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EXTENDED HADAMARD PRODUCTS, TRIGONOMETRIC INTEGRALS AND ASSOCIATED SUMS

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ABSTRACT. The notion of the Hadamard product is extended to evaluate an extensive number of trigonometric integrals in terms of sums. These sums are taken over index sets defined by a Diophantine equation which can be simplified in certain circumstances. The results obtained include generalizations of integrals defining sums of products of Bessel functions and integrals of powers of cosines evaluated in terms of sums of products of binomial coefficients. Generating functions for special polynomials are also called upon in some of the developments.

1. Introduction. Let $f(z_1)$ and $g(z_2)$ be a pair of analytic functions of z_1 and z_2 and let $f(z_1) \circ g(z_2) = (2\pi)^{-1} \int_0^{2\pi} f(z_1 e^{i\theta}) g(z_2 e^{-i\theta})\, d\theta$. If $f(z_1) = \sum_{n=0}^{\infty} a_n z_1^n$ and $g(z_2) = \sum_{n=0}^{\infty} b_n z_2^n$ for $|z_j| < R$, $j = 1, 2$, then it follows that $f(z_1) \circ g(z_2) = \sum_{n=0}^{\infty} a_n b_n z_1^n z_2^n$ for $|z_j| < R$. The product $\circ$ was introduced by Hadamard [**6**] to discuss the singularities of the analytic function having element $\sum_{n=0}^{\infty} a_n b_n z^n$ in terms of those of the functions $f(z)$ and $g(z)$. It has been variously referred to as the Hadamard product, the Schur product or the quasi inner product. It was employed in [**1**] to discuss some properties of special functions and in [**2**] and [**3**] to construct solution representations of Cauchy problems. Examples relating to combinatorics and trigonometric integral evaluations were considered in [**4**]. Also see [**7**] for additional applications. A generalized version of this product, namely ${}_p\circ_q$, was also introduced in [**1**]:

$$f(z_1)_p \circ_q g(z_2) = (2\pi)^{-1} \int_0^{2\pi} f(z_1 e^{pi\theta}) g(z_2 e^{-qi\theta})\, d\theta \tag{1.1}$$

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where p and q are relatively prime integers. In terms of series, this product leads to $f(z_1)_p \circ_q g(z_2) = \sum_{n=0}^{\infty} a_{nq} b_{np} z_1^{nq} z_2^{np}$.

In this paper we consider extended versions of (1.1) and make application to integral evaluations and summations. To describe these formally, let $f_j(z) = \sum_{k=0}^{\infty} a_k^j z^k$ for $|z| < R$ and for $j = 1, 2, \ldots, n$ where the a_k^j are the coefficients in the series. We have used the same variable in all of the functions f_j for simplicity. Take $\varepsilon_j = 1$ for $j = 1, 2, \ldots, m$ and $\varepsilon_j = -1$ for $j = m+1, \ldots, n$, and let $p_1, p_2, \ldots, p_n$ be a set of positive integers. Now consider the integral

$$(2\pi)^{-1} \int_0^{2\pi} \left(\prod_{j=1}^{n} f_j(z e^{\varepsilon_j p_j i\theta}) \right) d\theta. \tag{1.2}$$

Note that if $n = 2$, $(p_1, p_2) = 1$ and $\varepsilon_1 = 1$, $\varepsilon_2 = -1$, this reduces to the generalized Hadamard product (1.1) with $z_1 = z_2 = z$. Upon replacing the functions $f_j(z)$ in (1.2) by their series representations, that integral becomes

(1.3)
$$(2\pi)^{-1} \int_0^{2\pi} \left(\sum_{l_1, l_2, \ldots, l_n = 0}^{\infty} \left(\prod_{j=1}^{n} a_{l_j}^j \right) \cdot z^{l_1 + l_2 + \cdots + l_n} \right) \exp\left(\left(\sum_{j=1}^{n} \varepsilon_j p_j l_j \right) i\theta \right) d\theta.$$

Now the integral $(2\pi)^{-1} \int_0^{2\pi} ((\sum_{j=1}^{n} \varepsilon_j p_j l_j) i\theta)\, d\theta$ vanishes unless the Diophantine equation $\sum_{j=1}^{n} \varepsilon_j p_j l_j = 0$ holds. In this case it takes on the value 1. We therefore refer to the set D of all vectors $l = (l_1, l_2, \ldots, l_n)$ with nonnegative integer components that satisfy $\sum_{j=1}^{n} \varepsilon_j p_j l_j = 0$ as the Diophantine index set for the product (1.2). The expression (1.3) then reduces to $\sum_{l \in D} (\prod_{j=1}^{n} a_{l_j}^j) z^{|l|}$ where $|l| = l_1 + l_2 + \cdots + l_n$. Hence, we have

$$(2\pi)^{-1} \int_0^{2\pi} \left(\prod_{j=1}^{n} f_j(z e^{\varepsilon_j p_j i\theta}) \right) d\theta = \sum_{l \in D} \left(\prod_{j=1}^{n} a_{l_j}^j \right) z^{|l|}. \tag{1.4}$$

Observe that if all of the $\varepsilon_j = 1$, i.e., $m = n$, then the righthand member of (1.4) reduces to the product $a_0^1 \cdot a_0^2 \cdots a_0^n$. Depending upon the values of m and n and the choices for the p_j, it may be possible to simplify the description of the set D and evaluation the righthand member of (1.4). In other cases, the integral in the left member of (1.4) can have an evaluation that permits assigning a value to the sum in its right member. This approach was taken in [**4**] for the generalized Hadamard product. In the sections to follow, we consider a variety of choices for the functions $f_j(z)$, the integers p_j and the parameters m and n. Among the choices for the $f_j(z)$ will be exponential functions, polynomials including binomial expansions and generating functions for special polynomials. Of particular interest will be the generating functions for certain generalized Hermite polynomials [**5**] and the Tchebychev polynomials $U_n(x)$ [**8, 9**]. These will lead to an extensive variety of trigonometric integral evaluations in terms of combinatorial sums as well as sums involving special functions that include the standard and modified Bessel functions. In many of these examples, we replace the variable z by the value 1 to obtain identities involving arithmetic and combinatorial sums.

To fix ideas and for future reference, consider the particular example with $f_j(z) = e^z$ for $j = 1, 2, \dots, n$. The integral in the left member of (1.2), using the Euler relations, can be given as

$$\begin{aligned}(2\pi)^{-1}\int_0^{2\pi} e^{z\sum_{j=1}^{n}\cos(\varepsilon_j p_j)\theta}\Bigg(&\cos z\Big(\sum_{j=1}^{n}\sin(\varepsilon_j p_j)\theta\Big)\\ &+ i\sin z\Big(\sum_{j=1}^{n}\sin(\varepsilon_j p_j)\theta\Big)\Bigg)\,d\theta.\end{aligned} \tag{1.5}$$

Using the oddness of the integrand in the integral that is the coefficient of i, it follows that this coefficient has the value 0. On the other hand, the right member of (1.3) reduces to $\sum_{l\in D} z^{|l|}/(l_1!l_2!\dots l_n!)$ with D as defined earlier. Hence we see that (1.2), for the given choices and with z replaced by 1, yields the integral evaluation formula

$$(2\pi)^{-1}\int_0^{2\pi} e^{\sum_{j=1}^{s}\cos(\varepsilon_j p_j)\theta}\cos\left(\sum_{j=1}^{n}\sin(\varepsilon_j p_j)\theta\right)d\theta = \sum_{l\in D} 1/(l_1!l_2!\dots l_n!). \tag{1.6}$$

2. Some of the $p_j = 1$. Here we reconsider the example above with at least one of the $p_j = 1$. The integral or the sum in (1.6) leads to the evaluation of a modified Bessel function $I_0(x)$ or expression involving a sum of modified Bessel functions.

(a) $p_n = 1$. We assume that $\varepsilon_j = 1$ for $j = 1, 2, \ldots, n-1$ and $\varepsilon_n = -1$. Then formula (1.4) with z replaced by 1 can be written

$$(2\pi)^{-1} \int_0^{2\pi} \left(\prod_{j=1}^{n} f_j(e^{\varepsilon_j p_j i\theta}) \right) d\theta = \sum_{\substack{l_j \geq 0 \\ j=1,2,\ldots,n-1}} \left(\prod_{j=1}^{n-1} a_{l_j}^j \right) \cdot a^n_{l_1 p_1 + l_2 p_2 + \cdots + l_{n-1} p_{n-1}}.$$

If all of the $f_j(z) = e^z$, the reader can show that this becomes

(2.1)

$$(2\pi)^{-1} \int_0^{2\pi} \exp\left\{ \left(\sum_{j=1}^{n-1} \cos p_j \theta \right) + \cos\theta \right\} \cdot \cos\left\{ \sin\theta - \sum_{j=1}^{n-1} \sin p_j \theta \right\} d\theta = \sum_{\substack{l_j \geq 0 \\ j=1,2,\ldots,n-1}} \frac{1}{l_1! \cdot l_2! \cdots l_{n-1}! \cdot (p_1 l_1 + p_2 l_2 + \cdots + p_{n-1} l_{n-1})!}.$$

For this general case, the integral in the left member usually has no simple evaluation. However, we now examine a special case of this that leads to sums of modified Bessel function evaluations at 2. To obtain this, reconsider (2.1) with n replaced by $n+2$, $p_{n+1} = p_{n+2} = 1$ and $\varepsilon_j = 1$ for $j = 1, 2, \ldots, n+1$ and $\varepsilon_{n+2} = -1$. Then (2.1) becomes

$$(2.2) \quad (2\pi)^{-1} \int_0^{2\pi} \exp\left(\sum_{j=1}^{n} \cos p_j \theta + 2\cos\theta \right) \cdot \cos\left(\sum_{j=1}^{n} \sin p_j \theta \right) d\theta = \sum_{\substack{l_j \geq 0 \\ j=1,2,\ldots,n+1}} \frac{1}{l_1! \cdot l_2! \cdots l_{n+1}! \cdot (p_1 l_1 + p_2 l_2 + \cdots + p_n l_n + l_{n+1})!}.$$

But, by inserting the factor $2^{2l_{n+1} + p_1 l_1 + p_2 l_2 + \cdots + p_n l_n}$ into the numerator and denominator of this sum and the first summing on l_{n+1}, the reader can show that the right member of (2.2) becomes

$$\sum_{\substack{l_j \geq 0 \\ j=1,2,\ldots,n}} \frac{1}{l_1! \cdot l_2! \cdots l_n!} I_{p_1 l_1 + p_2 l_2 + \cdots + p_n l_n}(2)$$

where $I_v(z)$ denotes the modified Bessel function defined by the sum $\sum_{j=0}^{\infty} z^{v+2j}/(2^{v+2j} j! \cdot (j+v)!)$ for v a nonnegative integer. Hence, (2.2) becomes

$$
\begin{aligned}
&(2\pi)^{-1} \int_0^{2\pi} \exp\left(\sum_{j=1}^{n} \cos p_j\theta + 2\cos\theta\right) \cdot \cos\left(\sum_{j=1}^{n} \sin p_j\theta\right) d\theta \\
&\qquad = \sum_{\substack{l_j \geq 0 \\ j=1,2,\dots,n}} \frac{1}{l_1! \cdot l_2! \cdots l_n!} I_{p_1 l_1 + p_2 l_2 + \cdots + p_n l_n}(2).
\end{aligned} \tag{2.3}
$$

(b) All $p_j = 1$. Again, assume $\varepsilon_j = 1$ for $j = 1, 2, \dots, m$ and $\varepsilon_j = -1$ for $j = m+1, \dots, n$. The equation determining the Diophantine index set D in this situation becomes $l_1 + l_2 + \cdots + l_m - l_{m+1} - \cdots - l_n = 0$ with the $l_j \geq 0$, $j = 1, 2, \dots, n$. If $m = n-1$, we can rewrite this as $l_n = l_1 + l_2 + \cdots + l_{n-1}$. Then the integral relation (1.6) becomes

$$
(2\pi)^{-1} \int_0^{2\pi} e^{n\cos\theta} \cos((2m-n)\sin\theta)\, d\theta = \sum_{l \in D} 1/(l_1! l_2! \cdots l_n!). \tag{2.4}
$$

Consider the standard Hadamard product $e^{z_1} \circ e^{z_2} = (2\pi)^{-1} \int_0^{2\pi} e^{z_1 e^{i\theta}} \times e^{z_2 e^{-i\theta}}\, d\theta$. Using the Euler relations, this can be expressed as $(2\pi)^{-1} \times \int_0^{2\pi} e^{(z_1+z_2)\cos\theta} \cos((z_1 - z_2)\sin\theta)\, d\theta$. This agrees with the integral in the left member of (2.4) if $z_1 + z_2 = n$ and $z_1 - z_2 = 2m - n$. Thus, $z_1 = m$ and $z_2 = n - m$. On the other hand $e^{z_1} \circ e^{z_2} = \sum_{n=0}^{\infty} z_1^n z_2^n/(n! \cdot n!) = I_0(2\sqrt{z_1 z_2})$. Thus, the integral in the first member of (2.4) has the value $I_0(2\sqrt{m(n-m)})$ and we obtain the formula (again from (2.4))

$$
\sum_{l \in D} 1 \Big/ \left(\prod_{j=1}^{n} l_j!\right) = I_0(2\sqrt{m(n-m)}). \tag{2.5}
$$

If $m = n-1$, this can be written as

$$
\sum_{\substack{l_j \geq 0 \\ j=1,2,\dots,n-1}} \frac{1}{l_1! \cdot l_2! \cdots l_{n-1}! \cdot (l_1 + l_2 + \cdots + l_{n-1})!} = I_0(2\sqrt{n-1}). \tag{2.6}
$$

For $n \geq 3$, it is clear from this, by summing first on l_{n-1} and then calling on the definition of the modified Bessel function that

$$\sum_{\substack{l_j \geq 0 \\ j=1,2,\dots,n-2}} \frac{1}{l_1! \cdot l_2! \cdots l_{n-2}!} I_{l_1+l_2+\cdots+l_{n-2}}(2) = I_0(2\sqrt{n-1}). \tag{2.7}$$

3. Binomial expansions choices. In this section we develop generalizations of some of the special integral evaluations obtained in [**4**] that make use of binomial expansions. For this purpose, take $f_j(z) = (1+z)^{a_j}$, $j = 1, 2, \dots, n$, in which the a_j are positive integers. Then consider the extended Hadamard product

(3.1)

$$\begin{aligned}
&(2\pi)^{-1} \int_0^{2\pi} \bigg(\prod_{j=1}^{n} (1 + ze^{2p_j i\theta})^{a_j} \bigg) (1 + ze^{\pm 2p_n i\theta})^{a_n} \, d\theta \\
&\quad = (2\pi)^{-1} \int_0^{2\pi} \bigg(\prod_{j=1}^{n-1} \Big(\sum_{l_j=0}^{a_j} \binom{a_j}{l_j} z^{l_j} e^{2p_j l_j i\theta} \Big) \Big(\sum_{l_n=0}^{a_n} \binom{a_n}{l_n} z^{l_n} e^{\pm 2p_n l_n i\theta} \Big) \bigg) \, d\theta.
\end{aligned}$$

With some factoring of complex exponentials in the binomial terms, the first member of this can be written

$$\begin{aligned}
(2\pi)^{-1} \int_0^{2\pi} e^{(p_1a_1+p_2a_2+\cdots+p_{n-1}a_{n-1} \pm p_n a_n)i\theta} \bigg(\prod_{j=1}^{n-1} (ze^{p_j i\theta} + e^{-p_j i\theta})^{a_j} \bigg) \\
\times (ze^{\pm p_n i\theta} + e^{\mp p_n i\theta})^{a_n} \, d\theta.
\end{aligned}$$

Taking $z = 1$, it isn't difficult to show that this reduces to

$$\begin{aligned}
\frac{2^{a_1+a_2+\cdots+a_n}}{2\pi} \int_0^{2\pi} \bigg(\prod_{j=1}^{n} (\cos p_j\theta)^{a_j} \bigg) \\
\times \cos(p_1a_1 + p_2a_2 + \cdots + p_{n-1}a_{n-1} \pm p_n a_n)\theta \, d\theta
\end{aligned} \tag{3.2}$$

after replacing $(e^{p_j^{i\theta}} + e^{-p_j^{i\theta}})^{a_j}$ by $2^{a_j}(\cos p_j\theta)^{a_j}$. On the other hand, the second member of (3.1) with $z = 1$ can be rewritten

(3.3)
$$\sum_{\substack{l_j=0\\ j=1,2,\dots,n}}^{a_j} \left(\prod_{j=1}^{n}\binom{a_j}{l_j}\right)(2\pi)^{-1}\int_0^{2\pi}\exp\left(\left(2\sum_{j=1}^{n-1}p_ja_j \pm 2p_na_n\right)\cdot i\theta\right)d\theta = \sum_{l\in D}\left(\prod_{j=1}^{n}\binom{a_j}{l_j}\right)$$

where D is defined by $\sum_{j=0}^{n-1} p_jl_j \pm p_nl_n = 0$ with $0 \le l_j \le a_j$. Comparing this with the integral (3.2) and solving, we get

(3.4)
$$\int_0^{2\pi}\left(\prod_{j=1}^{n}(\cos p_j\theta)^{a_j}\right)\cos(p_1a_1 + p_2a_2 + \cdots + p_{n-1}a_{n-1} \pm p_na_n)\theta\, d\theta = \frac{2\pi}{2^{a_1+a_2+\cdots+a_n}}\sum_{l\in D}\left(\prod_{j=1}^{n}\binom{a_j}{l_j}\right).$$

Note that if a plus sign is selected in the exponent in the factor $(1 + ze^{\pm p_ni\theta})^{a_n}$ in (3.1), then $D = (0, 0, \dots, 0)$ and (3.4) reduces to

(3.5)
$$\int_0^{2\pi}\left(\prod_{j=1}^{n}(\cos p_j\theta)^{a_j}\right)\cdot\cos\big((a_1p_1 + a_2p_2 + \cdots + a_np_n)\theta\big)\,d\theta = \frac{2\pi}{2^{a_1+a_2+\cdots+a_n}}.$$

However, suppose a minus sign is selected in the exponent of the factor $(1 + ze^{\pm 2p_ni\theta})^{a_n}$ in (3.1). Then with the choices $p_n = 1$ and $a_n = \sum_{j=1}^{n-1} a_jp_j$, the formula (3.4) becomes

$$\text{(3.6)}\quad \int_0^{2\pi}\left(\prod_{j=1}^{n-1}(\cos^{a_j} p_j\theta)\right)\cdot(\cos^{a_1p_1+a_2p_2+\cdots+a_{n-1}p_{n-1}}\theta)\,d\theta$$
$$= \frac{2\pi}{2^{a_1(p_1+1)+a_2(p_2+1)+\cdots+a_{n-1}(p_{n-1}+1)}}$$
$$\times \sum_{\substack{0\le l_j\le a_j\\ j=1,2,\dots,n-1}}\left(\prod_{j=1}^{n-1}\binom{a_n}{l_j}\right)\cdot\binom{a_1p_1 + a_2p_2 + \cdots + a_{n-1}p_{n-1}}{l_1p_1 + l_2p_2 + \cdots + l_{n-1}p_{n-1}}.$$

If n in this is replaced by $n+1$ and all of the $p_j = 1$, the lefthand member can be evaluated by using the beta function and then expressed as a binomial coefficient. With this hint, we leave it to the reader to establish the combinatorial formula

$$\sum_{\substack{0\le l_j\le a_j\\ j=1,2,\dots,n}} \binom{a_1}{l_1}\binom{a_2}{l_2}\cdots\binom{a_n}{l_n}\binom{a_1+a_2+\cdots+a_n}{l_1+l_2+\cdots+l_n} = \binom{2(a_1+a_2+\cdots+a_n)}{a_1+a_2+\cdots+a_n}.$$

4. Binomials with exponentials. We consider an example of an extended Hadamard product that involves four functions and uses notions for the generalized Hadamard product. For this, take $f_1(z) = f_2(z) = e^{z^2/2}$, $f_3(z) = f_4(z) = (1+z)^m$ and consider the integral

$$(2\pi)^{-1}\int_0^{2\pi} e^{(z^2e^{2pi\theta})/2}e^{(z^2e^{-2pi\theta})/2}(1+ze^{2qi\theta})^m(1+ze^{-2qi\theta})^m\,d\theta \tag{4.1}$$

where we assume $(p,q)=1$. Using some of the approaches of Section 3, we can rewrite the integrand of this to obtain the integral

$$(2\pi)^{-1}\int_0^{2\pi} e^{z^2\cos 2p\theta}(ze^{qi\theta}+e^{-qi\theta})^m(e^{qi\theta}+ze^{-qi\theta})^m\,d\theta. \tag{4.2}$$

Alternatively, if we expand each of the factors in the integrand in (4.2) in powers of z, that integral becomes

$$\sum_{\substack{l_j\ge 0\\ j=1,2,3,4}} \frac{z^{2l_1+2l_2+l_3+l_4}}{l_1!\cdot l_2!\cdot 2^{l_1+l_2}}\binom{m}{l_3}\cdot\binom{m}{l_4}\cdot(2\pi)^{-1}\int_0^{2\pi} e^{\{2p(l_1-l_2)+2q(l_3-l_4)\}i\theta}\,d\theta \tag{4.3}$$

in which the binomial coefficient $\binom{m}{l_j}$ is assigned the value 0 if $l_j > m$. The complex integral in this sum takes on the value 1 only if $2p(l_1-l_2) = 2q(l_4-l_3)$ (and 0 otherwise). Since $(p,q)=1$, this conditions holds only if $l_1 - l_2 = q_k$ and $l_4 - l_3 = p_k$ for $k = 0, 1, 2, \ldots$. Thus $l_1 = l_2 + q_k$ and $l_4 = l_3 + pk$ and (4.3) becomes

$$\sum_D \frac{z^{4l_2+2l_3+2qk+pk}}{l_2! \cdot (l_2+qk)! \cdot 2^{2l_2+qk}} \binom{m}{l_3} \cdot \binom{m}{l_3+pk}. \tag{4.4}$$

In this sum, $l_3 + qk$ must be restricted so that $l_3 + qk \le m$ or $k \le [(m - l_3)/p]$. Now take $z = 1$ in (4.2) and (4.4) and replace the summation variable l_2 by j and the summation variable l_3 by n. Then equating (4.2) and (4.4), we obtain the formula

$$\begin{aligned}(4.5)\quad &\int_0^{2\pi} e^{\cos 2p\theta} \cos^{2m} q\theta \, d\theta \\ &= \frac{\pi}{2^{2m-1}} \sum_{\substack{j,k,n \ge 0 \\ 0 \le n+pk \le m}} \frac{1}{2^{2j+qk} j! \cdot (j+qk)!} \binom{m}{n} \cdot \binom{m}{n+pk}.\end{aligned}$$

The relabeling of variables was introduced to avoid subscripts in this final relation.

5. Exponentials and Bessel relations.

Again, suppose that $(p, q) = 1$ and consider the integral

$$(2\pi)^{-1} \int_0^{2\pi} e^{xe^{pi\theta}/2} e^{xe^{-pi\theta}/2} e^{ye^{qi\theta}/2} e^{ye^{-qi\theta}/2} \, d\theta.$$

This clearly reduces to the real integral

$$(2\pi)^{-1} \int_0^{2\pi} e^{x \cos p\theta + y \cos q\theta} \, d\theta. \tag{5.1}$$

Upon expanding the integrand in the starting integral in powers of x and y, that integral can also be written as

$$\begin{aligned}(5.2)\quad &(2\pi)^{-1} \int_0^{2\pi} \Bigg(\sum_{\substack{l_j \ge 0 \\ j=1,2}} \frac{x^{l_1+l_2} e^{p(l_1-l_2)i\theta}}{2^{j_1+j_2} l_1! \cdot l_2!} \Bigg) \cdot \Bigg(\sum_{\substack{l_j \ge 0 \\ j=3,4}} \frac{y^{l_3+l_4} e^{q(l_3-l_4)i\theta}}{2^{l_3+l_4} l_3! \cdot l_4!} \Bigg) \cdot d\theta \\ &= \sum_{\substack{l_j \ge 0 \\ j=1,2,3,4}} \frac{x^{l_1+l_2} y^{l_3+l_4}}{2^{l_1+l_2+l_3+l_4} l_1! l_2! l_3! l_4!} \cdot (2\pi)^{-1} \int_0^{2\pi} e^{\{p(l_1-l_2)+q(l_3-l_4)\}i\theta} \, d\theta.\end{aligned}$$

Just as in the previous example, the integral in the last member of this takes on the value 1 only if $l_1 = l_2 + qk$, $l_4 = l_3 + pk$ for $k = 0, 1, 2, \dots$. The last member of (5.2) then becomes

$$\sum_{k\geq 0}\bigg(\sum_{l_2\geq 0}\frac{x^{2l_2+qk}}{2^{2l_2+qk}l_2!(l_2+qk)!}\bigg)\bigg(\sum_{l_3\geq 0}\frac{y^{2l_3+pk}}{2^{2l_3+pk}l_3!(l_3+pk)!}\bigg) = \sum_{k=0}^{\infty} I_{qk}(x)I_{pk}(y)$$

where $I_v(x)$ denotes, as earlier, a modified Bessel function of index v. A comparison of this with (5.1) yields the integration formula

$$(5.3) \qquad (2\pi)^{-1}\int_0^{2\pi} e^{x\cos p\theta + y\cos q\theta}\,d\theta = \sum_{k=0}^{\infty} I_{qk}(x)I_{pk}(y).$$

Suppose one were to replace the function $e^{ye^{-qi\theta}/2}$ in the starting integral by the function $e^{-ye^{-qi\theta}/2}$. Then the types of reductions, as considered above, lead to an integral formula somewhat analogous to (5.3), namely,

$$(5.4) \qquad (2\pi)^{-1}\int_0^{2\pi} e^{x\cos p\theta}\cos(y\sin q\theta)\,d\theta = \sum_{k=0}^{\infty}(-1)^{pk} I_{qk}(x)J_{pk}(y).$$

6. Some generating function. In this final section, we consider two examples that involve generating functions of special polynomials. The first of these uses one for a class of generalized Hermite polynomials while the second calls on the generator for the Tchebychev polynomials of the second kind, $U_n(x)$.

(a) *A generalized Hermite generator.* Let $f(a,x) = e^{qxa - a^q}$. Then $f(a,x) = \sum_{n=0}^{\infty} H_n^q(x)a^n/n!$ where $H_n^q(x)$ is a generalized Hermite polynomial [**5**]. Observe that if $q = 2$, then $f(a,x)$ generates the classical Hermite polynomials. Take $g(y) = e^{y/2}$ and $h(y) = e^{-y/2}$ and consider the following extended Hadamard product

$$(6.1) \qquad P(a) = (2\pi)^{-1}\int_0^{2\pi} f(ae^{pi\theta})g(ye^{i\theta})h(ye^{-i\theta})\,d\theta$$

with a replaced by 1. A straightforward calculation, using the Euler relations, permits showing that

$$P(1) = (2\pi)^{-1} \int_0^{2\pi} e^{qx\cos p\theta - \cos pq\theta} \cos(qx \sin p\theta - \sin pq\theta + y\sin\theta)\, d\theta. \tag{6.2}$$

Following our earlier discussions, it is left to the reader to show that $P(1)$ can also be expressed as

$$P(1) = \sum_{\ell,m,n=0}^{\infty} a_n b_m c_\ell \left((2\pi)^{-1} \int_0^{2\pi} e^{(pn+m-\ell)i\theta}\, d\theta \right) \tag{6.3}$$

in which $a_n = H_n^q(x)/n!$, $b_n = y^n/(2^n n!)$ and $c_n = (-1)^n y^n/(2^n n!)$. Making the usual restriction on the exponent in the integral in (6.3), we can solve to get $\ell = m + np$. Reinserting this into (6.3) and using the noted formulas for the coefficients, we find

$$P(1) = \sum_{n=0}^{\infty} \frac{H_n^q(x)}{n!} \left(\sum_{m=0}^{\infty} \frac{(-1)^{m+pn} y^{2m+pn}}{2^{2m+pn} m!(m+pn)!} \right). \tag{6.4}$$

The inner sum in this simplifies to $(-1)^{pn} J_{pn}(y)$ where, again, $J_v(y)$ denotes the familiar Bessel function. A comparison of (6.4) with (6.2) then leads to the following integral result

$$\begin{aligned} (2\pi)^{-1} \int_0^{2\pi} & e^{qx\cos p\theta - \cos pq\theta} \cos(qx\sin p\theta - \sin pq\theta + y \sin\theta)\, d\theta \\ &= \sum_{n=0}^{\infty} (-1)^{pn} H_n^q(x) J_{pn}(y)/n!. \end{aligned} \tag{6.5}$$

(b) *A Tchebychev polynomial generator.* Take $f_1(z) = f_2(z) = (1 - 2xz + z^2)^{-1}$ for $|x| \leq a$ for some $a < 1$ and $f_3(z) = f_4(z) = (1+z)^m$. Now $f_1(z) = \sum_{n=0}^{\infty} U_n(x) z^n$ in which $U_n(x)$ is a Tchebychev polynomial of the second kind. Let p and q be positive integers with $(p, 2q) = 1$ and consider the extended Hadamard product

$$(2\pi)^{-1} \int_0^{2\pi} f_1(ze^{pi\theta}) f_2(ze^{-pi\theta}) f_3(ze^{2qi\theta}) f_4(ze^{-2qi\theta})\, d\theta. \tag{6.6}$$

Let us note that

$$f_1(ze^{pi\theta})f_2(ze^{-pi\theta}) = \big((1+4x^2z^2+z^4)-(xz+xz^3)\cos p\theta+(z^2/2)\cos 2p\theta\big)^{-1}$$

and

$$f_3(ze^{2qi\theta})f_4(ze^{-2qi\theta}) = (ze^{qi\theta}+e^{-qi\theta})^m(e^{qi\theta}+ze^{-qi\theta})^m.$$

Hence, the integral in (6.6) becomes

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{(ze^{qi\theta}+e^{-qi\theta})^m(e^{qi\theta}+ze^{-qi\theta})^m}{(1+4x^2z^2+z^4)-(xz+xz^3)\cos p\theta+(z^2/2)\cos 2p\theta}\,d\theta. \tag{6.7}$$

Again, if we expand the functions in the integrand of (6.6) in powers of z following the approaches of Sections 4 and 5 with our bounds on x, it is not difficult to show that (6.6) can be expressed as

$$\sum_{\substack{j\ge 0,k\ge 0,\ell\ge 0\\ 0\le j+pk\le m}}\binom{m}{j}\binom{m}{j+pk}U_\ell(x)U_{\ell+2qk}(x)z^{2\ell+2j+2qk+pk}. \tag{6.8}$$

Then setting $z=1$ in (6.7) and (6.8) and equating them, we can obtain as a final formula

$$\int_0^{2\pi}\frac{\cos^{2m} q\theta}{4+8x^2-4x\cos p\theta+\cos 2p\theta}\,d\theta = \frac{\pi}{2^{2m}}\cdot\sum_{\substack{j\ge 0,k\ge 0,\ell\ge 0\\ 0\le j+pk\le m}}\binom{m}{j}\binom{m}{j+pk}U_\ell(x)U_{\ell+2qk}(x). \tag{6.9}$$

Using the double angle formula for $\cos 2p\theta$, one can show that the denominator in the integrand of the integral (6.7) can be expressed as $2(\cos p\theta - x)^2+3+6x^2$ and this doesn't vanish for $-1<x<1$. This shows that the integral in (6.7) exists for all such x. However, the expansions, integration and evaluation of (6.8) at $z=1$ leading to the right member of (6.9) requires the stronger restriction $|x|\le a$.

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