}R_\mathfrak{q} \leq \operatorname{proj.dim} M_\mathfrak{q} + \dim M_\mathfrak{q}/\mathfrak{p}M_\mathfrak{q} = \operatorname{proj.dim} M_\mathfrak{q}.$$

Since $\mathfrak{p}R_\mathfrak{q} \in \operatorname{Supp} R_\mathfrak{q}/\mathfrak{p}R_\mathfrak{q}$ we have that

$$\begin{aligned} \operatorname{depth} N_\mathfrak{p} &\geq \operatorname{grade} (R_\mathfrak{q}/\mathfrak{p}R_\mathfrak{q}, N_\mathfrak{q}) \\ &\geq \operatorname{depth} N_\mathfrak{q} - \dim R_\mathfrak{q}/\mathfrak{p}R_\mathfrak{q} \text{ (Proposition 1.2)} \\ &\geq \operatorname{depth} N_\mathfrak{q} - \operatorname{proj.dim} M_\mathfrak{q} \\ &= \operatorname{depth} M_\mathfrak{q} \otimes_{R_q} N_\mathfrak{q}. \end{aligned}$$

If $\operatorname{depth} M_\mathfrak{q} \otimes_{R_\mathfrak{q}} N_\mathfrak{q} < n$, then $M_\mathfrak{q} \otimes_{R_\mathfrak{q}} N_\mathfrak{q}$ is Cohen-Macaulay and, from Theorem 2.3, we will have that $N_\mathfrak{q}$ is Cohen-Macaulay, then so is $N_\mathfrak{p} \cong (N_\mathfrak{q})_{\mathfrak{p}R_\mathfrak{q}}$.

If $\operatorname{depth} M_{\mathfrak{q}} \otimes_{R_{\mathfrak{q}}} N_{\mathfrak{q}} \geq n$, then the above inequality guarantees that $\operatorname{depth} N_{\mathfrak{p}} \geq n$. $\square$

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