ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 34, Number 2, Summer 2004

TRANSVERSALITY THEOREMS FOR HARMONIC FORMS

KO HONDA

ABSTRACT. We prove that harmonic 1-forms and 2-forms for generic metrics on a closed manifold M enjoy the standard transversality properties with respect to the stratification of $\wedge^k T^*M$ which is induced from the action of $O(n)$ on $\wedge^k(\mathbf{R}^n)^*$.

1. Introduction. This paper is motivated by the following (related) questions on the regularity of the zero sets of solutions to Laplace's equation:

(A) *The Dirichlet problem.* Let Ω be a domain in $\mathbf{R}^n$ and Δ_g be the Laplacian on Ω with respect to the (Riemannian) metric g. Consider the solution u to the equation $\Delta_g u = 0$, with Dirichlet boundary condition $u|_{\partial\Omega} = f$, where f is a fixed function on $\partial\Omega$. Then is the zero set of u regular for generic choices of g?

(B) *Harmonic forms.* Let M be a closed, oriented n-manifold and Δ_g be the Laplacian (Laplace-Beltrami operator) on M with respect to the (Riemannian) metric g. If g is a generic metric and ω is a harmonic k-form with respect to the metric g (i.e., $\Delta_g\omega = 0$), is the zero set of ω regular? Moreover, is the generic harmonic form ω transverse to the various strata of $\bigwedge^k T^*M$ under the action of $SO(n)$?[1]

We will call a harmonic form ω with respect to the metric g a *g-harmonic form.* The goal of this paper is to prove affirmative results for (A) and certain special cases of (B). On a closed, oriented n-manifold M we prove transversality results for 1-forms and also for 2-forms (provided n is even). Dually, we obtain transversality for $(n-1)$-forms on any n-manifold and $(n-2)$-forms for n even. As we shall see, 4-manifolds exhibit unusual behavior in the dichotomy between the self-dual (SD) (or anti-self-dual (ASD)) 2-forms and the non-SD (and non-ASD) 2-forms. The following is a sampling of the results which are proved:

Received by the editors on August 28, 2001, and in revised form on April 10, 2002.

Theorem 1.1. *Let M be a closed, oriented 4-manifold with $b_2^+(M) > 0$, and $Met^l(M)$ be the space of C^l-Hölder metrics on M, for a sufficiently large non-integer l. Let $Q^+ \subset H^2(M;\mathbf{R}) \times Met^l(M)$ be the Banach submanifold consisting of pairs $([\omega], g)$, where $g \in Met^l(M)$ and ω is a SD g-harmonic 2-form with $[\omega] \neq 0$. Then there exists a dense open set $\mathcal{U} \subset Q^+$ such that if $([\omega], g) \in \mathcal{U}$, then ω has regular zeros. This means that the zeros of ω consist of disjoint circles for generic g.*

Theorem 1.2. *Let M be a closed, oriented 4-manifold with $b_2^{\pm}(M) > 0$, and $Met^l(M)$ be the space of C^l-Hölder metrics on M, for a sufficiently large non-integer l. Then there exists a dense open set $\mathcal{U} \subset H^2(M;\mathbf{R}) \times Met^l(M)$ such that if $(\omega, g) \in \mathcal{U}$ and ω is a g-harmonic form representing its cohomology class $[\omega] \in H^2(M;\mathbf{R})$, then ω is neither SD nor ASD, and ω has no zeros, has full rank away from a submanifold of codimension 1, and is SD/ASD on a union of disjoint circles.*

Theorem 1.3. *Let M be a closed, oriented 6-manifold with $H^2(M;\mathbf{R}) \neq 0$, and $Met^l(M)$ be the space of C^l-Hölder metrics on M, for a sufficiently large non-integer l. If we fix a nonzero cohomology class $\alpha \in H^2(M;\mathbf{R})$, then there exists a dense open set $\mathcal{U} \subset Met^l(M)$ such that if $g \in \mathcal{U}$, then the g-harmonic 2-form $\omega \in \alpha$ has no zeros, has isolated points where it has rank 2, and, away from the rank 2 points, has rank 4 on a submanifold of codimension 1.*

In Theorem 1.3, by *rank* we mean the rank of a skew-symmetric bilinear form.

A few words on Theorem 1.1: although this theorem on self-dual 2-forms was conceived and proven independently by the author, circa 1995, later conversations with Taubes and Eliashberg revealed that this had been known to specialists for a long time, cf. Appendix to the preprint [**14**], excised from the published version. The first detailed proof appeared in LeBrun's paper [**10**]. We include Theorem 1.1 here because it is readily proved with the transversality machinery developed in this paper. By now, Taubes has extensively studied properties of holomorphic curves with respect to these self-dual harmonic 2-forms in a series of papers [**16, 17, 18**]. (Some very basic remarks can be found in [**7**].)

In Theorem 1.2, the nonduality result is well-known (cf., Section 4.3.5 of [**4**]), and we prove the part of interest, namely the transversality. A study of such non-SD/non-ASD forms appears in [**8**].

We remark here that the proofs of the transversality theorems are rather nontrivial because of the harmonicity – one must work much harder than in the case of closed forms, as was carried out by Martinet in [**11**]. This is primarily due to the fact that a local metric perturbation gives rise to global effects on the manifold. The key ingredient in the proofs of the theorems, therefore, is the control afforded from using the asymptotics of the Green's function. The idea of using Green's functions appears in Uhlenbeck's paper [**19**] on eigenfunctions for generic metrics.

In Section 2, we study generic harmonic forms on compact manifolds. After reviewing the basic setup of transversality theory in Section 2.1 and the crucial facts from the asymptotics of Green's functions in Section 2.2, we make the necessary computations in Sections 2.3, 2.4 and 2.5. Subsequently, 1-forms are treated in Section 2.6, SD 2-forms on 4-manifolds (Theorem 1.1) in Section 2.7, the non-SD/non-ASD case (Theorem 1.2) in Section 2.8, and 2-forms on $2n$-dimensional manifolds with $2n > 4$ in Section 2.9. Section 3 is devoted to transversality results for solutions to the Dirichlet Problem.

Remark 1.4. It appears at the moment that, if transversality still holds for k-forms on an n-dimensional manifold with $2 < k < n-2$, the proof would require substantially more work. This is due to two complications. The first, and probably the less serious, is that the orbits of the action of $SO(n)$ on $\wedge^k(\mathbf{R}^n)$ become more complicated. The second is that the map i_ω, defined in Section 2.5, is *not* surjective, and we need to consider all the points near the point at which we are trying to perturb. This would require considerations similar to that of Section 2.9, only much more involved.

2. Harmonic forms on compact manifolds. In what follows, M will be a closed, oriented n-manifold and $\mathrm{Met}^l(M)$ the space of C^l-Hölder metrics on M, for sufficiently large $l \in \mathbf{R}^+ - \mathbf{Z}$. Let

$P = H^k(M;\mathbf{R}) \times \mathrm{Met}^l(M)$. Given $(\alpha, g) \in P$, there exists a unique g-harmonic form $\omega_{(\alpha,g)}$ in the class α by the Hodge theorem. This gives rise to the natural map

$$j : P \to \Omega^k_m(M) \times \mathrm{Met}^l(M) \tag{1}$$

which sends (α, g) to the corresponding $(\omega_{(\alpha,g)}, g)$. Here $\Omega^k_m(M)$ is the space of C^m-Hölder k-forms with m sufficiently large.

Proposition 2.1. *The map j gives an isomorphism of P with a Banach submanifold Q of the Banach manifold $\Omega^k_m(M) \times Met^l(M)$.*

The proof will be given in the Appendix.

2.1 Transversality theory. We would like to prove that there is a dense open set $\mathcal{U} \subset Q$ such that if $(\omega_{(\alpha,g)}, g) \in \mathcal{U}$, then the harmonic form $\omega_{(\alpha,g)}$ has regular zeros.

Let us describe the general transversality theory setup. We start with the following "full" evaluation map:

$$\begin{aligned} ev : Q \times M &\to \wedge^k T^*M \\ [(\omega, g), x] &\longmapsto \omega(x). \end{aligned} \tag{2}$$

Here ω is a g-harmonic form.

In subsequent sections, it will be shown that ev is *regular*, i.e., transverse to the zero section of $\wedge^k T^*M$, for appropriate k and $n =$ dim M. This means that, for $[(\omega, g), x]$ fixed,

$$ev_* : T_{(\omega,g)}Q \times T_xM \longrightarrow \wedge^k T^*_x M \tag{3}$$

is surjective whenever $\omega(x) = 0$. If ev is regular, then by Proposition 2.2 below, there exists a dense G_δ-subset of Q for which ω has regular zeros. Since transversality is an open condition on our C^l-function spaces with $l >> 0$, this implies that there exists an open dense subset $\mathcal{U} \subset Q$ for which ω has regular zeros.

Proposition 2.2. *Let X be a Banach manifold, M, N finite-dimensional manifolds, and $f : X \times M \to N$ be a C^l-map for l*

sufficiently large. Suppose f is transverse to a submanifold Z of N. Then for a dense G_δ-set in X, $f_x : M \to N$ is transverse to Z, where $f_x(m) = f(x, m)$.

For a well-written account of Proposition 2.2, consult Section 4.3 of [**4**].

Consider the following "partial" evaluation map

$$ev_x : Q \longrightarrow \wedge^k T_x^* M, \tag{4}$$

which maps $(\omega, g) \mapsto \omega(x)$. In other words, this is the evaluation map with $x \in M$ held constant. In our present situation, it suffices to show that

$$(ev_x)_* : T_{(\omega,g)} Q \longrightarrow \wedge^k T_x^* M \tag{5}$$

is surjective whenever $\omega(x) = 0$, that is, it is not necessary to let x vary in M. The following lemma shows that we are not sacrificing any extra degrees of freedom by restricting to $(ev_x)_*$.

Lemma 2.3. $(ev_x)_*(T_{(\omega,g)}Q) = ev_*(T_{(\omega,g)}Q \times T_x M)$.

Proof. If $\tilde{v} \in T_x M$, then extend $\tilde{v}$ to a vector field V on M. If ϕ_t is the 1-parameter family of diffeomorphisms generated by V, then

$$ev_*((\omega, g), x)(0, \tilde{v}) = \mathcal{L}_V \omega(x) = \frac{d}{dt} \phi_t^* \omega(x) \bigg|_{t=0}, \tag{6}$$

where $\mathcal{L}_V$ is the Lie derivative in the direction of V. Finally, we observe that $\phi_t^* \omega$ is harmonic for the metric $\phi_t^* g$. □

2.2 Green's functions. We now start to compute $(ev_x)_*(T_{(\omega,g)}Q)$. Consider the family $(\omega_t, g_t) \in Q$, $t \in (-\varepsilon, \varepsilon)$, $\varepsilon > 0$ small, with $(\omega_0, g_0) = (\omega, g)$. Since $\Delta_{g_t} \omega_t = 0$, by differentiating the equation with respect to t at $t = 0$, we see that if $(v, h) \in T_{(\omega,g)}Q$, then

$$\Delta_g v + \frac{d}{dt}(\Delta_{g+th}) \bigg|_{t=0} \omega = 0. \tag{7}$$

Here, we will often write $g_t = g + th$, where $h \in C^l(Sym^2T^*M)$ is $dg_t/dt|_{t=0}$. This presents no loss of generality since (1) all our computations concern g_t only up to first order in t and (2) $g + th$ is a (positive definite) metric, provided t is sufficiently small.

Conversely, we have the following:

Claim 2.4. *If (v, h) satisfies Equation 7, then $(v, h) \in T_{(\omega,g)}Q$.*

Proof. First note that $d/dt(\Delta_{g+th})|_{t=0}\omega$ is an exact form. Indeed,

$$\frac{d}{dt}(\Delta_{g+th})\bigg|_{t=0}\omega = \frac{d}{dt}(dd^*_{g+th} + d^*_{g+th}d)\bigg|_{t=0}\omega = d\left(\frac{d}{dt}(d^*_{g+th}\omega)\bigg|_{t=0}\right), \tag{8}$$

since $d\omega = 0$. Now recall the Hodge decomposition, which states that

$$\Omega^k(M) = d\Omega^{k-1}(M) \oplus d^*\Omega^{k+1} \oplus \mathcal{H}^k_g, \tag{9}$$

where $\mathcal{H}^k_g$ is the set of g-harmonic k-forms. In particular,

$$\frac{d}{dt}(\Delta_{g+th})\bigg|_{t=0}\omega$$

is L^2-orthogonal to $\mathcal{H}^k_g$, and there is a unique solution $v \perp \mathcal{H}^k_g$ which solves Equation 7 with fixed h. Here $\perp$ refers to L^2-orthogonality. Since for each $g+th$ there must exist a $(g+th)$-harmonic form ω_t in the same cohomology class as ω, this unique solution $v \perp \mathcal{H}^k_g$ must necessarily arise as $d\omega_t/dt\big|_{t=0}$. All other solutions v' to Equation 7 with fixed h differ from v by a g-harmonic form, and clearly $(v - v', 0) \in T_{(\omega,g)}Q$. □

In order to write Equation 7 in integral form, i.e., to invert Equation 7 and solve for v, we make use of the Green's function $G(x, y)$. We now collect some facts that we need on the Green's function. They will be presented without proof; see [**2, 12**] for Laplacians on functions - the generalization to forms is straightforward. Let π_i, $i = 1, 2$, be the ith projection of $M \times M$ onto M, and let $\Delta \subset M \times M$ be the diagonal.

Proposition 2.5. *There exists a section $G_g(x,y)$ of $\pi_1^*(\wedge^k T^*M) \otimes \pi_2^*(\wedge^k T^*M)$ over $M \times M - \Delta$, called the* Green's function*, with the following properties*:

1. *If g is of class C^l with $l >> 0$, then G_g of class C^l on $M \times M - \Delta$.*
2. $G_g(x,y) = G_g(y,x)$.
3. *If the k-form ω of class C^l is L^2-orthogonal to $ker\Delta_g$, then*

$$\int_M \langle G_g(x,y), \Delta_g \omega(y)\rangle_g dv_g(y) = \omega(x).$$

Here $\langle , \rangle_g$ is the fiber metric on $\wedge^k T^*M$ induced from g, $dv_g(y)$ is the volume form for the metric g with respect to the variable y, and the dy_i terms get paired with respect to $\langle , \rangle$, while the dx_i terms are left untouched.

4. *If $\omega \in ker\,\Delta_g$, then*

$$\int_M \langle G_g(x,y), \omega(y)\rangle_g dv_g(y) = 0.$$

Recall the *Green's operator* $\widetilde{G}_g : (\mathcal{H}_g^k)^\perp \to (\mathcal{H}_g^k)^\perp$ which is the bounded inverse of $\Delta_g : (\mathcal{H}_g^k)^\perp \to (\mathcal{H}_g^k)^\perp$, where $(\mathcal{H}_g^k)^\perp$ is the L^2-orthogonal complement of $\mathcal{H}_g^k$. Then on $(\mathcal{H}_g^k)^\perp$, integrating against the Green's function is the same as applying the Green's operator.

Lemma 2.6. *Let $\mathcal{H}_g^k$ be the space of harmonic k-forms for the metric g. Then $(v,h) \in T_{(\omega,g)}Q$ and $v \perp \mathcal{H}_g^k$ if and only if*

$$(ev_x)_*((v,h)) = v(x) = \pm \int_M \langle dd^* G_g(x,y), *(D_h *)\omega(y)\rangle_g dv_g(y). \tag{10}$$

Here $D_h *_g$ or $D_h *$ is shorthand for $d/dt(*_{g+th})|_{t=0}$.

Proof. By Equation 7 and Proposition 2.5, if $(v,h) \in T_{(\omega,g)}Q$ and $v \perp \mathcal{H}_g^k$, then we have

(11)
$$(ev_x)_*((v,h)) = v(x) = -\int_M \langle G_g(x,y), \frac{d}{dt}(\Delta_{g+th})\Big|_{t=0} \omega(y)\rangle_g dv_g(y).$$

We compute $v(x)$:

$$
\begin{aligned}
v(x) &= -\int \frac{d}{dt}\langle G_g(x,y), \Delta_{g+th}\omega(y)\rangle_g dv_g(y)\\
&= -\int \frac{d}{dt}\{\langle dG_g(x,y), d\omega(y)\rangle_g + \langle d^*_{g+th}G_g(x,y), d^*_{g+th}\omega(y)\rangle_g\} dv_g(y)\\
&= -\int \frac{d}{dt}\langle *_{g+th}d *_{g+th} G_g(x,y), *_{g+th}d *_{g+th} \omega(y)\rangle_g dv_g(y)\\
&= -\int \langle *_g d *_g G_g(x,y), *_g d\ (D_h *_g)\omega(y)\rangle_g dv_g(y)\\
&= \pm\int \langle * d * d * G_g(x,y), (D_h *)\omega(y)\rangle_g dv_g(y),
\end{aligned}
$$

keeping in mind that $d\omega = 0$, $d^*_g\omega = 0$. This yields Equation 10.

On the other hand, if we define v according to Equation 10, then the steps can be reversed so that v satisfies Equation 11. Observe that

$$
\frac{d}{dt}\left(\Delta_{g+th}\right)\bigg|_{t=0}\omega
$$

is exact from the proof of Claim 2.4 and is therefore in $(\mathcal{H}^k_g)^\perp$. Then

$$
v = \widetilde{G}_g\left(\frac{d}{dt}(\Delta_{g+th})\bigg|_{t=0}\omega\right),
$$

and, equivalently, $v \perp \mathcal{H}^k_g$ and

$$
\Delta_g v = \frac{d}{dt}(\Delta_{g+th})\bigg|_{t=0}\omega.
$$

The lemma then follows from Claim 2.4. $\square$

Remark 2.7. Perturbations (v,h) satisfying Equation 10 have the nice additional property that v is *exact*. Suppose we can show that, given $(\omega, g) \in Q$,

$$
\wedge^k T^*_x M = \{v(x) \mid (v,h) \text{ satisfies Equation 10}\},
$$

whenever $\omega(x) = 0$. This would then imply a stronger transversality result, namely the regularity of:

$$ev_\alpha : \mathrm{Met}^l(M) \times M \longrightarrow \wedge^k T^* M, \quad (g, x) \longmapsto \omega(x), \tag{12}$$

where ω is the g-harmonic representative of the class $\alpha \in H^k(M; \mathbf{R})$. We will use the following evaluation map in Section 2.6:

$$ev_{\alpha,x} : \mathrm{Met}^l(M) \to \wedge^k T_x^* M, \quad g \mapsto \omega(x), \tag{13}$$

where ω is the g-harmonic representative in α.

Although we do not have a good grasp of $G_g(x, y)$ in general, we can still take advantage of the asymptotics of $G_g(x, y)$ near the diagonal Δ. This is because the perturbations h of the metric g we will be using are the ones supported arbitrarily close to x.

Conventions. We use multi-indices $I = (i_1, \cdots, i_k)$ to write

$$\omega_I = \omega_{(i_1,\cdots,i_k)} = \omega_{i_1} \wedge \cdots \wedge \omega_{i_k}.$$

Also, we will often omit the $\wedge$ symbol in expressions for forms.

We then have the following:

Proposition 2.8. *Let g_0 be the flat metric on $\mathbf{R}^n$. Then*

$$G_{g_0}(x, y) = \begin{cases} \sum_{\substack{I=(i_1,\cdots,i_k) \\ i_1<\cdots<i_k}} c_n/|x-y|^{n-2} \cdot dx_I \otimes dy_I, & \text{if } n > 2 \\ \sum_{\substack{I=(i_1,\cdots,i_k) \\ i_1<\cdots<i_k}} c_n \log|x-y| \cdot dx_I \otimes dy_I, & \text{if } n = 2, \end{cases} \tag{14}$$

where c_n is a dimensional constant.

For an arbitrary metric g, pick local geodesic coordinates $U \subset \mathbf{R}^n$ for g, centered at $x = 0$. Then we have the following, which is Theorem 2.2

of [**12**] (some ingredients are also found in [**13**]) and roughly states that G_g and G_{g_0} are asymptotic near the diagonal:

Proposition 2.9. *On a sufficiently small* U, $G_g(0,y)$ *has the form*:

$$G_g(0,y) = G_{g,2-n}(y) + G_{g,4-n}(y) + \cdots + \begin{cases} G_{g,-2}(y) + G_{g,0}(y), & n \text{ even}, \\ G_{g,-1}(y) + G_{g,\log}(y) + G_{g,0}(y), & n \text{ odd}, \end{cases} \tag{15}$$

where

1. $G_{g,2-n}(y) = G_{g_0}(0,y)$,

2. $G_{g,i}(y)$ *are* k*-forms whose coefficients have a singularity at* $y=0$ *of the type* $P_{i+s}(y)/|y|^s$, *where* $P_{i+s}(y)$ *is a polynomial in* $y_1,\cdots,y_n$ *of degree* $i+s$,

3. $G_{g,\log}$ *has coefficients with logarithmic singularities about* $y=0$,

4. $G_{g,0}(y)$ *is bounded, and*

5. *the coefficients of all the* $G_{g,*}(y)$ *are sufficiently differentiable away from* $y=0$.

In what follows we write $F(y) = c_n^{-1} G_{g_0}(0,y)$ for g_0 flat.

2.3 Computation of $dd^*F(y)$ for g flat. The goal of Sections 2.3, 2.4 and 2.5 is to assemble the preliminary computations to determine $\{v(x) | (v,h) \text{ satisfies Equation 10}\} \subset (ev_x)_*(T_{(\omega,g)}Q)$, whenever $\omega(x)=0$. In Section 2.3, we compute the left-hand term of the inner product in the integrand of Equation 10 for the flat metric g. In Section 2.4 we compute the right-hand term $*(D_h*)\omega(y)$, and in Section 2.5 we compute the space $\{*(D_h*)\omega(y) | h \in Sym^2 T_y^* M\} \subset \wedge^k T_y^* M$, where $Sym^2 T_y^* M$ is the second symmetric power of $T_y^* M$. With Proposition 2.9 in mind, these ingredients will be put together starting with Section 2.6.

Consider $\mathbf{R}^n$ with the standard inner product $\langle , \rangle$. Given a nonzero $y \in \mathbf{R}^n$, we define a linear map $R_y : \mathbf{R}^n \to \mathbf{R}^n$ which maps $y \mapsto (1-n/k)\,y$ and $v \mapsto v$ for $v \perp y$. The map R_y is nearly a reflection along the hyperplane $\{x \in \mathbf{R}^n | \langle x,y \rangle = 0\}$ and is one when $n=2k$. Using the inner product we identify $\mathbf{R}^n \simeq (\mathbf{R}^n)^*$, and then naturally extend R_y to $\wedge^* R_y : \wedge^*(\mathbf{R}^n)^* \to \wedge^*(\mathbf{R}^n)^*$ by mapping $v_1 \wedge \cdots \wedge v_k \mapsto R_y v_1 \wedge \cdots \wedge R_y v_k$.

Proposition 2.10. *$\langle dd^* F(y), \cdot \rangle = (C/|y|^n) \wedge^* R_y$, where C is a nonzero constant.*

Proof. There are two cases: either $n > 2$ and $k \geq 1$, or $n = 2$ and $k = 1$. The difference is due to the different forms of the Green's functions. Throughout the proof, C denotes a nonzero constant which may vary from line to line.

Case $n > 2$, $k \geq 1$. We compute:

$$dd^* F(y) = \sum_{\substack{I=(i_1,\cdots,i_k)\\ i_1<\cdots<i_k}} dx_I \otimes dd^* \left(\frac{1}{|y|^{n-2}} dy_I \right) \tag{16}$$

$$= C \sum_{\substack{I=(i_1,\cdots,i_k)\\ i_1<\cdots<i_k}} dx_I \otimes \left\{ \left(\frac{k}{|y|^n} - \frac{n}{|y|^{n+2}} (y_{i_1}^2 + \cdots + y_{i_k}^2) \right) dy_I \right. \tag{17}$$

$$\left. + \sum_{\substack{j=1,\cdots,k\\ i\neq i_1,\cdots,i_k}} (-1)^{j+k} \left(- \frac{n}{|y|^{n+2}} y_{i_j} y_i \right) dy_{i_1} \cdots \widehat{dy_{i_j}} \cdots dy_{i_k} dy_i \right\}. \tag{18}$$

Claim 2.11. *$dd^* F(y)$ is invariant under the action of $A \in SO(n)$.*

Proof of claim. We first define the action of $A \in SO(n)$ to be:

$$A^*(dd^* F(y)) = \sum_I A_x^* dx_I \otimes A_y^* \left(dd^* \left(\frac{1}{|y|^{n-2}} dy_I \right) \right), \tag{19}$$

where A_x^* is the pullback with respect to the coordinates x and A_y^* is with respect to the coordinates y. Since d and $*$ commute with A^*, we have

$$A^*(dd^* F(y)) = \sum_I A_x^* dx_I \otimes dd^* \left(\frac{1}{|y|^{n-2}} A_y^* dy_I \right). \tag{20}$$

Finally, we compute that $\sum_I A_x^* dx_I \otimes A_y^* dy_I = \sum_I dx_I \otimes dy_I$ (check this!), which proves the claim. $\square$

It therefore suffices to choose $y = (y_1, 0, \cdots, 0)$, so that

$$dd^* F(y_1, 0, \cdots, 0) = \frac{C}{|y|^n} \sum_{\substack{I=(i_1,\cdots,i_k)\\ i_1<\cdots<i_k}} \xi(I) dx_I \otimes dy_I, \tag{21}$$

where

$$\xi(i_1, \cdots, i_k) = \begin{cases} 1 - n/k & \text{if } i_1 = 1 \\ 1 & \text{otherwise.} \end{cases}$$

Case $n = 2$, $k = 1$. We compute likewise that

$$dd^* F(y) = C \cdot \sum_i dx_i \otimes \left\{ \left(\frac{1}{|y|^2} - \frac{2}{|y|^4} y_i^2 \right) dy_i - \sum_{j \neq i} \frac{2}{|y|^4} y_i y_j dy_j \right\}. \tag{22}$$

Choosing $y = (y_1, 0)$, we specialize to

$$dd^* F(y_1, 0) = \frac{C}{|y_1|^2} (-dx_1 \otimes dy_1 + dx_2 \otimes dy_2). \tag{23}$$

This proves Proposition 2.10. $\square$

2.4 Computation of $*(D_h *)$. Consider the family of metrics $g_t = g + th$. In this section we compute $*(D_h *)\omega(y)$, which is a term on the right-hand side of Equation 10. (Recall that $D_h * = d/dt(*_{g_t})|_{t=0}$.) Given an oriented orthonormal basis $(e_1, \cdots, e_n)$ for $T_y^* M$ with respect to $g = g_0$, we first find a convenient oriented basis $(e_1(t), \cdots, e_n(t))$ for $T_y^* M$ which satisfies:

1. $e_i(0) = e_i$,
2. $e_i(t)$ is smooth in t,
3. the basis is orthonormal with respect to g_t, up to first order in t.

Claim 2.12. *If* $\langle e_i, e_j \rangle_{g+th} = \delta_{ij} + th_{ij}$, *then* $e_i(t) = e_i - (1/2)t \sum_j h_{ij} e_j$ *suffices.*

Proof. We easily compute

$$\langle e_i(t), e_j(t)\rangle_{g_t} = \left\langle e_i - \frac{1}{2}t\sum_l h_{il}e_l, e_j - \frac{1}{2}t\sum_m h_{jm}e_m \right\rangle_{g_t} \tag{24}$$

$$= \delta_{ij} + th_{ij} - \frac{1}{2}t\left(\sum_l h_{il}\delta_{lj} + \sum_m h_{jm}\delta_{mi}\right) = \delta_{ij}, \tag{25}$$

up to first order in t. □

Proposition 2.13. *Let $\omega = \sum_{i_1,\cdots,i_k} \omega_{i_1\cdots i_k} e_{i_1}\cdots e_{i_k}$ be a fixed k-form on M. Then*

$$*(D_h*)\omega = C\cdot\left\{ \sum_{i_1,\cdots,i_k,j} \left(h_{i_1 j}\omega_{j i_2\cdots i_k} + \cdots + h_{i_k j}\omega_{i_1\cdots i_{k-1} j}\right) e_{i_1}\cdots e_{i_k} - \frac{1}{2}tr(h)\cdot\omega \right\}. \tag{26}$$

Here, C is a nonzero constant.

Proof. For a 1-form ω we will compute

$$*(D_h*)\omega = C\cdot\left\{\sum_{i,j} h_{ij}\omega_j e_i - \frac{1}{2}tr(h)\omega\right\}, \tag{27}$$

leaving the harder verifications to the reader.

With $e_i(t)$ as in Claim 2.12, we write $\omega = \sum_{i=1}^n \omega_i(t)e_i(t)$. Then

$$*_{g_t}\omega = \sum_i (-1)^{i-1}\omega_i(t)e_1(t)\cdots\widehat{e_i(t)}\cdots e_n(t).$$

Using $\dot{e}_i = -1/2\sum_{j=1}^n h_{ij}e_j$ from Claim 2.12 (a dot over a function or form η indicates $d\eta/dt|_{t=0}$), we compute

$$\begin{aligned}
(D_h*)\omega &= \frac{d}{dt}(*_{g_t}\omega)\bigg|_{t=0} = \sum_i(-1)^{i-1}\frac{d}{dt}\left\{\omega_i(t)e_1(t)\cdots\widehat{e_i(t)}\cdots e_n(t)\right\}\\
&= \sum_i(-1)^{i-1}\dot{\omega}_i e_1\cdots\widehat{e_i}\cdots e_n + \sum_{i,j\neq i}(-1)^{i-1}\omega_i e_1\cdots\dot{e}_j\cdots\widehat{e_i}\cdots e_n\\
&= \sum_i(-1)^{i-1}\dot{\omega}_i e_1\cdots\widehat{e_i}\cdots e_n + \sum_{i,j\neq i}(-1)^{i}\frac{1}{2}\omega_i\cdot\{h_{jj}e_1\cdots\widehat{e_i}\cdots e_n\\
&\quad +(-1)^{i-j-1}h_{ji}e_1\cdots\widehat{e_j}\cdots e_n\right\}.
\end{aligned}$$

Now, since $\dot{\omega} = \sum_{i=1}^n(\dot{\omega}_i e_i + \omega_i\dot{e}_i) = 0$, we have:

$$\begin{aligned}
(D_h)\omega &= (-1)^{n-1}\frac{1}{2}\sum_{i,j}\omega_i h_{ij}e_j + \sum_{i,j\neq i}(-1)^i\frac{1}{2}\\
&\qquad\times\omega_i\left\{(-1)^{n-i}h_{jj}e_i + (-1)^{n-i-1}h_{ji}e_j\right\}\\
&= (-1)^{n-1}\sum_{i,j\neq i}\frac{1}{2}\left(\omega_i h_{ji}e_j - \omega_i h_{jj}e_i\right) + (-1)^{n-1}\frac{1}{2}\sum_{i,j}\omega_i h_{ij}e_j\\
&= (-1)^{n-1}\left\{\sum_{i,j}h_{ij}\omega_j e_i - \frac{1}{2}tr(h)\omega\right\}. \qquad \square
\end{aligned}$$

In particular,

Corollary 2.14. *If $\omega = \sum_{i,j}\omega_{ij}e_ie_j$ is a 2-form on M, then*

$$*(D_h*)\omega = C\left\{\sum_{i,j,k}\left(h_{ik}\omega_{kj} + \omega_{ik}h_{kj}\right)e_ie_j - \frac{1}{2}tr(h)\cdot\omega\right\}. \tag{28}$$

2.5 Computation of $i_{\omega(y)}$. In this section we compute the image of the map

$$i_{\omega(y)} : Sym^2T_y^*M \longrightarrow \wedge^kT_y^*M, \quad h\longmapsto *(D_h*)\omega(y), \tag{29}$$

for certain values of k and n, using Proposition 2.13. Here ω is a k-form and $\mathrm{Sym}^2 T_y^* M$ is the second symmetric power of $T_y^* M$.

$k = 1$. If $\omega(y) = 0$, then $\mathrm{Im}\ i_{\omega(y)} = 0$. If $\omega(y) \neq 0$, then $\mathrm{Im}\ i_{\omega(y)} = T_y^* M$ by using Equation 27.

$k = 2$. Let $\omega = \sum_{i,j} \omega_{ij} e_i \wedge e_j$ be a 2-form which has been normalized so that $\omega_{ji} = -\omega_{ij}$. Then we associate to $\omega(y)$ the skew-symmetric matrix A whose ij-th entry is $\omega_{ij}(y)$. To the variation in the metric $h(y)$, we associate the symmetric matrix $H = (h_{ij}(y))$. By Corollary 2.14, $*(D_h *)\omega(y)$ corresponds to the skew-symmetric matrix $\{H, A\} - (1/2)tr(H) \cdot A$, where $\{H, A\} = HA + AH$. Hence, in terms of matrices, $i_\omega(y)$ becomes

$$i_A : \mathcal{S} \longrightarrow \mathcal{A}, \qquad (30)$$
$$i_A(H) = \{H, A\} - \frac{1}{2} tr(H) \cdot A,$$

where $\mathcal{S}$ is the set of symmetric $n \times n$ matrices and $\mathcal{A}$ is the set of skew-symmetric $n \times n$ matrices.

Consider $C \in O(n)$. If $B = C^t AC$, then we can write $i_B(H) = C^t i_A(CHC^t)C$, where $CHC^t \in \mathcal{S}$. Therefore, it suffices to compute i_A for representatives A of each orbit of $\mathcal{A}$ under the action of $O(n)$ by conjugation.

Let $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. We will use the convention that a blank matrix entry signifies a matrix block consisting of all zeros. The following is an exercise in linear algebra:

Fact 2.15. Suppose $n = 2m$. Then an $n \times n$ skew-symmetric matrix A can always be put into the form

$$\begin{pmatrix} \lambda_1 J & & & \\ & \lambda_2 J & & \\ & & \ddots & \\ & & & \lambda_m J \end{pmatrix} \qquad (31)$$

with $\lambda_i \geq 0$, $i = 1, \cdots, m$, via an orthonormal change of basis.

If A has the form as in Equation 31 with $\lambda_i \geq 0$, $i = 1, \cdots, m$, then it is said to be in *normal form*, and will be denoted by $A =$

$(\lambda_1, \lambda_2, \cdots, \lambda_m)$. Note that we may take all λ_i to be nonnegative, since we are allowing conjugation by elements in $O(n)$, rather than $SO(n)$. If $\lambda_i > 0$ for all $i = 1, \cdots, m$ and $\lambda_i \neq \lambda_j$ for all $i \neq j$, then A is said to be of *generic type*.

We can make a further simplification when $n \neq 4$:

Claim 2.16. *If $n \neq 4$, then Im i_A = Im $\{\cdot, A\}$. Here $\{\cdot, A\}$ maps $H \mapsto \{H, A\}$.*

Proof. Provided $n \neq 4$, if we take $H = I$, then we obtain $A \in \text{Im } i_A$. By taking $H = I$ we also have $A \in \text{Im } \{\cdot, A\}$. The equality follows from the linearity of the two maps. □

The situation for $n = 4$ is quite different ($n = 4$ is the only anomaly), and this is the first indication of the differences between $n = 4$ and $n > 4$.

We have the following useful rule:

Rule 2.17. *Let $A = (\lambda_1, \cdots, \lambda_m)$ be in normal form, where $n = 2m$. If $\lambda_i \neq \lambda_j$, then Im $\{\cdot, A\}$ contains*

$$\begin{pmatrix} \ddots & & & & \\ & \lambda_i * & & * & \\ & & \ddots & & \\ & * & & \lambda_j * & \\ & & & & \ddots \end{pmatrix}.$$

Here $*$ is an arbitrary 2×2 block which is consistent with the skew-symmetry of A. The $*$'s are placed in the (i,i)-th, (i,j)-th, (j,i)-th, and (j,j)-th 2×2 blocks. On the other hand, if $\lambda_i = \lambda_j > 0$, then

Im $\{\cdot, A\}$ *will contain*

$$\begin{pmatrix} \ddots & & & & \\ & * & & X & \\ & & \ddots & & \\ & -X^t & & * & \\ & & & & \ddots \end{pmatrix}.$$

Here X *is a* 2×2 *block of the form* $\left(\begin{smallmatrix} c & -d \\ d & c \end{smallmatrix}\right)$, *with* $c, d \in \mathbf{R}$.

$k = 2$, $n = 4$. We have the following four possibilities for the 4×4 matrix A in normal form:

(1) $A_1 = 0$.

(2) $A_2 = (\lambda, 0)$, $\lambda > 0$.

(3) $A_3 = (\lambda, \lambda)$, $\lambda > 0$.

(4) $A_4 = (\lambda_1, \lambda_2)$, where $\lambda_1, \lambda_2 > 0$, and $\lambda_1 \neq \lambda_2$.

We can easily compute Im i_A for each of the four cases, with the help of Rule 2.17.

(1) Im $i_{A_1} = 0$.

(2) Im $i_{A_2} = \left\{ \left(\begin{smallmatrix} A & B \\ -B^t & 0 \end{smallmatrix}\right) \middle| \begin{smallmatrix} A=2\times 2 \text{ skew-symmetric matrix,} \\ B=2\times 2 \text{ matrix} \end{smallmatrix} \right\}$. Hence, dim Im $i_{\omega(y)} = 5$ and Im $i_{\omega(y)} = (*\omega(y))^{\perp}$.

(3) Im $i_{A_3} = \left\{ \left(\begin{smallmatrix} 0 & -a & b & c \\ a & 0 & -c & b \\ -b & c & 0 & a \\ -c & -b & -a & 0 \end{smallmatrix}\right) \middle| a, b, c \in \mathbf{R} \right\}$. Hence, if $\omega(y)$ is SD, then Im $i_{\omega(y)}$ is the space of ASD 2-forms.

(4) Im $i_{A_4} = \left\{ \left(\begin{smallmatrix} \lambda_1 A & B \\ -B^t & -\lambda_2 A \end{smallmatrix}\right) \middle| \begin{matrix} A = 2\times 2 \text{ skew-symmetric matrix} \\ B = 2\times 2 \text{ matrix} \end{matrix} \middle| \right\}$. Just as in (2), dim Im $i_{\omega(y)} = 5$ and Im $i_{\omega(y)} = (*\omega(y))^{\perp}$. Observe that $i_{\omega(y)}$ is not surjective even if $\omega(y)$ is of generic type - this is in sharp contrast with the cases $n \geq 6$.

$k = 2$, $n \geq 6$. Using Rule 2.17, it is easy to prove the following:

Proposition 2.18. *If an $n \times n$ skew-symmetric matrix A with $n = 2m$ even is of* generic type *(i.e., $A = (\lambda_1, \cdots, \lambda_m)$, $\lambda_i > 0$ for $i = 1, \cdots, m$, and $\lambda_i \neq \lambda_j$ if $i \neq j$), then* $\operatorname{Im} i_A = \operatorname{Im}\{\cdot, A\} = \mathcal{A}$.

$k = 2$**,** $n = 6$**:** The following are the possible types of orbits, where A is in normal form:

(1) $A = 0$. Im $i_A = 0$.

(2) $A = (\lambda, 0, 0)$. Im $i_A = \left\{ \begin{pmatrix} * & * & * \\ * & & \\ * & & \end{pmatrix} \right\}$.

(3) $A = (\lambda, \lambda, 0)$. Im $i_A = \left\{ \begin{pmatrix} * & X & * \\ -X^t & * & * \\ * & * & 0 \end{pmatrix} \right\}$.

(4) $A = (\lambda_1, \lambda_2, 0)$, where $\lambda_1 \neq \lambda_2$. Im $i_A = \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & 0 \end{pmatrix} \right\}$.

(5) $A = (\lambda_1, \lambda_1, \lambda_2)$, where $\lambda_1 \neq \lambda_2$. Im $i_A = \left\{ \begin{pmatrix} * & X & * \\ -X^t & * & * \\ * & * & * \end{pmatrix} \right\}$.

(6) $A = (\lambda_1, \lambda_2, \lambda_3)$, where all the λ_i are distinct. Im $i_A = \mathcal{A}$.

2.6 Harmonic 1-forms. In this section we prove the following theorem:

Theorem 2.19. *Let M be a closed, oriented n-dimensional manifold with $H^1(M; \mathbf{R}) \neq 0$. Fix a nonzero cohomology class $\alpha \in H^1(M; \mathbf{R})$. Then there exists a dense open set $\mathcal{U} \subset Met^l(M)$ for which every g-harmonic form ω in the class α has regular zeros, if $g \in \mathcal{U}$. Hence a generic harmonic 1-form ω in a fixed cohomology class α has isolated zeros.*

Remark 2.20. Provided all the zeros of ω are isolated, the number of zeros of ω, counted with sign, is the Euler characteristic of M.

Before we begin the proof of Theorem 2.19, we state (without proof) an important result of Aronszajn [**1**]:

Theorem 2.21.

1. ("Weak" Unique Continuation) *Let L be a second order linear elliptic operator with C^l coefficients, for sufficiently large l. Suppose $Lu = 0$ on a domain Ω and $u = 0$ on a nonempty open subset of Ω. Then $u = 0$ on all of Ω.*

2. ("Strong" Unique Continuation) *Let L be a second order linear elliptic operator with C^∞ coefficients. Suppose $Lu = 0$ on a domain Ω, $u(x) = 0$, and all the partial derivatives of all orders vanish at x. Then $u = 0$ on all of Ω.*

We now prove Theorem 2.19:

Proof of Theorem 2.19. We will prove the regularity of the map $ev_{\alpha,x}$ given in Equation 13. Suppose that $\omega(x) = ev_{\alpha,x}(g)(x) = 0$. Without loss of generality, take geodesic normal coordinates about $x = 0$; this identifies $T_y^*M \simeq (\mathbf{R}^n)^*$ for all y near 0. According to Equation 10, the perturbation $v(0)$ of $\omega(0)$ corresponding to the perturbation h of the metric g has the form

$$
\begin{aligned}
(ev_{\alpha,x=0})_*(g)(h) &= v(0) \\
&= \pm \int_M \langle dd^* G_g(0,y), *(D_h*)\omega(y)\rangle_g dv_g(y).
\end{aligned}
\tag{32}
$$

Here, $h(y)$ is of class C^l for $l >> 0$, and $c_n^{-1} G_g(0,y) \sim F(y)$ asymptotically, as $y \to 0$.

For $k = 1$ we showed the following:

- $\langle dd^{*_0} F(y), \cdot\rangle_{g_0} = (C/|y|^n) R_y$ for the flat metric g_0 and the adjoint d^{*_0} of d with respect to g_0.
- $*(D_h*)\omega(y) : (\mathbf{R}^n)^* \to (\mathbf{R}^n)^*$ is surjective, whenever $\omega(y) \neq 0$.

By the "Weak" Unique Continuation theorem, there exists a sequence of points $y_i \to 0$ for which $\omega(y_i) \neq 0$. Suppose $\eta \in (\mathbf{R}^n)^*$. By the surjectivity of $i_{\omega(y_i)}$ (since $\omega(y_i) \neq 0$), there exists a variation $h_i(y_i)$ of the metric at y_i for which $*(D_{h_i(y_i)}*)\omega(y_i) = \eta$. By taking a sequence of functions $f(y)$ with small support approaching $h_i(y_i) \cdot \delta_{y_i}(y)$ (here

$\delta_{y_i}(y)$ is the delta function with support at y_i), we obtain:

$$\lim_{f \to h_i(y_i)\cdot\delta_{y_i}} (ev_{\alpha,x=0})_*(g)(f) \tag{33}$$

$$= \lim_{f \to h_i(y_i)\cdot\delta_{y_i}} \pm \int_M \langle dd^* G_g(0,y), *(D_f *_g)\omega(y)\rangle_g dv_g(y)$$

$$= \pm\langle dd^* G_g(0,y_i), \eta\rangle. \tag{34}$$

The following claim implies that $\langle dd^* G_g(0,y_i),\eta\rangle \in \mathrm{Im}\ (ev_{\alpha,0})_*(g)$ for all $\eta \in (\mathbf{R}^n)^*$.

Claim 2.22. *Let $\phi : V \to W$ be a linear map with W finite-dimensional. If $\{w_i\}_{i=1}^{\infty}$ is a sequence in $\phi(V) \subset W$ which converges to $w \in W$, then $w \in \phi(V)$.*

Proof. This is a restatement of the fact that $\phi(V)$ is a subspace of W and that all vector subspaces of a finite-dimensional vector space are closed. □

By taking a subsequence, we may assume further that $y_i \to 0$ and $y_i/|y_i| \to \beta$. Now, as $y_i \to 0$, we have:

$$\frac{\langle dd^* G_g(0,y_i),\eta\rangle_g}{|\langle dd^* G_g(0,y_i),\eta\rangle_g|} \to \frac{\langle dd^{*_0} F(y_i),\eta\rangle_{g_0}}{|\langle dd^{*_0} F(y_i),\eta\rangle_{g_0}|} = \frac{R_\beta(\eta)}{|R_\beta(\eta)|}, \tag{35}$$

by Proposition 2.9. Since $\langle dd^* G_g(0,y_i),\eta\rangle_g \in \mathrm{Im}\ (ev_{\alpha,0})_*(g)$ for all $\eta \in (\mathbf{R}^n)^*$, it follows from Claim 2.22 that $R_\beta(\eta) \in \mathrm{Im}\ (ev_{\alpha,0})_*(g)$ for all $\eta \in (\mathbf{R}^n)^*$. Finally, since R_β is an isomorphism, we have proved that $\mathrm{Im}\ (ev_{\alpha,0})_*(g) = T^*_{x=0}M$. Theorem 2.19 follows from Remark 2.7 and the discussion of Section 2.1. □

2.7 SD/ASD harmonic 2-forms on a 4-manifold. We will now prove Theorem 1.1. Note that we may substitute ASD 2-forms for SD 2-forms with the same result.

Proof of Theorem 1.1. Consider the evaluation map

$$\begin{gathered} ev_+ : Q^+ \times M \to \wedge^+, \\ ((\omega,g),x) \mapsto (\omega(x),(g,x)) \end{gathered} \tag{36}$$

where $\wedge^+ \to \mathrm{Met}^l(M) \times M$ is the universal vector bundle whose fiber $\wedge_g^+ T_x^* M$ over the point $(g, x) \in \mathrm{Met}^l(M) \times M$ is the set of g-SD 2-forms at x. Recall that Q^+ is the set of pairs (ω, g) where $g \in \mathrm{Met}^l(M)$ and ω is an SD g-harmonic form with $[\omega] \neq 0$. (Here we are viewing $Q^+ \subset \Omega_m^2(M) \times \mathrm{Met}^l(M)$.)

We will show that ev_+ is transverse to the zero section of $\wedge^+$, i.e.,

$$(ev_{+,x})_* : T_{(\omega,g)} Q^+ \to \wedge_g^+ T_x^* M \tag{37}$$

is surjective, whenever $\omega(x) = 0$.

The two necessary and sufficient conditions for $(v, h) \in T_{(\omega,g)} Q^+$ are:

A. $\Delta_g(v) + d/dt(\Delta_{g+th})|_{t=0}\omega = 0$,

B. $*_{g+th}(\omega + tv) = \omega + tv$, up to first order in t.

Differentiating (B), we obtain that $v = *_g v + d/dt(*_{g+th})|_{t=0}\omega$. At a point x where $\omega(x) = 0$, this gives $v(x) = *_g v(x)$. Hence, in order to determine $(ev_{+,x})_*$ at (ω, g) when $\omega(x) = 0$, it suffices to compute the $v(x)$'s as in Equation 10, and project onto $\wedge_g^+ T_x^* M$, i.e., take $v(x) + *_g v(x)$.

Take geodesic normal coordinates about $x = 0$ and identify $T_y^* M \simeq (\mathbf{R}^4)^*$ for all y near 0. We have the following:

(i) $\langle dd^{*_0} F(y), \cdot \rangle_{g_0} = (C/|y|^4) R_y$ for g_0 flat. Here d^{*_0} is the adjoint of d with respect to g_0.

(ii) If $\omega(y) \neq 0$ is SD, then Im $i_{\omega(y)} = \wedge^-(\mathbf{R}^4)^*$.

(iii) If $y \neq 0$, then R_y swaps SD forms and ASD forms, i.e., $R_y : \wedge^\pm(\mathbf{R}^4)^* \xrightarrow{\simeq} \wedge^\mp(\mathbf{R}^4)^*$.

The rest of the proof proceeds in the same fashion as the proof of Theorem 2.19. Let $(\omega, g) \in Q^+$ with $\omega(0) = 0$. Since $[\omega] \neq 0$, there exists a sequence $y_i \to 0$ such that $\frac{y_i}{|y_i|} \to \beta$ and $\omega(y_i) \neq 0$ by the "Weak" Unique Continuation theorem. Combining (ii) and (iii), we obtain $R_{y_i}(\mathrm{Im}\ i_{\omega(y_i)}) = \wedge^+(\mathbf{R}^4)^*$. Now, using a limiting argument as in Theorem 2.19 (and Claim 2.22), Im $(ev_{+,x})_* \supset R_\beta(\wedge^-(\mathbf{R}^4)^*)$, implying the surjectivity of $(ev_{+,x})_*$. $\square$

Remark 2.23. An SD form ω is nondegenerate at all points x where $\omega(x) \neq 0$. This is because $\omega^2(x) = \omega(x) \wedge *\omega(x) > 0$, if $\omega(x) \neq 0$.

Hence if $b_2^+(M) > 0$, we can construct an SD harmonic form which is nearly symplectic, that is, is nondegenerate away from a collection of disjoint circles.

Let (ω_0, g_0), (ω_1, g_1) be regular points in Q^+ with respect to the evaluation map ev_+ given by Equation 36. (In particular, $[\omega_0]$, $[\omega_1]$ are nonzero cohomology classes.) Define

$$P(Q^+) = \big\{\gamma : [0,1] \to Q^+ | \gamma(0) = (\omega_0, g_0); \gamma(1) = (\omega_1, g_1); \\ \gamma \text{ is of class } C^l \text{ for } l >> 0\big\}. \tag{38}$$

We write $\gamma(t) = (\omega_t, g_t)$. Consider

$$\begin{aligned} \widetilde{ev}_+ : P(Q^+) \times M \times [0,1] &\to \wedge^+, \\ (\gamma, x, t) &\mapsto (\omega_t(x), (g_t, x)). \end{aligned} \tag{39}$$

The following is the one-parameter family version of Theorem 1.1:

Theorem 2.24. *There exists a dense open set $\mathcal{U} \subset P(Q^+)$ such that if $\gamma \in \mathcal{U}$ and $\gamma(t) = (\omega_t, g_t)$, then $\{(x,t) \in M \times [0,1] | \omega_t(x) = 0\}$ gives a cobordism inside $M \times [0,1]$ between the zeros of ω_0 and the zeros of ω_1.*

Proof. By Proposition 2.2, it suffices to show that $(\widetilde{ev}_{+,x,t})_* : T_\gamma P(Q^+) \to \wedge^+_{g_t} T^*_x M$ is surjective whenever $\omega_t(x) = 0$. Since $[\omega_t] \neq 0$ from the definition of Q^+, the proof of Theorem 1.1 gives enough perturbations (v_t, h_t) of (ω_t, g_t) *at the point* t such that $\{v_t(x) | (v_t, h_t) \in T_{([\omega_t],g)} Q^+\} = \wedge^+_{g_t} T^*_x M$. It is easy to incorporate (v_t, h_t) into a one-parameter family, thereby proving the surjectivity of $(\widetilde{ev}_{+,x,t})_*$. □

2.8 Non-SD/ASD harmonic 2-forms on a 4-manifold. In this section assume that M is a closed, oriented 4-manifold with $b_2^+(M) > 0$ and $b_2^-(M) > 0$. If the intersection pairing on $H^2(M; \mathbf{R})$ is definite, then all the harmonic 2-forms are automatically SD or ASD, and the considerations of Section 2.7 apply instead.

Before discussing genericity results for non-SD/ASD harmonic 2-forms, we first explain the stratification of $\wedge^2(\mathbf{R}^4)^*$ under the action of $SO(4)$, where $\mathbf{R}^4 \simeq (\mathbf{R}^4)^*$ via the inner product on $\mathbf{R}^4$. Since rank is invariant under the action of $SO(4)$, we let $V_i = \{\omega \in \wedge^2(\mathbf{R}^4)^* \mid \operatorname{rank}\omega = i\}$. In particular, $V_0 = \{0\}$. Note that V_4 has two substrata, namely the SD 2-forms and the ASD 2-forms, which we denote $V_{4,+}$ and $V_{4,-}$, respectively. For convenience, we assemble the relevant data in a chart. In the chart, $\{e_1, e_2, e_3, e_4\}$ is an orthonormal basis for $(\mathbf{R}^4)^*$.

Stratum	Typical Element	Dim Orbit	Dim Stratum
V_0	0	0	0
V_2	$\lambda e_1 e_2$	4	5
$V_{4,\pm}$	$\lambda(e_1 e_2 \pm e_3 e_4)$	2	3
$V_4 - (V_{4,+} \cup V_{4,-})$	$\lambda_1 e_1 e_2 + \lambda_2 e_3 e_4$, $\lambda_1 \neq \pm\lambda_2$	4	6

We prove the following key theorem:

Theorem 2.25. *Consider the evaluation map ev given by Equation 2. There exists a dense open set* $\mathcal{U} \subset Q - D$, *where* $D = \{(\omega, g) \in Q | *\omega = \pm\omega\}$, *such that if* $(\omega, g) \in \mathcal{U}$, *then* $(ev_x)_* : T_{(\omega,g)}Q \to \wedge^2 T_x^* M$ *is surjective for all* $x \in M$. *Moreover,* $\mathcal{U}$ *contains* $Q' = \{(\omega, g) \in Q - D \mid g$ *is of class* $C^\infty\}$.

Proof. Let $(\omega, g) \in Q'$. We can write $\omega = \omega_+ + \omega_-$ with ω_+ SD and ω_- ASD, and $\omega_\pm \not\equiv 0$. The set of points where ω is SD is the zero set of ω_-. Since $\omega_\pm$ are harmonic, by the "Weak" Unique Continuation theorem, ω is SD or ASD away from a dense open subset of M. The locus $\{x | \omega(x) \neq 0\}$ is also a dense open subset of M. In what follows, take geodesic normal coordinates y on a suitably small ball D^n about $x = 0$, and identify $T_y^* M \simeq (\mathbf{R}^4)^*$ over D^n.

Case 1. Suppose $\omega(0) \in V_4 - V_{4,\pm}$. Then $\omega(y) \in V_4 - V_{4,\pm}$ for all $y \in D^n$, after possibly shrinking D^n, and $\operatorname{Im} i_{\omega(y)} = (*\omega(y))^\perp$, a five-dimensional subspace of $\wedge^2(\mathbf{R}^4)^*$. Now, taking a sequence $y_i \to 0$ with $y_i/|y_i| \to \beta$,

$$\langle dd^* G(0, y_i), \operatorname{Im}\, i_{\omega(y_i)} \rangle_g \to (*_0 R_\beta(\omega(0)))^{\perp_0}, \tag{40}$$

with $*_0$, g_0, $\perp_0$ with respect to the inner product at $y = 0$. But the subspaces $(*_0 R_\beta(\omega(0)))^{\perp_0}$ do not approach the same five-dimensional space from the various directions parametrized by $S^3 = \{\beta \in \mathbf{R}^4 \mid |\beta| = 1\}$. Using the limiting argument from the proof of Theorem 2.19, $(ev_{x=0})_*$ must be surjective.

Case 2. Suppose $\omega(0) \in V_{4,+}$. Since rank is upper semi-continuous, we may assume that $\omega(y) \in V_4$ for all $y \in D^n$. Moreover, $\omega(y) \in V_4 - V_{4,\pm}$ on a dense open subset of D^n, and $(ev_{x=0})_*$ is surjective by the same argument as in Case 1.

Case 3. Suppose $\omega(0) \in V_2$. We may assume that $\omega(y) \in V_2 \cup (V_4 - V_{4,\pm})$ for all $y \in D^n$. Im $i_{\omega(y)} = (*\omega(y))^\perp$ is still five-dimensional, and $(ev_{x=0})_*$ is surjective as in Case 1.

Case 4. Suppose $\omega(0) \in V_0$. Here we use the condition that the metric g is of class C^∞, and hence ω is also of class C^∞ by the standard elliptic theory. We argue by contradiction. Suppose $(ev_{x=0})_*$ is not surjective. Then, for y in a dense open set of D^n, $\omega(y) \in V_2 \cup (V_4 - V_{4,\pm})$ and $\langle dd^* G(0,y), \mathrm{Im}\ i_{\omega(y)}\rangle = \langle dd^* G(0,y), (*\omega(y))^\perp\rangle$ must be a five-dimensional subspace of $\wedge^2(\mathbf{R}^4)^*$, independent of y. Since ω is C^∞, we are able to write

$$\omega(y) = \omega_r(y) + \text{h.o.}, \tag{41}$$

where $\omega_r(y) = \sum_{i,j} p_r^{ij}(y) dy_i dy_j$ with $p_r^{ij}(y)$ a homogeneous polynomial of degree r, and 'h.o.' is the remainder consisting of terms of degree $> r$ in y. The "Strong" Unique Continuation theorem (Theorem 2.21) ensures that there exists some $r < \infty$ for which $\omega_r(y) \neq 0$.

Now, take a sequence $y_i \to 0$ such that $y_i/|y_i| \to \beta$, $\beta \in S^3$, and $\omega(y_i) \in V_2 \cup (V_4 - V_{4,\pm})$. If $*, \perp, d^*$ are with respect to g and $*_0, \perp_0, d^{*_0}$ are with respect to the flat metric g_0, then we have:

(42)
$$\langle dd^* G(0, y_i), \mathrm{Im}\ i_{\omega(y_i)}\rangle_g \to R_\beta((*_0\omega_r(\beta))^{\perp_0}) = (R_\beta(*_0\omega_r(\beta)))^{\perp_0},$$

since $\omega(y)$ is dominated by $\omega_r(y)$ near $y = 0$, and the homogeneity of $\omega_r(y)$ implies that $\omega_r(ty)$ is a multiple of $\omega_r(y)$. Therefore,

$$R_\beta(*_0\omega_r(\beta)) = f(\beta)\widetilde{\omega}_0, \tag{43}$$

where $\widetilde{\omega}_0$ is (nonzero and) constant and f is a function of $\beta \in S^3$.

Next, observe that the coefficients of $|y|^2 R_y$ are polynomials of degree 2. Then we can write:

$$|y|^2 R_y(*_0\omega_r(y)) = \sum_{i,j} f^{ij}_{r+2}(y)dy_i dy_j, \tag{44}$$

where f^{ij}_{r+2} is a homogeneous polynomial of degree $r+2$ in y. By combining Equations 43 and 44, we obtain:

$$|y|^2 R_y(*_0\omega_r(y)) = f_{r+2}(y)\widetilde{\omega}_0, \tag{45}$$

with $f_{r+2}(y)$ homogeneous in y. Hence

$$*_0\omega_r = \frac{f_{r+2}(y)}{|y|^2} R_y(\widetilde{\omega}_0) = f_{r+2}|y|^2\langle dd^{*_0}F(y), \widetilde{\omega}_0\rangle_{g_0}, \tag{46}$$

$$\omega_r = f_{r+2}|y|^2\langle dd^{*_0}F(y), *_0\widetilde{\omega}_0\rangle_{g_0}. \tag{47}$$

Note that $\omega_r \neq 0$.

We readily calculate that

$$d(\langle dd^{*_0}F(y), \eta\rangle_{g_0}) = d\left(\frac{C}{|y|^4}R_y(\eta)\right) = 0, \tag{48}$$

for all constant 2-forms η. (Here C is a nonzero constant.) Coupled with the fact that $d\omega = 0$ and $d*\omega = 0$ imply $d\omega_r = 0$ and $d*_0\omega_r = 0$, we obtain:

$$d\omega_r = 0 \Longrightarrow d(f_{r+2}|y|^2) \wedge R_y(*_0\widetilde{\omega}_0) = 0 \tag{49}$$

$$d(*_0\omega_r) = 0 \Longrightarrow d(f_{r+2}|y|^2) \wedge R_y(\widetilde{\omega}_0) = 0. \tag{50}$$

Now, if $0 \neq \xi \in (\mathbf{R}^4)^*$, $0 \neq \zeta \in \wedge^2(\mathbf{R}^4)^*$, and $\xi \wedge \zeta = 0$, then ζ is decomposable and ξ lies on the 2-plane given by ζ. If $\xi \wedge *\zeta = 0$ as well, then ξ also lies on a 2-plane orthogonal to ζ, a contradiction. Therefore, $d(f_{r+2}|y|^2) = 0$ and $f_{r+2} = 0$, contradicting the nonsurjectivity of $(ev_{x=0})_*$.

We have now shown that for all $(\omega, g) \in Q'$, $(ev_x)_*$ is surjective at all points $x \in M$. Since the surjectivity of the derivative is an open condition, we can cover $\{(\omega, g)\} \times M$ with a finite number of open sets $\mathcal{U}_i \times V_i$, $i = 1, \cdots, l$. Then $\cap_{i=1}^{l} \mathcal{U}_i$ is still open and $(ev_x)_*$ is surjective at all points of $(\cap_{i=1}^{l} \mathcal{U}_i) \times M$. □

We now complete the proof of Theorem 1.2.

Proof of Theorem 1.2. This immediately follows from Theorem 2.25 and the table before Theorem 2.25, since the evaluation map ev, when restricted to $\mathcal{U} \times M$, is transverse to all the strata. □

2.9 Harmonic 2-forms on even-dimensional manifolds. We will prove the following theorem, which has Theorem 1.3 as a special case.

Theorem 2.26. *Let M be a closed, oriented n-manifold, where $n = 2m > 4$ is even. Fix a nonzero cohomology class $\alpha \in H^2(M; \mathbf{R})$. Then there exists a dense open set $\mathcal{U} \subset Q'' = Met^l(M)$ on which $(ev_\alpha)_*$ given by Equation* 12 *is surjective at all points in $\mathcal{U} \times M$.*

Proof. Suppose for the moment that we have already proved the following proposition.

Proposition 2.27. *Starting from any $(g, x) \in Q'' \times M$ with corresponding g-harmonic 2-form $\omega \in \alpha$, we can find an arbitrarily small perturbation $g + h \in Q''$ of g so that the corresponding $(g+h)$-harmonic form $\omega + v \in \alpha$ has* generic type *at x.*

Proposition 2.27 implies the following lemma:

Lemma 2.28. *Let S be the set of $g \in Q''$ for which the g-harmonic form $\omega \in \alpha$ is of generic type on a dense open set in M. Then S is dense in Q''.*

Proof of Lemma 2.28. We will exhibit a $g \in S$ arbitrarily close to $g_0 \in Q''$. Pick a countable dense subset of M, say $\{x_i\}_{i=1}^{\infty}$. Let $\mathcal{V}_0 \ni g_0$ be a closed ball in Q'' of small radius, centered about g_0. We pick a closed ball $\mathcal{V}_i \subset \mathcal{V}_{i-1}$ and an open ball $U_i \ni x_i$ inductively, as follows: Given x_i, there exists a point $g_i \in \mathcal{V}_{i-1}$ such that the g_i-harmonic $\omega_i \in \alpha$ is of generic type at x_i. Then there exists $\mathcal{V}_i \times U_i \ni (g_i, x_i)$, on which $\omega(x)$ is of generic type, since the generic type condition is an open condition.

Now let $g \in \cap_{i=1}^{\infty} \mathcal{V}_i$, which is nonempty because of completeness. By our construction, the g-harmonic $\omega \in \alpha$ is of generic type on a dense open subset $\cup_{i=1}^{\infty} U_i$ of M. This proves Lemma 2.28. □

By Lemma 2.28, for any $g \in S$ and $x \in M$, there exist points $y_i \to x$ such that the g-harmonic $\omega \in \alpha$ is of generic type at y_i. According to Proposition 2.18, $i_{\omega(y_i)} = \wedge^2 T^*_{y_i} M$. Therefore, by using the approximation technique from Section 2.6, it follows that $(ev_{\alpha,x})_*$ is surjective for all $x \in M$ and $g \in S$.

We now argue as in the last paragraph of Theorem 2.25. The surjectivity of $(ev_{\alpha,x})_*$ is an open condition on $Q'' \times M$. Combining this with the compactness of M, we obtain that the condition "$(ev_{\alpha,x})_* : T_g Q'' \to \wedge^2 T^*_x M$ is surjective for all $x \in M$" is an open condition on Q''. But S is dense, so hence there is an open dense set $\mathcal{U} \subset Q''$ for which $(ev_{\alpha,x})_*$ is surjective for all $x \in M$. This completes the proof of Theorem 2.26. □

Proof of Proposition 2.27. Let us now proceed to show that ω can be perturbed at x so that $(\omega + v)(x)$ has generic type. If $\omega(x) = 0$, pick a point y arbitrarily close to x such that $\omega(y) \neq 0$ – such a sequence exists by the "Weak" Unique Continuation theorem. If $\omega(x) \neq 0$, pick a point y arbitrarily near x such that rank $\omega(y) \geq$ rank $\omega(x)$. This is possible because the rank is upper semi-continuous. Using the notation introduced in the paragraph after Fact 2.15 for matrices in normal form, we represent $\omega(y)$ by the matrix $(\lambda_1, \cdots, \lambda_s, 0, \cdots, 0)$, after an orthonormal change of basis. Here the $\lambda_i > 0$ are not necessarily distinct.

Then, by Rule 2.17, Im $i_{\omega(y)}$ contains an element η of higher rank than $\omega(y)$, if $\omega(y)$ does not have maximal rank already. For example, Im $i_{\omega(y)}$ contains any element of the form

$$\begin{pmatrix} * & & & & & \\ & * & & & & \\ & & \ddots & & & \\ & & & * & * & \\ & & & * & 0 & 0 \\ & & & & 0 & 0 \end{pmatrix}. \tag{51}$$

As before, $*$ is an arbitrary 2×2 matrix block, consistent with the skew-symmetry of the matrix. Now form $\omega'(x) = \omega(x) + tR_y\eta$. Recall that, using the approximation technique from Section 2.19, $R_y\eta \in$ Im $(ev_{\alpha,x})_*$ at $g \in Q''$, so there exists a perturbation $g+th$ of the metric g which gives rise to $\omega'(x)$, up to 1st order in t. Since $R_y\eta$ preserves the rank of η, $\operatorname{rank} R_y\eta > \operatorname{rank}\omega(x)$ and $\operatorname{rank}\omega'(x) > \operatorname{rank}\omega(x)$ for small enough t. Continue this process until we obtain a harmonic form $\omega'' \in \alpha$ (for a metric g'' close to g) such that the representation $\omega''(x) \leftrightarrow (\lambda_1, \cdots, \lambda_m)$ has λ_i all nonzero.

The next step is to perturb $\omega''(x)$ until the λ_i become distinct, while keeping them nonzero. Using our notation for matrices in normal form, let J_s be the $2s \times 2s$ matrix $(1, \cdots, 1)$, and let $(\lambda_1 J_{k_1}, \cdots, \lambda_r J_{k_r})$ denote the $n \times n$ matrix $(\lambda_1, \cdots, \lambda_1, \lambda_2, \cdots, \lambda_2, \cdots, \lambda_r, \cdots, \lambda_r)$, where λ_i appears k_i times and $2\sum_{i=1}^r k_i = n$. After an orthonormal change of basis, we write $\omega''(x) = (\lambda_1 J_{k_1}, \cdots, \lambda_r J_{k_r})$. Let

$$\lambda = \left\{ BAB^{-1} \,\middle|\, A = (\lambda_1 J_{k_1}, \cdots, \lambda_r J_{k_r}), \begin{array}{c} \lambda_i \neq \lambda_j, \lambda_i > 0, B \in O(n), \\ k_1 \geq \cdots \geq k_r, 2\displaystyle\sum_{i=1}^r k_i = n \end{array} \right\}. \tag{52}$$

λ is the stratum consisting of orbits of skew-symmetric matrices of the form $A = (\lambda_1 J_{k_1}, \cdots, \lambda_r J_{k_r})$. Before perturbing, we make the following dimensional computations, Lemmas 2.29 and 2.30.

Lemma 2.29. $\dim \lambda = \dim O(n) - \sum_{i=1}^r k_i^2 + r$.

Proof of Lemma 2.29. Let $A = (\lambda_1 J_{k_1}, \cdots, \lambda_r J_{k_r})$. Then the dimension of the orbit of A is given by:

$$\dim O(n)\cdot A = \dim O(n) - \dim \mathrm{Stab}(A) = \dim O(n) - \dim\ \ker(ad(A)), \tag{53}$$

where $ad(A)(B) = [A,B] = AB - BA$. Let us write $B = (B_{ij})$, where B_{ij} is a $2k_i \times 2k_j$ block. Then $[A,B]=0$ gives

$$\lambda_i J_{k_i} B_{ij} = \lambda_j B_{ij} J_{k_j}. \tag{54}$$

One computes that if $i=j$, then $B_{ii} \in \mathfrak{gl}(k_i, \mathbf{C})$, and if $i \neq j$, then $B_{ij} = 0$. Thus,

$$\ker(ad(A)) \simeq (\mathfrak{gl}(k_1, \mathbf{C}) \oplus \cdots \oplus \mathfrak{gl}(k_r, \mathbf{C})) \cap \mathfrak{o}(n) \tag{55}$$

$$= \mathfrak{u}(k_1) \oplus \cdots \oplus \mathfrak{u}(k_r), \tag{56}$$

where $\mathfrak{gl}(s,\mathbf{C})$, $\mathfrak{o}(s)$, and $\mathfrak{u}(s)$ are Lie algebras of $Gl(s,\mathbf{C})$, $O(s)$, and $U(s)$, respectively. Then

$$\dim\ \ker(ad(A)) = k_1^2 + \cdots + k_r^2, \tag{57}$$

and

$$\dim \lambda = \dim O(n)\cdot A + r = \dim O(n) - \sum_{i=1}^{r} k_i^2 + r. \qquad \square \tag{58}$$

Lemma 2.30. $\dim \mathrm{Im}\ i_A = \dim O(n) - \sum_{i=1}^r k_i^2 + n/2$.

Proof of Lemma 2.30. Using Rule 2.17, one computes that

$$\mathrm{Im}\ i_A = \begin{pmatrix} A_1 & * & * & * \\ * & A_2 & * & * \\ * & * & \ddots & * \\ * & * & * & A_r \end{pmatrix}, \quad \text{with} \quad A_i = \begin{pmatrix} * & X & X & X \\ X & * & X & X \\ X & X & \ddots & X \\ X & X & X & * \end{pmatrix},$$

where $*$ is an arbitrary block of the correct size consistent with the skew-symmetry, and X is a 2×2 block of the form $\left(\begin{smallmatrix} c & -d \\ d & c \end{smallmatrix}\right)$. Therefore,

$$\dim \mathrm{Im}\ i_A = \dim O(n) - 2\sum_{i=1}^{r} \frac{k_i(k_i-1)}{2} = \dim O(n) - \sum_{i=1}^{r} k_i^2 + \frac{n}{2}. \qquad \square \tag{59}$$

Lemmas 2.29 and 2.39 prove that if $r < n/2$, then dim Im $i_A >$ dim Λ, for $A \in \Lambda_{k_1,\cdots,k_r}$. We now claim that there exists a small perturbation of (ω'', g'') to (ω''', g''') such that $\omega'''(x) \in \Lambda_{l_1,\cdots,l_s}$ with dim $\Lambda_{l_1,\cdots,l_s} >$ dim $\Lambda_{k_1,\cdots,k_r}$, as long as not all $k_i = 1$, i.e., $\omega''(x)$ is not already of *generic type*. Indeed, pick y arbitrarily close to x such that $\omega''(y) \in \Lambda_{m_1,\cdots,m_p}$ with $p \geq r$ and $\sum_{i=1}^{p} m_i^2 \leq \sum_{i=1}^{r} k_i^2$. Then dim Im $i_{\omega''(y)} >$ dim $\Lambda_{k_1,\cdots,k_r}$. This means that there exists η such that $\omega''(x)+tR_y\eta$ exits $\Lambda_{k_1,\cdots,k_r}$, as well as avoids other $\Lambda_{k'_1,\cdots,k'_{r'}}$ with dim $\Lambda_{k'_1,\cdots,k'_{r'}} \leq$ dim $\Lambda_{k_1,\cdots,k_r}$, by dimension count.

Therefore, we can perturb in stages until we finally obtain $(\widetilde{\omega}, \widetilde{g})$ close to (ω, g) with $\widetilde{\omega}(x)$ of generic type. This concludes the proof of Proposition 2.27. □

A consequence of Theorem 2.26 is the general principle that a harmonic 2-form, as regards generic transversality issues, behaves just like an ordinary closed 2-form, which, in turn, behaves like an ordinary 2-form with no differential condition. (See Martinet [**11**] for a study of generic *closed* forms.)

Corollary 2.31. *Fix a nonzero cohomology class $\alpha \in H^2(M; \mathbf{R})$. Then there exists a dense open subset of $Q'' = Met^l(M)$ on which the g-harmonic 2-form ω in the class α has no zeros.*

Remark 2.32. Any symplectic form ω on a closed, oriented, even-dimensional manifold M is *intrinsically harmonic*, i.e., there exists a metric g on M for which ω is harmonic. This suggests that we may be able to study generic harmonic two-forms as *degenerate symplectic forms*, and conversely symplectic geometry from the point of view of harmonic forms. The following question arises naturally: is it possible to use results on generic harmonic 2-forms to construct symplectic forms on M?

3. The Dirichlet problem. Using a setup similar to that for harmonic forms on a compact manifold, one can prove an analogous theorem for solutions to the Dirichlet problem. We refer the reader to [**5**] for standard facts on the Dirichlet problem. Let $\Omega \subset \mathbf{R}^n$ be a bounded domain with a smooth boundary $\partial\Omega$. The following is well-known:

Fact 3.1. *There exists a unique solution u to the Dirichlet problem $\Delta_g u = 0$, $u|_{\partial\Omega} = f$, where Δ_g is the Laplacian with respect to the C^l-metric g with $l \gg 0$, $l \notin \mathbf{Z}$, and f is a fixed C^l-function on the boundary.*

Let $\mathrm{Met}^l(\overline{\Omega})$ be the space of C^l-metrics on $\overline{\Omega}$, i.e., defined and C^l on some open set containing $\overline{\Omega}$. This is a Banach manifold, which we also view as $\{(u,g)|\Delta_g u = 0, u|_{\partial\Omega} = f\} \subset C^m(\overline{\Omega}) \times \mathrm{Met}^l(\overline{\Omega})$, $m >> 0$, $m \notin \mathbf{Z}$. We shall prove the following theorem:

Theorem 3.2. *Fix a C^l-function $f \not\equiv 0$ on $\partial\Omega$. Then there is a dense open set in $Q = Met^l(\overline{\Omega})$ for which the solution to the Dirichlet problem $\Delta_g u = 0$, $u|_{\partial\Omega} = f$, has regular zeros inside Ω.*

Remark 3.3. No claims are being made about the behavior of zeros as we approach $\partial\Omega$.

Proof. Consider the evaluation map:

$$ev : Q \times \Omega \to \mathbf{R}, \quad ((u,g),x) \mapsto u(x). \tag{60}$$

We will show that ev is regular, i.e., $ev_*((u,g),x)$ is surjective (nonzero), whenever $u(x) = 0$.

As before, computing $ev_*((u,g),x)$ is equivalent to differentiating the conditions $\Delta_{g_t} u_t = 0$, $u_t|_{\partial\Omega} = f$, where $g_t = g + th$ and $u_t = u + tv$, up to 1st order in t. Differentiating, we get

$$\frac{d}{dt}(\Delta_{g_t} u_t)\Big|_{t=0} = \Delta_g v + \frac{d}{dt}(\Delta_{g+th})\Big|_{t=0} u = 0, \tag{61}$$

and $v|_{\partial\Omega} = 0$.

Next, we convert Equation 61 into an integral involving the Green's function.

Fact 3.4. *If Ω is a bounded domain with C^∞-boundary $\partial\Omega$, then a Green's function $G : \overline{\Omega} \times \overline{\Omega} \to \mathbf{R}$ satisfying $G(\cdot, \partial\Omega) = G(\partial\Omega, \cdot) = 0$ exists.*

As before, we can write

$$v(x) = -\int_\Omega G(x,y)\ \frac{d}{dt}(\Delta_{g+th})\bigg|_{t=0} u(y)\,dv_g(y) \tag{62}$$

$$= \pm \int_\Omega G(x,y)(*d(D_h*))\,du(y)\,dv_g(y), \tag{63}$$

using $d*du = 0$. Here $D_h* = d/dt(*_{g+th})|_{t=0}$.

Pick geodesic normal coordinates on a small open disk U about $x = 0$. For perturbations h of g with support inside U, we have

$$v(0) = \pm \int_U G(0,y)(*d(D_h*))du(y)dv_g(y). \tag{64}$$

Fact 3.5. *$G(0,y)$ is asymptotic to*

$$F(y) = \begin{cases} c_n/|y|^{n-2}, & \text{if } n > 2; \\ c_n \log|y|, & \text{if } n=2, \end{cases} \tag{65}$$

as $y \to 0$. Here c_n is a nonzero constant. The derivatives of $G(0,y)$ are also asymptotic to the derivatives of $F(y)$ as $y \to 0$.

Now set $\omega(y) = du(y)$, and write $\omega = \sum_i \omega_i(t)e_i(t)$. We can choose an orthonormal basis $e_i(t) = e_i - (1/2)t\sum_j h_{ij}e_j$ on U for $g_t = g + th$, where $\{e_i\}$ is an orthonormal frame with respect to g. Then,

$$v(0) = \pm \int_U G(0,y)d(D_h*)du(y) \tag{66}$$

$$= \pm \left\{ \int_U d(G(0,y)(D_h*)du(y)) - \int_U dG(0,y) \wedge (D_h*)du(y) \right\} \tag{67}$$

$$= \pm \int_{\partial U} G(0,y)(D_h*)du(y) \pm \int_U \langle dG(0,y), *(D_h*)du(y)\rangle_g dv_g(y) \tag{68}$$

$$= \pm \int_U \langle dG(0,y), *(D_h*)du(y)\rangle_g dv_g(y), \tag{69}$$

since we are taking h with small support near x.

If du is identically 0 near y, then u is constant near y, and u must be constant on all of Ω by the Unique Continuation theorem. For u constant, Theorem 3.2 is trivially true. Therefore assume that u is not constant. Then there exist points $y_i \to 0$ such that $du(y_i) \neq 0$. When $du(y_i) \neq 0$, $i_{du(y_i)}$ is surjective, and, just as in the case of harmonic 1-forms, a limiting argument proves the surjectivity of $ev_*((u,g),x)$. This concludes the proof of Theorem 3.2. □

Acknowledgments. This work is part of the author's Ph.D. Thesis. He expresses his gratitude to his advisor Phillip Griffiths for support and encouragement, and Cliff Taubes for generously sharing his ideas. The author is also extremely grateful (and rather apologetic) to the anonymous referee who made 12 pages of comments!

Appendix

We give a proof of Proposition 2.1. Let Met^l be the set of C^l-Hölder metrics on M, $U \subset \mathrm{Met}^l$ be a small open set containing g_0, Ω_j^k be the set of C^j-Hölder k-forms on M, and $\mathcal{H}_{g_0}^k$ be the set of g_0-harmonic k-forms. Then consider the map

$$\begin{aligned} \Psi : \Omega_m^k \times U &\longrightarrow \Omega_{m-2}^k/\mathcal{H}_{g_0}^k, \\ (\omega, g) &\longmapsto [\Delta_g \omega]. \end{aligned} \tag{70}$$

If $\omega \in \Omega_m^k$, then $\Delta_g\omega \in \Omega_{m-2}^k$, provided $l >> m$, since $|\Delta_g\omega|_{m-2} \leq C|\omega|_m$. Here $|\cdot|_j$ is the Hölder j-norm and C is a constant. Since $\Delta_g\omega$ can be expressed in terms of derivatives of g and ω up to 2nd order, it follows that Ψ is smooth.

Claim. *Suppose $\Psi(\omega_0, g_0) = 0$. Then $D\Psi(\omega_0, g_0) : \Omega_m^k \times T_{g_0}Met^l \to \Omega_{m-2}^k/\mathcal{H}_{g_0}^k$ is surjective and has a bounded right inverse.*

Proof. $D\Psi(\omega_0, g_0)$ maps $(v, h) \mapsto \Delta_{g_0}v + d/dt\Delta_{g_0+th}|_{t=0}\omega_0$. We use the following Hölder estimate, which can be found, for example, in [**6**]:

$$|\widetilde{G}_{g_0}v|_m \leq C|v|_{m-2}, \tag{71}$$

where $v \in \Omega^k_{m-2}$ and $\widetilde{G}_{g_0}$ is the Green's operator for Δ_{g_0}. In particular, if v is L^2-orthogonal to $\mathcal{H}^k_{g_0}$, then $\Delta_{g_0} \circ \widetilde{G}_{g_0}(v) = v$. Since Equation 71 implies that $v \in \Omega^k_{m-2} \Rightarrow \widetilde{G}_{g_0} v \in \Omega^k_m$, we see that $D\Psi(\omega_0, g_0)$ is surjective. The Hölder estimate also immediately implies the existence of a bounded right inverse $L : \Omega^k_{m-2}/\mathcal{H}^k_{g_0} \to \Omega^k_m \times T_{g_0}\mathrm{Met}^l$ which maps $[v] \mapsto (\widetilde{G}_{g_0} v, 0)$. □

Using the Claim, we define:

$$
\begin{aligned}
\Pi : \Omega^k_m \times T_{g_0}\mathrm{Met}^l &\longrightarrow \mathcal{H}^k_{g_0} \times T_{g_0}\mathrm{Met}^l, \\
(\omega, g) &\mapsto (proj_{g_0}(\omega), g),
\end{aligned} \tag{72}
$$

where $proj_{g_0}$ is the L^2-projection onto $\mathcal{H}^k_{g_0}$. Π is the quotient map of $\Omega^k_m \times T_{g_0}\mathrm{Met}^l$ by the image of L. By identifying a small neighborhood of $g_0 \in \mathrm{Met}^l$ with a neighborhood of $0 \in T_{g_0}\mathrm{Met}^l$, we have:

$$
(\Psi, \Pi) : \Omega^k_m \times U \to (\Omega^k_{m-2}/\mathcal{H}^k_{g_0}) \times \mathcal{H}^k_{g_0} \times U. \tag{73}
$$

Now, $D(\Psi, \Pi)(\omega_0, g_0)$ is an isomorphism, and the Inverse Function theorem for Banach spaces (see [**9**]) implies that (Ψ, Π) is a local diffeomorphism near (ω_0, g_0). The zero set $Q|_{N(\omega_0,g_0)}$ of Ψ, restricted to an open neighborhood $N(\omega_0, g_0)$ of (ω_0, g_0), is therefore a Banach submanifold locally isomorphic to an open ball in $\mathcal{H}^k_{g_0} \times U$.

It remains to show that $Q|_{N(\omega_0,g_0)} = \{(\omega, g) \in N(\omega_0, g_0) | \Delta_g \omega = 0\}$. Clearly, $Q|_{N(\omega_0,g_0)} \supset \{(\omega, g) \in N(\omega_0, g_0) | \Delta_g \omega = 0\}$. The inclusion in the other direction follows from the fact that each slice of $Q|_{N(\omega_0,g_0)}$ with g held constant (g near g_0) has the same dimension as $\dim \mathcal{H}^k_g$, since $\dim \mathcal{H}^k_g$ is independent of g by Hodge theory.

ENDNOTE

1. Note that the action of $SO(n)$ on $\wedge^k \mathbf{R}^n$ and the metric g on M induce an action of $SO(n)$ on $\wedge^k T^*M$ and hence a stratification $\wedge^k T^*M = \sqcup_\alpha N_\alpha$ into submanifolds N_α related to the orbit type. These submanifolds N_α are fiber-subbundles of $\wedge^k T^*M$, and we can consider whether the section ω of T^*M is transverse to the fiber-subbundle N_α.

REFERENCES

1. N. Aronszajn, *A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of the second order*, J. Math. Pures Appl. **36** (1957), 235–249.

2. T. Aubin, *Nonlinear analysis on manifolds. Monge-Ampère equations*, Grundlehren Math. Wiss., vol. 252, Springer-Verlag, New York, 1982.

3. S.K. Donaldson, *Connections, cohomology, and the intersection forms of* 4*-manifolds*, J. Differential Geom. **24** (1986), 275–341.

4. S.K. Donaldson and P.B. Kronheimer, *The geometry of four-manifolds*, Oxford Math. Monographs, Oxford Univ. Press, New York, 1990.

5. G. Folland, *Introduction to partial differential equations*, Princeton Univ. Press, Princeton, NJ, 1976.

6. D. Fujiwara, *Elliptic partial differential operators on a manifold*, Appendix to K. Kodaira, *Complex manifolds and deformation of complex structures*, Grundlehren Math. Wiss., vol. 283, Springer-Verlag, New York, 1986.

7. K. Honda, *Local properties of self-dual harmonic* 2*-forms on a* 4*-manifold*, preprint 1997. `ArXiv: dg-ga/9705010`.

8. ———, *An openness theorem for harmonic* 2*-forms on* 4*-manifolds*, Illinois J. Math. **44** (2000), 479–495.

9. S. Lang, *Real and functional analysis*, 3rd ed., Graduate Texts in Math., vol. 142, Springer-Verlag, New York, 1993.

10. C. LeBrun, *Yamabe constants and the perturbed Seiberg-Witten equations*, Comm. Anal. Geom. **5** (1997), 535–553.

11. J. Martinet, *Sur les singularités des formes différentielles*, Ann. Inst. Fourier (Grenoble) **20** (1970), 95–178.

12. T. Parker and S. Rosenberg, *Invariants of conformal Laplacians*, J. Differential Geom. **25** (1987), 199–222.

13. V.K. Patodi, *Curvature and the eigenforms of the Laplace operator*, J. Differential Geom. **5** (1971), 233–249.

14. C.H. Taubes, *Self-dual connections on 4-manifolds with indefinite intersection matrix*, J. Differential Geom. **19** (1984), 517–560.

15. ———, *The Seiberg-Witten and Gromov invariants*, Math. Res. Lett. **2** (1995), 221–238.

16. ———, *The structure of pseudo-holomorphic subvarieties for a degenerate almost complex structure and symplectic form on* $S^1 \times B^3$, Geom. Topol. **2** (1998), 221–332 (electronic).

17. ———, *Seiberg-Witten invariants and pseudo-holomorphic subvarieties for self-dual, harmonic* 2*-forms*, Geom. Topol. **3** (1999), 167–210 (electronic).

18. ———, *Moduli spaces and Fredholm theory for pseudoholomorphic subvarieties associated to self-dual, harmonic* 2*-forms*, in *Sir Michael Atiyah: A great mathematician of the twentieth century*, Asian J. Math. **3** (1999), 275–324.

19. K. Uhlenbeck, *Generic properties of eigenfunctions*, Amer. J. Math. **98** (1976), 1059–1078.

University of Southern California, Los Angeles, CA 90089
E-mail address: khonda@math.usc.edu
URL: http://math.usc.edu/~khonda