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ON THE IMPROVEMENT OF LINEAR DISCRETE SYSTEM STABILITY: THE MAXIMAL SET OF THE F-ADMISSIBLE INITIAL STATES

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ABSTRACT. The linear controlled discrete system

$$\begin{cases} x(i+1) = Ax(i) + Bu(i) & i \geq 0, \\ x(0) = x_0, \end{cases}$$

$$u(i) = Fx(i), \quad i \geq 0,$$

$$y(i) = Cx(i), \quad i \geq 0,$$

is supposed to be output stabilizable, i.e., $\lim_{i\to+\infty} y(i) = 0$. To improve the stability of the system we propose in this paper a theoretical and algorithmic characterization of all the initial states x_0 for which $y(i) \in B(0, \alpha_i)$, for all $i \geq 0$, where $B(0, \alpha_i)$ is the ball of center 0 and radius α_i, the sequence $(\alpha_i)_i$ is appropriately chosen "$(\alpha_i)_i$ can be interpreted as a desired degree of stability." The case of discrete delayed systems is also considered.

1. Introduction. Nowadays it is not more to justify the important role of the stability and the stabilizability in the theory of systems. Many works that have been dedicated to this topic are very varied; we mention as examples [**1–3, 6, 13, 14, 19, 20, 22**] and [**23**]. The central problem in this work will be the output stabilization of the discrete linear system described by

$$(1) \qquad \begin{cases} x(i+1) = Ax(i) + Bu(i), & i \geq 0, \\ x(0) = x_0, \end{cases}$$

where $x(i) \in \mathbf{R}^n$, $u(i) \in \mathbf{R}^m$ are respectively the state variable and the control variable, while A, B are matrices of appropriate dimensions. The associated output is

$$(2) \qquad y(i) = Cx(i), \quad i \geq 0,$$

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where $C \in \mathcal{L}(\mathbf{R}^n, \mathbf{R}^p)$. More precisely, we suppose the existence of a feedback

$$u(i) = Fx(i), \quad i \geq 0, \quad F \in \mathcal{L}(\mathbf{R}^n, \mathbf{R}^m), \tag{3}$$

stabilizing the output of a system, that means

$$\lim_{i \to +\infty} y_i = 0, \quad \forall\, x_0 \in \mathbf{R}^n.$$

Given a real positive sequence $(\alpha_i)_{i \geq 0}$, $\alpha_i = (1/i), (1/i^2), e^{-i}, \dots,$ for a better stabilizability of the system it will be very interesting to construct feedback controls whose role is not only to stabilize the system but to achieve a predefined mode of stabilization. Motivated by the difficulty that presents the realization of such controls, we propose in this paper a partial answer to the question, i.e., we consider the control law $(u_i)_{i \geq 0}$ defined by (3) whose role is only to stabilize the system and we determine, under certain hypotheses, all initial states for whose

$$\|y(i)\| \leq \alpha_i, \quad i \geq 0.$$

That means we focus our interest to determine the set

$$\mathcal{S}(F) = \{x_0 \in \mathbf{R}^n / \|y(i)\| \leq \alpha_i, \quad i \geq 0\}.$$

Inspired by what was done in the area of the maximal set, see [**7, 15–17**], and [**26**], we call $\mathcal{S}(F)$ the maximal set of the F-admissible initial states, and we propose a theoretic and algorithmic characterization of the set $\mathcal{S}(F)$. The case of discrete delayed systems is also considered and some examples are given.

2. Preliminary results. Consider the linear controlled system described by the difference-equation

$$\begin{cases} x(i+1) = Ax(i) + Bu(i), & i \geq 0, \\ x(0) = x_0 \in \mathbf{R}^n; \end{cases} \tag{4}$$

the corresponding output is

$$y(i) = Cx(i), \quad i \geq 0, \tag{5}$$

where $x(i) \in \mathbf{R}^n$ is the state variable, $u(i) \in \mathbf{R}^m$ is the input variable and $y(i) \in \mathbf{R}^p$. A, B and C are constant matrices of appropriate dimensions.

We suppose that the system (4) is stabilizable by a state-feedback control law

$$u(i) = Fx(i), \quad \forall\, i \geq 0. \tag{6}$$

Note that the closed-loop system (4) and (6) is given by

$$\begin{cases} x(i+1) = \hat{A}x(i), \quad \forall\, i \geq 0, \\ \hat{A} = A + BF. \end{cases}$$

Let $(\alpha_i)_{i\geq 0}$ be a positive decreasing sequence which verifies

$$\frac{\alpha_i}{\alpha_{i+1}} \leq \frac{\alpha_{i-1}}{\alpha_i}, \quad \forall\, i \geq 1. \tag{7}$$

As examples of such sequences we cite

$$\alpha_i = \frac{1}{i+1}; \quad \alpha_i = \frac{1}{(i+1)^s}, \quad s \in [1, +\infty[\,; \quad \alpha_i = \rho^i, \quad \rho < 1.$$

In this paper, we propose, under some conditions on the matrix F, to determine the set of all initial states for which the resulting output function satisfies the pointwise-in-time conditions

$$y(i) \in B(0, \alpha_i), \quad \forall\, i \in \mathbf{N},$$

where $B(0, \alpha_i)$ is the ball with center 0 and radius α_i. More precisely we investigate the set

$$\mathcal{S}(F) = \{x_0 \in \mathbf{R}^n,\ y(i) \in B(0, \alpha_i),\ i \geq 0\}.$$

We call $\mathcal{S}(F)$ the maximal set of F-admissible initial states. Since $y(i) = C\hat{A}^i x_0$, this set can also be represented by

$$\mathcal{S}(F) = \{x_0 \in \mathbf{R}^n,\ C\hat{A}^i x_0 \in B(0, \alpha_i), \quad i \geq 0\}.$$

In the following proposition, we propose some properties of the set $\mathcal{S}(F)$.

Proposition 1. i) *For every* $F \in \mathcal{L}(\mathbf{R}^n, \mathbf{R}^m)$*,* $\mathcal{S}(F)$ *is a closed, convex and symmetric set.*

ii) *If we suppose that* $\limsup_{k\to+\infty} \|\hat{A}^k\|/\alpha_k < \varepsilon$*, where* $\varepsilon \geq 0$*, then* $0 \in \operatorname{int} \mathcal{S}(F)$*.*

Proof. The results in i) are easily checked from the definition of $\mathcal{S}(F)$. The assumption in ii) implies that there exists a constant $\gamma > 0$ such that, for all $x \in \mathbf{R}^n$ and $i \in \mathbf{N}$, $\|C\hat{A}^i x\| \leq \gamma\alpha_i \|x\|$. Then, $x \in B(0, (1/\gamma))$ implies $C\hat{A}^i x \in B(0, \alpha_i)$ for all $i \in \mathbf{N}$. Hence $B(0, (1/\gamma)) \in \mathcal{S}(F)$ and consequently $0 \in \operatorname{int} \mathcal{S}(F)$. □

Remark 1. The condition $\limsup_{k\to+\infty} \|\hat{A}^k\|/\alpha_k < \varepsilon$ in the previous proposition is equivalent to $\|C\hat{A}^i\| \leq \beta\alpha_i$, for all $i \geq 0$, where β is a positive constant.

3. Algorithmic characterization of the set $\mathcal{S}(F)$. In order to give a simple structure of the set $\mathcal{S}(F)$, we propose in this section sufficient conditions on F which allow us to characterize the set $\mathcal{S}(F)$ by a finite number of inequations. For this, let $\mathcal{S}_k(F)$ be the family of sets defined by

$$\mathcal{S}_k(F) = \{x_0 \in \mathbf{R}^n,\ C\hat{A}^i x_0 \in B(0, \alpha_i),\ 0 \leq i \leq k\}.$$

We have the following result

Proposition 2. *Suppose that* $\mathcal{S}_k(F) = \mathcal{S}_{k+1}(F)$ *for some integer* k*. Then the set* $\mathcal{S}(F)$ *given by* (8) *is described by a finite number of equations; more precisely, we have* $\mathcal{S}(F) = \mathcal{S}_k(F)$*. Conversely, if* $\mathcal{S}(F) = \mathcal{S}_k(F)$ *for some integer* k*, then* $\mathcal{S}_k(F) = \mathcal{S}_{k+1}(F) = \mathcal{S}_j(F)$*, for all* $j \geq k$*.*

Proof. Suppose the existence of an integer k such that $\mathcal{S}_k(F) = \mathcal{S}_{k+1}(F)$. Then $x \in \mathcal{S}_k(F)$ implies that

$$C\hat{A}^{k+1} x \in B(0, \alpha_{k+1});$$

thus,

$$C\hat{A}^k \left(\frac{\alpha_k}{\alpha_{k+1}} \hat{A}x \right) \in B(0, \alpha_k) \tag{9}$$

and, for $i \in \{0, \ldots, k-1\}$, we have

$$C\hat{A}^i\left(\frac{\alpha_k}{\alpha_{k+1}}\hat{A}x\right) = \frac{\alpha_k}{\alpha_{k+1}}C\hat{A}^{i+1}x \in B\left(0, \frac{\alpha_k \alpha_{i+1}}{\alpha_{k+1}}\right).$$

Since $(\alpha_j)_{j\geq 0}$ verifies $(\alpha_j/\alpha_{j+1}) \leq (\alpha_{j-1}/\alpha_j)$, for all $j \geq 1$, then

$$\frac{\alpha_k}{\alpha_{k+1}} \leq \frac{\alpha_i}{\alpha_{i+1}}, \quad \forall\, i \in \{0, \ldots, k-1\},$$

which implies that

$$C\hat{A}^i\left(\frac{\alpha_k}{\alpha_{k+1}}\hat{A}x\right) \in B(0, \alpha_i), \quad \forall\, i \in \{0, \ldots, k-1\}. \tag{10}$$

Consequently, from (9) and (10) we deduce that

$$\frac{\alpha_k}{\alpha_{k+1}}\hat{A}x \in \mathcal{S}_k(F)$$

and, by iteration, $(\alpha_k/\alpha_{k+1})^j \hat{A}^j x \in \mathcal{S}_k(F)$, for all $j \geq 0$, then

$$C\hat{A}^{i+j}x \in B\left(0, \frac{\alpha_{k+1}^j \alpha_i}{\alpha_k^j}\right), \quad \forall\, i \in \{0, \ldots, k\}, \quad \forall\, j \geq 0.$$

So, for $i = k$, we have

$$C\hat{A}^{k+j}x \in B\left(0, \frac{\alpha_{k+1}^j}{\alpha_k^{j-1}}\right), \quad \forall\, j \geq 1,$$

as $(\alpha_i)_{i\geq 0}$ verify (7), then we easily establish that

$$\frac{\alpha_{k+1}^j}{\alpha_k^{j-1}} \leq \alpha_{k+j}, \quad \forall\, j \geq 1;$$

thus,

$$C\hat{A}^{k+j}x \in B(0, \alpha_{k+j}), \quad \forall\, j \geq 1.$$

Therefore $x \in \mathcal{S}(F)$, hence $\mathcal{S}_k(F) \subset \mathcal{S}(F)$. But $\mathcal{S}(F)$ is a subset of $\mathcal{S}_k(F)$, consequently $\mathcal{S}(F) = \mathcal{S}_k(F)$. Conversely, if $\mathcal{S}_k(F) = \mathcal{S}(F)$ for some integer k, then we deduce that $\mathcal{S}_k(F) \subset \mathcal{S}_{k+1}(F)$ which implies that $\mathcal{S}_k(F) = \mathcal{S}_{k+1}(F)$ (because $\mathcal{S}(F) \subset \mathcal{S}_{j_1}(F) \subset \mathcal{S}_{j_2}(F)$, $j_1 \geq j_2$). $\square$

In order to determine the smallest integer k^*, if there exists, such that $\mathcal{S}(F) = \mathcal{S}_k^*(F)$, we suggest an algorithm stated as follows:

Let $\mathbf{R}^p$ be endowed with the following norm

$$\|x\| = \max_{1\leq i\leq p} |x_i| \quad \forall\, x = (x_1, \dots, x_p) \in \mathbf{R}^p.$$

The set $\mathcal{S}_k(F)$ is then described as follows

$$\mathcal{S}_k(F) = \Big\{x \in \mathbf{R}^n / h_j\Big(\frac{1}{\alpha_i} C\hat{A}^i x\Big) \leq 0 \quad \text{for} \quad j = 1, 2, \dots, 2p \quad \text{and} \quad i = 0, 1, \dots, k\Big\},$$

where $h_j : \mathbf{R}^p \to \mathbf{R}$, are defined for every $x = (x_1, \dots, x_p) \in \mathbf{R}^p$ by

$$h_{2m-1}(x) = x_m - 1, \quad \text{for } m \in \{1, 2, \dots p\},$$
$$h_{2m}(x) = -x_m - 1, \quad \text{for } m \in \{1, 2, \dots p\}.$$

It follows from $\mathcal{S}_{k+1}(F) \subset \mathcal{S}_k(F)$, for all $k \geq 0$, that

$$\mathcal{S}_{k+1}(F) = \mathcal{S}_k(F) \Longleftrightarrow \mathcal{S}_k(F) \subset \mathcal{S}_{k+1}(F),$$

so

$$\mathcal{S}_{k+1}(F) = \mathcal{S}_k(F) \Longleftrightarrow \forall\, x \in \mathcal{S}_k(F),$$
$$h_j\Big(\frac{1}{\alpha_{k+1}} C\hat{A}^{k+1} x\Big) \leq 0, \quad \forall\, j \in \{1, 2, \dots, 2p\},$$

or equivalently,

$$\sup_{x\in\mathcal{S}_k(F)} h_j\Big(\frac{1}{\alpha_{k+1}} C\hat{A}^{k+1} x\Big) \leq 0, \quad \forall\, j \in \{1, 2, \dots, 2p\}.$$

Finally, we deduce the algorithm

Algorithm.

Step 1 : Let $k = 0$;

Step 2 : For $i = 1, \dots, 2p$, do:

$$\text{Maximize } J_i(x) = h_i[(1/\alpha_{k+1}) C\hat{A}^{k+1} x]$$

$$\begin{cases} h_i[(1/\alpha_l) C\hat{A}^l x] \le 0, & \\ i = 1, \dots, 2p & l = 0, \dots, k. \end{cases}$$

Let J_i^* be the maximum value of $J_i(x)$.

If $J_i^* \le 0$, for $i = 1, \dots, 2p$ then set $k^* := k$ and stop.

Else continue.

Step 3 : Replace k by $k+1$ and return to Step 2.

The optimization problem cited in Step 2 is a mathematical programming problem and can be solved by standard methods.

4. Conditions for finite characterization of $\mathcal{S}(F)$. It is clear that the above algorithm converges if and only if there exists an integer k such that $\mathcal{S}_{k+1}(F) = \mathcal{S}_k(F)$. So it is desirable to establish simple conditions which allows us to characterize the set $\mathcal{S}(F)$ by a finite number of equations. Our main result in this direction is the following

Theorem 1. *Suppose the following assumptions hold*

i) *the pair $(C, \hat{A})$ is observable, i.e., $[C^\top | \hat{A}^\top C^\top | \cdots | (\hat{A}^\top)^{n-1} C^\top]$ has rank n.*

ii) $\limsup_{k \to +\infty} \|\hat{A}^k\| / \alpha_k < \lambda_0 / (\|C\| \|H\| \alpha_0)$, *where*

$$\lambda_0 = \inf_{\lambda \in \sigma(H^\top H)} \lambda \quad \text{and} \quad H = \begin{bmatrix} C \\ C\hat{A} \\ \vdots \\ C\hat{A}^{n-1} \end{bmatrix}.$$

Then there exists an integer k such that $\mathcal{S}(F) = \mathcal{S}_k(F)$.

Proof. It follows from the definition of $\mathcal{S}_{n-1}(F)$ that

$$x \in \mathcal{S}_{n-1}(F) \Longrightarrow Hx \in \overbrace{\mathcal{B}(0, \alpha_0) \times \cdots \times \mathcal{B}(0, \alpha_{n-1})}^{n\text{-times}}. \tag{11}$$

On the other hand the observability of $(C, \hat{A})$ implies that the matrix $H^\top H$ is invertible, consequently there exists a constant $c = \inf_{\lambda \in \sigma(H^\top H)} \lambda > 0$ such that

$$c\|x\|^2 \leq \langle H^\top Hx, x\rangle, \quad \forall\, x \in \mathbf{R}^n, \quad \text{see [24]},$$

which implies that

$$c\|x\|^2 \leq \|H^\top\|, \|Hx\|\,\|x\|, \quad \forall\, x \in \mathbf{R}^n,$$

then it follows from (11) that

$$c\|x\|^2 \leq \alpha_0 \|H^\top\|\,\|x\|, \quad \forall\, x \in \mathcal{S}_{n-1}(F)$$

(because $\|Hx\| \leq \max_{0 \leq i \leq n-1}(\alpha_i) = \alpha_0$ and $\|x\| = \max_{1 \leq i \leq p}|x_i|$, $\forall\, x \in \mathbf{R}^p$). So

$$\|x\| \leq \gamma = \frac{\|H^\top\|\alpha_0}{c}, \quad \forall\, x \in \mathcal{S}_{n-1}(F).$$

Hence

$$\mathcal{S}_{n-1}(F) \subset \mathcal{B}(0, \gamma) = \{x \in \mathbf{R}^n / \|x\| \leq \gamma\}.$$

The fact that $\limsup_{k \to +\infty} \|\hat{A}^k\|/\alpha_k = \varepsilon$ implies that

$$\forall\, \beta > 0 \quad \exists k_0 \quad \forall\, k \geq k_0 : \sup_{i \geq k} \frac{\|\hat{A}^i\|}{\alpha_i} \leq \beta + \varepsilon,$$

then for $\beta = 1/\gamma\|C\| - \varepsilon > 0$ there exists an integer $k_0 \geq n-1$ such that

$$\|C\hat{A}^{k_0+1}\| \leq \frac{\alpha_{k_0+1}}{\gamma}.$$

For every $x \in \mathcal{S}_{k_0}(F)$ we have

$$\|C\hat{A}^{k_0+1}x\| \leq \|C\hat{A}^{k_0+1}\|\|x\|,$$

but $\mathcal{S}_{k_0}(F) \subset \mathcal{S}_{n-1}(F) \subset B(0, \gamma)$, so we deduce that

$$\|C\hat{A}^{k_0+1}x\| \leq \alpha_{k_0+1}, \quad \forall\, x \in \mathcal{S}_{k_0}(F);$$

consequently, $C\hat{A}^{k_0+1}x \in B(0, \alpha_{k_0+1})$, for all $x \in \mathcal{S}_{k_0}(F)$. Thus, $\mathcal{S}_{k_0}(F) \subset \mathcal{S}_{k_0+1}(F)$, which implies that $\mathcal{S}_{k_0}(F) = \mathcal{S}_{k_0+1}(F) = \mathcal{S}(F)$. □

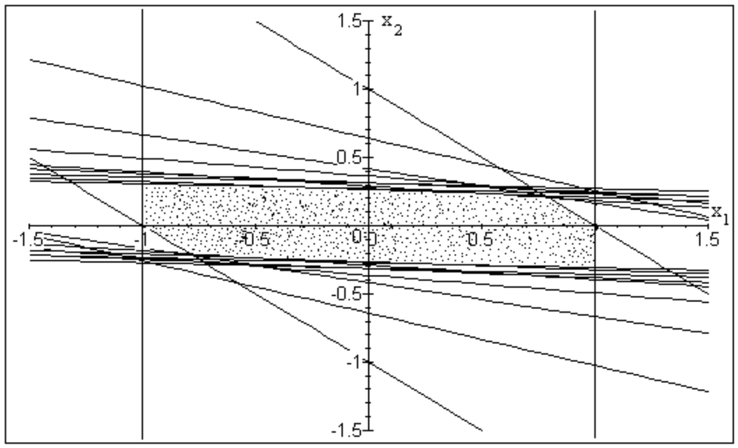

Figure 1. The dotted region is the set $\mathcal{S}(F)$ corresponding to Example 1.

5. Examples.

Example 1. Consider systems (4) and (5) with data

$$A = \begin{pmatrix} 0.8 & -1 \\ 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} -4 & 0.5 \\ 0.6 & -0.2 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}.$$

Let (α_i) be the sequence defined by $\alpha_i = 1/2^i$ and consider the feedback $u(i) = Fx(i)$ where

$$F = \begin{pmatrix} 0.2 & 0.7 \\ 0.6 & 10. \end{pmatrix}$$

Using the algorithm defined in Section II.3, we establish $k^* = 4$. Figure 1 gives the representation of the maximal set of F-admissible initial states $\mathcal{S}(F)$.

Example 2. For

$$A = \begin{pmatrix} 1.4 & -0.6 \\ -2.2 & 1.5 \end{pmatrix}, \quad B = \begin{pmatrix} -1 \\ 2 \end{pmatrix},$$
$$C = (-1,\, 0.2), \quad u(i) = (1,\, -0.5)x(i)$$

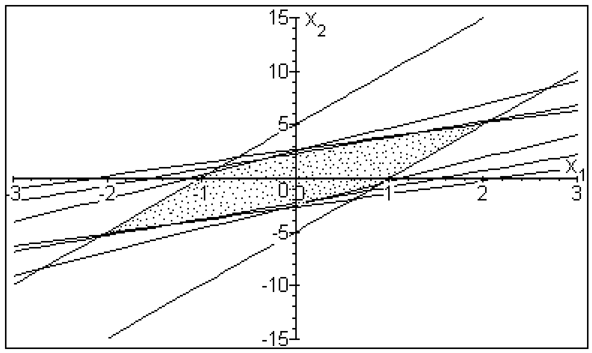

Figure 2. The dotted region is the set $\mathcal{S}(F)$ corresponding to Example 2.

and $\alpha_i = 1/(i+1)$, we use the algorithm of Section II.3 to establish that $k^* = 3$. Figure 2 gives a representation of the maximal set of F-admissible initial states $\mathcal{S}(F)$ corresponding to Example 2.

6. Discrete-time controlled delayed system. In this section we consider the discrete controlled delayed system given by

$$(12) \qquad \begin{cases} x(i+1) = \sum_{j=0}^{d} A_j x(i-j) + Bu(i) & i \geq 0, \\ x(0) = x_0, \\ x(k) = \theta_k \quad \text{for } k \in \{-d, -d+1, \dots, -1\}, \end{cases}$$

the corresponding delayed output function is

$$(13) \qquad y(i) = \sum_{j=0}^{t} C_j x(i-j), \quad i \geq 0,$$

where $A_j \in \mathcal{L}(\mathbf{R}^n)$, $B \in \mathcal{L}(\mathbf{R}^m, \mathbf{R}^n)$ and $C_j \in \mathcal{L}(\mathbf{R}^n, \mathbf{R}^p)$. d and t are integers such that $t \leq d$. $x(i) \in \mathbf{R}^n$ is the state variable, $u(i) \in \mathbf{R}^m$ the input variable and the observation $y(i) \in \mathbf{R}^p$ satisfies the output constraints

$$(14) \qquad y(i) \in B(0, \alpha_i), \quad \forall\, i \in \mathbf{N},$$

where $(\alpha_i)_{i\geq 0}$ is a positive decreasing sequence which verifies equation (7).

We suppose that the system (15) is stabilizable by a state-feedback control law

$$u(i) = \mathcal{F}(x(i), x(i-1), \dots, x(i-d)), \quad \forall\, i \in \mathbf{N}.$$

The initial condition $(x_0, \theta_{-1}, \dots, \theta_{-d+1}, \theta_{-d}) \in \mathbf{R}^{n(d+1)}$ is said to be an $\mathcal{F}$-admissible condition if the resulting output function (13) satisfies the constraints (14). The set of all such initial conditions is the maximal set of $\mathcal{F}$-admissible initial states $\mathcal{T}(\mathcal{F})$. In order to characterize the set $\mathcal{T}(\mathcal{F})$, we define the new state variable $e(i) \in \mathbf{R}^{n(d+1)}$ for $i \in \mathbf{N}$ by

$$e(i) = (x(i), x(i-1), \dots, x(i-d))^{\top}, \quad i \geq 0$$

and the matrices $\mathcal{A} \in \mathcal{L}(\mathbf{R}^{n(d+1)})$ and $\mathcal{B} \in \mathcal{L}(\mathbf{R}^{n}, \mathbf{R}^{n(d+1)})$ by

$$\mathcal{A} = \begin{pmatrix} A_0 & A_1 & \cdots & \cdots & A_d \\ I_n & 0_n & \cdots & \cdots & 0_n \\ 0_n & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0_n & \cdots & 0_n & I_n & 0_n \end{pmatrix} \quad \text{and} \quad \mathcal{B} = \begin{pmatrix} B \\ 0_{n\times m} \\ \vdots \\ \vdots \\ 0_{n\times m} \end{pmatrix},$$

where I_n is the $n\times n$-unit matrix, O_n is the $n\times n$-zero matrix and $0_{n\times m}$ is the $n \times m$-zero matrix. Then the system (12) can be equivalently rewritten in the form

$$\begin{cases} e(i+1) = \mathcal{A}e(i) + \mathcal{B}u(i), \quad i \geq 0, \\ e(0) = (x_0, \theta_{-1}, \dots, \theta_{-d+1}, \theta_{-d}). \end{cases} \tag{15}$$

If we define the $p \times [n(d+1)]$-matrix C by

$$\mathcal{C} = (C_0|C_1|\cdots|C_t|\underbrace{O_{p\times n}|\cdots|O_{p\times n}}_{d-t\text{-times}}) \in \mathcal{L}(\mathbf{R}^{n(d+1)}, \mathbf{R}^{p}).$$

The observation (13) can be expressed as follows

$$y(i) = \mathcal{C}e(i) = \mathcal{C}(\mathcal{A} + \mathcal{B}\mathcal{F})^{i} e(0), \quad \text{for all } i \in \mathbf{N}. \tag{16}$$

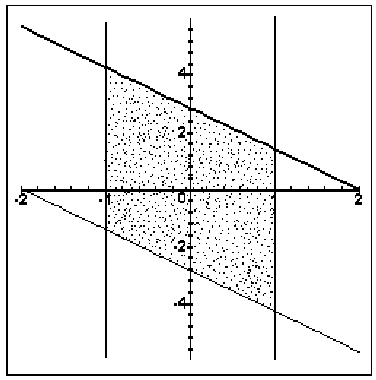

FIGURE 3. The representation of the maximal set of $\mathcal{F}$-admissible initial states $\mathcal{T}(\mathcal{F})$ corresponding to Example 3.

Thus, the set of all $\mathcal{F}$-admissible initial states is formally given by

$$\mathcal{T}(\mathcal{F}) = \{e \in \mathbf{R}^{n(d+1)} / \mathcal{C}(\mathcal{A} + \mathcal{B}\mathcal{F})^i e \in B(0, \alpha_i),\ \forall\, i \in \mathbf{N}\}.$$

So it is obvious that Theorem 1 gives sufficient conditions to characterize the set $\mathcal{T}(\mathcal{F})$ by a finite number of functional inequalities.

Example 3. Consider the discrete-time delayed system described by

$$\begin{cases} x(i+1) = 1.2x(i) - 0.5x(i-1) + u(i), & i \geq 0, \\ x(0) = x_0, \\ x(-1) = \theta_{-1}. \end{cases}$$

The corresponding output is

$$y(i) = x(i), \quad i \geq 0.$$

If we take $\alpha_i = (4/5)^i$ and $u(i) = \mathcal{F}(x(i), x(i-1))^\top$ where $\mathcal{F} = \left(\begin{smallmatrix} -0.8 & 0.78 \\ 0 & 0.08 \end{smallmatrix}\right)$, then we use the algorithm described in Section II.3 to establish that $k^* = 1$. Figure 3 gives the representation of the maximal set of $\mathcal{F}$-admissible initial states $\mathcal{T}(\mathcal{F})$.

6.1 Other sufficient conditions for discrete delayed systems. We assume that $(\alpha_i)_{i \geq 0}$ is a positive decreasing sequence which verifies equation (7); the linear control law is a delayed feedback control given by

$$u(i) = \sum_{j=0}^{d} F_j x(i-j), \quad i \geq 0, \tag{17}$$

where $F_j \in \mathcal{L}(\mathbf{R}^n, \mathbf{R}^m)$, for all $j \in \{0, \dots, d\}$, and we establish other sufficient conditions to characterize the maximal set $\mathcal{T}(\mathcal{F})$ of all admissible states $(x_0, \theta_{-1}, \dots, \theta_{-d+1}, \theta_{-d})$ for which the resulting output function (13) satisfies the constraints (14) and which are adapted to delayed discrete systems. This set can be formally rewritten by

$$\mathcal{T}(\mathcal{F}) = \{(x_0, \theta_{-1}, \dots, \theta_{-d+1}, \theta_{-d}) \in \mathbf{R}^{n(d+1)} / y(i) \in B(0, \alpha_i), \\ \forall\, i \in \mathbf{N}\}.$$

To solve the problem, we consider the two following situations

a) In the first case we suppose that $p = n$, i.e., the observation space and the state space have the same dimension.

b) In the second case we suppose that $p < n$.

First case, $p = n$**.** In this case every C_i is an $n \times n$ matrix

Proposition 3. *Suppose the following assumptions hold*

i) C_i *commute with* $A_j + BF_j$ *for every* $i \in \{0, \dots, t\}$ *and for every* $k \in \{0, \dots, d\}$*.*

ii) $\|\sum_{i=0}^{d} (A_i + BF_i) z_i\| \leq \alpha_{d+1}$ *for every* $z_i \in B(0, \alpha_{d-i})$*.*

Then

$$\mathcal{T}(\mathcal{F}) = \{(x_0, \theta_{-1}, \dots, \theta_{-d}) \in \mathbf{R}^{n(d+1)} / y(i) \in B(0, \alpha_i),\ \forall\, i \in \{0, \dots, d\}\},$$

where d *is the number of delays in the state variable of system* (12)*.*

Proof. If every C_i commutes with every $A_j + BF_j$ for every $i \in \{0, \dots, t\}$ and for every $k \in \{0, \dots, d\}$, then it follows for every $i \geq d+1$

that

$$\begin{aligned} y(i) &= \sum_{k=0}^{t} C_k x_{i-k} = \sum_{k=0}^{t} C_k \sum_{j=0}^{d} (A_j + BF_j) x_{i-k-j-1} \\ &= \sum_{j=0}^{d} (A_j + BF_j) \sum_{k=0}^{t} C_k x_{i-k-j-1} \\ &= \sum_{j=0}^{d} (A_j + BF_j) y(i-j-1). \end{aligned}$$

Now, let $e \in \mathcal{T}_d(\mathcal{F}) = \{(x_0, \theta_{-1}, \dots, \theta_{-d}) \in \mathbf{R}^{n(d+1)} / y(i) \in B(0, \alpha_i)$, for all $i \in \{0, \dots, d\}\}$ then

$$y(i) \in B(0, \alpha_i), \quad \forall\, i \in \{0, \dots, d\}.$$

From hypothesis ii) of Proposition 3, it follows that

$$y(d+1) = \sum_{j=0}^{d} (A_j + BF_j) y(d-j) \in B(0, \alpha_{d+1}),$$

which implies that $e \in \mathcal{T}_{d+1}(\mathcal{F})$ where

$$\mathcal{T}_{d+1}(\mathcal{F}) = \{(x_0, \theta_{-1}, \dots, \theta_{-d}) \in \mathbf{R}^{n(d+1)} / \|y(i)\| \alpha_i, \forall\, i \in \{0, \dots, d+1\}\},$$

hence

$$\mathcal{T}_d(\mathcal{F}) \subset \mathcal{T}_{d+1}(\mathcal{F}).$$

Consequently, from Proposition 2, we deduce that

$$\mathcal{T}_d(\mathcal{F}) = \mathcal{T}_{d+1}(\mathcal{F}) = \mathcal{T}(\mathcal{F}). \qquad \square$$

Second case, $p < n$. Since every C_i is a $p \times n$ matrix, we define $\hat{C}_i$ by

$$\hat{C}_i = \begin{pmatrix} C_i \\ 0 \end{pmatrix},$$

if we introduce the new observation $\hat{y}(i)$ by

$$\hat{y}(i) = \begin{pmatrix} y(i) \\ 0 \end{pmatrix},$$

then clearly we have $\hat{y}(i) = \sum_{k=0}^{t} \hat{C}_k x_{i-k}$ and we easily verify that

$$\|\hat{y}(i)\| \le \alpha_i \Longleftrightarrow \|y(i)\| \le \alpha_i.$$

So the set $\mathcal{T}(\mathcal{F})$ is given by

$$\mathcal{T}(\mathcal{F}) = \{(x_0, \theta_{-1}, \dots, \theta_{-d+1}, \theta_{-d}) \in \mathbf{R}^{n(d+1)} / \hat{y}(i) \in B(0, \alpha_i), \quad \forall\, i \in \mathbf{N}\},$$

since $\hat{C}_i$ are $n \times n$-matrices, then results of case a) can be applied to deduce the following proposition.

Proposition 4. *Suppose the following hypotheses hold*:

i) $\hat{C}_i$ *commute with* $A_j + BF_j$ *for every* $i \in \{0, \dots, t\}$ *and for every* $k \in \{0, \dots, d\}$.

ii) $\|\sum_{i=0}^{d} (A_i + BF_i) z_i\| \le \alpha_{d+1}$ *for every* $z_i \in B(0, \alpha_{d-i})$.

Then $\mathcal{T}(\mathcal{F})$ *is characterized by a finite number of equations*; *more precisely, we have*

$$\mathcal{T}(\mathcal{F}) = \{(x_0, \theta_{-1}, \dots, \theta_{-d}) \in \mathbf{R}^{n(d+1)} / y(i) \in B(0, \alpha_i), \quad \forall\, i \in \{0, \dots, d\}\}.$$

Example 4. Consider the system

$$\begin{cases} x(i+1) = \sum_{j=0}^{d} A_j x(i-j) + Bu(i) \quad i \ge 0, \\ x(0) = x_0 \text{ is given}, \\ x(k) = \theta_k, \quad -d \le k \le -1, \end{cases} \tag{18}$$

with the output function

$$y(i) = x(i), \quad i \ge 0.$$

We suppose that the input is a delayed feedback control $u(i) = \sum_{j=0}^{d} F_j x(i-j)$ and the output satisfies the constraints

$$\|y(i)\| \le \alpha_i, \quad i \ge 0,$$

where $(\alpha_i)_{i\geq 0}$ is a positive decreasing sequence which verifies equation (7). Then we have the following result

Proposition 5. *If* $\sum_{j=0}^{d} \|A_i + BF_i\|^2 \leq (\alpha_{d+1}^2 / \sum_{j=0}^{d} \alpha_i^2)$, *then the maximal set of* $\mathcal{F}$*-admissible initial states* $\mathcal{T}(\mathcal{F})$ *is characterized by* $\mathcal{T}(\mathcal{F}) = \{(x_0, \theta_{-1}, \dots, \theta_{-d}) \in \mathbf{R}^{n(d+1)} / \|y(i)\| \leq \alpha_i$, *for all* $i \in \{0, \dots, d\}\}$.

Proof. We show that the hypotheses of Proposition 3 are verified. Indeed, for every $z_i \in B(0, \alpha_{d-i})$, $i = 0, \dots, d$, we have

$$\begin{aligned} \|\sum_{j=0}^{d} \|(A_i + BF_i) z_i\| &\leq \Big(\|\sum_{j=0}^{d} \|A_i + BF_i\|^2\Big)^{1/2} \Big(\|\sum_{j=0}^{d} \|z_i\|^2\Big)^{1/2} \\ &\leq \Big(\|\sum_{j=0}^{d} \|A_i + BF_i\|^2|\Big)^{1/2} \Big(\|\sum_{j=0}^{d} \alpha_i^2\Big)^{1/2} \\ &\leq \alpha_{d+1}. \qquad \square \end{aligned}$$

7. Conclusion. In this paper, we consider a discrete system output-stabilizable by a feedback $(u_i)_i$, and we focus our interest in this work to characterize the set of all initial states for which the output function satisfies the constraints $\|y(i)\| \leq \alpha_i$, for all $i \geq 0$, where α_i is appropriately chosen. An efficient algorithm for constructing the set of such initial states is given and numerical simulations have been done for some examples. The case of controlled discrete time-delayed systems has also been investigated.

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