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PERTURBATIONS OF p-ADIC LINEAR OPERATORS

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ABSTRACT. In this paper we give the perturbation theory for p-adic continuous linear operators. In particular, we deal with the gap between ranges of linear operators, the gap between kernels of linear operators and the gap of the solution sets of linear operator equations.

1. Introduction. The problems of perturbations of p-adic linear operators have been studied by many authors. Many of them dealt with the perturbations of the index of linear operators, cf. [**1, 8, 9**]. In this paper we deal with the perturbation of the gaps between the closed convex subsets which are defined by using the continuous linear operators with a generalized inverse.

Let E and F be non-Archimedean Banach spaces, let T and A be continuous linear operators from E to F, and let $R(T)$ be closed. Let b and $\bar{b}$ be fixed elements of $R(T)$ and $R(T+A)$, respectively, and set $X(T,b) = \{x \in E : Tx = b\}$. If T has a generalized inverse S, then under some conditions we show that the gap between $R(T)$ and $\overline{R(T+A)}$ is estimated by $\|S\|\,\|A\|$, the gap between $\operatorname{Ker}(T)$ and $\operatorname{Ker}(T+A)$ is estimated by $\|SA\|$ and the gap between $X(T,b)$ and $X(T+A,\bar{b})$ is also estimated by $\|SA\|$.

2. Preliminaries. Throughout, K is a non-Archimedean valued field that is complete under the metric induced by the nontrivial valuation $|\cdot|$ and E, F are Banach spaces over K. Let $L(E,F)$ denote the set of all continuous linear operators from E to F. For $B \in L(E,F)$, $R(B)$ and $\operatorname{Ker}(B)$ are the range and the kernel of B, respectively. If M is a linear subspace of E, $B \mid M$ is the restriction of B to M. The identity map on E is denoted by I_E. A subset V of E is said to be convex if, for every $x, y, z \in V$ and for every $\alpha, \beta, \gamma \in K$ with

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$\alpha+\beta+\gamma=1$, $\alpha x+\beta y+\gamma z\in V$. The set of closed convex subsets of E is denoted by $C(E)$. A closed linear subspace X of E is said to be topologically complemented in E if there exists a closed linear subspace Y such that $E=X\oplus Y$. In this case it is known that there exists a real number t, $0<t\leqq 1$, such that X and Y are t-orthogonal (see [**9**]). For more basic facts on non-Archimedean Banach spaces, we refer to [**9**].

3. Basic facts on the generalized inverse of a linear operator.

Let $L(E,F)$ be the set of all continuous linear operators from E to F and put $L(E)=L(E,E)$. Let T be a given element in $L(E,F)$.

Throughout this paper for all $U\in L(E,F)$, we take the following norm:

$$\|U\|=\sup\left\{\frac{\|Ux\|}{\|x\|}:x\in E,x\neq 0\right\}.$$

Definition. If there exists a linear operator $S\in L(F,E)$ such that $TST=T$, then S is said to be a pseudoinverse of T. In addition, if the pseudoinverse S of T satisfies the condition $STS=S$, then S is called a generalized inverse of T (see [**6**]).

In this paper we assume that $R(T)$ is closed. For the pseudoinverse linear operator, the following proposition is seen in [**1**].

Proposition 1. *The following statements are equivalent.*

(1) *There exist linear projections $P\in L(E)$ and $Q\in L(F)$ such that*

$$R(P)=\mathrm{Ker}\,(T),\qquad R(Q)=R(T);$$

(2) *There exist closed subspaces $W\subset E$ and $Z\subset F$ such that*

$$E=\mathrm{Ker}\,(T)\oplus W,\qquad F=Z\oplus R(T);$$

(3) *T has a pseudoinverse.*

Suppose that $R(T)$ and $\mathrm{Ker}\,(T)$ are complemented, and set $E=\mathrm{Ker}\,(T)\oplus W$ and $F=R(T)\oplus Z$. Then $T|W:W\to R(T)$ is a homeomorphism. For any $y \in F$ there exist the unique elements $w_y \in W$ and $z_y \in Z$ such that $y = T(w_y) + z_y$. Define a linear operator T^+ from F to E by $T^+(y) = w_y$. We recall that there exists a real number t, $0 < t \leqq 1$, such that $R(T)$ and Z are t-orthogonal (see [**9**]).

Lemma 2. *Suppose that $R(T)$ and Z are t-orthogonal. Then it holds that*

$$\|(T|_W)^{-1}\| \leqq \|T^+\| \leqq \frac{1}{t}\|(T|_W)^{-1}\|.$$

Proof. For any $y \in F$, let $w_y \in W$ and $z_y \in Z$ be as in the above discussion. Then it follows that

$$\begin{aligned}\sup_{w\in W\setminus\{0\}} \frac{\|w\|}{\|T(w)\|} &= \sup_{w\in W\setminus\{0\}} \frac{\|T^+T(w)\|}{\|T(w)\|}\\ &\leqq \sup_{y\in F\setminus\{0\}} \frac{\|T^+(y)\|}{\|y\|}\\ &\leqq \sup_{y\in F\setminus\{0\}} \frac{\|w_y\|}{t\max(\|T(w_y)\|, \|z_y\|)}\\ &\leqq \sup_{y\in F\setminus\{0\}} \frac{\|w_y\|}{t\|T(w_y)\|}\\ &\leqq \frac{1}{t}\sup_{w\in W\setminus\{0\}} \frac{\|w\|}{\|T(w)\|}. \qquad \square\end{aligned}$$

It is easy to see that $TT^+T = T$ and $T^+TT^+ = T^+$. Hence, we have the following proposition:

Proposition 3. *T^+ is a generalized inverse of T.*

Corollary 4. *T has a pseudoinverse if and only if T has a generalized inverse.*

Definition. The operator T^+ is said to be the generalized inverse of T with respect to W and Z.

Let S be a pseudoinverse of T and put $W_0 = ST(E)$ and $Z_0 = (I_F - TS)(F)$. Then, W_0 and Z_0 are complemented of $\mathrm{Ker}\,(T) = (I_E - ST)(E)$ and $R(T) = TS(F)$, respectively.

Proposition 5. *Let S be a generalized inverse of T. Then S consists of the generalized inverse of T with respect to W_0 and Z_0.*

Proof. Let T^+ be a generalized inverse of T with respect to W_0 and Z_0. For any $y \in F$,

$$y = TS(y) + (I_F - TS)(y) = TSTS(y) + (I_F - TS)(y),$$

where $STS(y) \in W_0$ and $(I_F - TS)(y) \in Z_0$. Hence, by the definition of T^+,

$$T^+(y) = STS(y) = S(y). \qquad \square$$

4. The gaps between closed convex subsets. In order to continue our discussion, we need the following definitions.

Definition. Let $U \in L(E, F)$. The minimum modulus of U, written $\gamma(U)$, is defined by

$$\gamma(U) = \inf \left\{ \frac{\|U(x)\|}{\mathrm{dist}\,(x, \mathrm{Ker}\,(U))} : x \in E \right\},$$

where $0/0$ is defined to be ∞ (see [**4**, p. 96]).

Definition. For any $V, W \in C(E)$, we set

$$\eta(V, W) = \sup \left\{ \frac{\mathrm{dist}\,(x, W)}{\|x\|} : x \in V, x \neq 0 \right\},$$

and

$$d(V, W) = \max(\eta(V, W), \eta(W, V)),$$

where $\eta(\{0\}, W)$ is defined to be 0. $d(V, W)$ is said to be the gap between V and W (see [**5**, p. 197]).

We now give the results on $\gamma(U), \eta(V,W)$ and $d(V,W)$. The following lemma is shown in the same way as the proof of real or complex Banach spaces (see [**4**, p. 98]).

Lemma 6. *Let $U \in L(E,F)$. Then $R(U)$ is closed if and only if $\gamma(U) > 0$.*

Proposition 7. *Suppose $T \neq 0$. Then*

$$\gamma(T) \leqq \|T\|.$$

In addition, if T has a pseudoinverse S, then

$$\frac{1}{\|S\|} \leqq \gamma(T).$$

Furthermore, if S is a generalized inverse, then

$$\frac{1}{\|S\|} \leqq \gamma(T) \leqq \min\left(\frac{\|ST\|\|TS\|}{\|S\|}, \|T\|\right).$$

Proof. For any $x \in E \setminus \mathrm{Ker}\,(T)$ and for any $z \in \mathrm{Ker}\,(T)$ it holds that $\|T(x)\| \leqq \|T\|\|x-z\|$. Hence $\|T(x)\|/\mathrm{dist}\,(x, \mathrm{Ker}\,(T)) \leq \|T\|$. From this it follows that $\gamma(T) \leqq \|T\|$. Next, for any $x \in E$, it holds

$$\mathrm{dist}\,(x, \mathrm{Ker}\,(T)) \leqq \|x - (I_E - ST)(x)\| \leqq \|S\|\|T(x)\|.$$

Hence $1/\|S\| \leqq \gamma(T)$. Furthermore, for any $y \in F$, it follows that

$$\begin{aligned}\|TS(y)\| &\geqq \gamma(T)\mathrm{dist}\,(S(y), \mathrm{Ker}\,(T)) \\ &\geqq \gamma(T)\frac{\|STS(y)\|}{\|ST\|} = \gamma(T)\frac{\|S(y)\|}{\|ST\|}.\end{aligned}$$

Hence we have that $\|ST\|\|TS\| \geqq \gamma(T)\|S\|$. Thus we complete the proof. □

For closed convex subsets of E, the following lemma holds:

Lemma 8. *Let* $V, W \in C(E)$.

(1) $0 \leqq \eta(V, W) \leqq 1$.

(2) $V \subset W$ *if and only if* $\eta(V, W) = 0$.

(3) *If* V *and* W *are closed linear subspaces in* E *such that* $W \subset V$, $V \neq W$, *then* $\eta(V, W) = 1$.

Proof. (1) and (2) are trivial. (3) For every t, $0 < t < 1$, there exists an $x_0 \in V \setminus W$ such that $\operatorname{dist}(x_0, W) \geqq t\|x_0\|$ (see [**9**, p. 66]). Hence $\eta(V, W) \geqq \operatorname{dist}(x_0, W)/\|x_0\| \geqq t$. Since t is arbitrary, we conclude $\eta(V, W) = 1$. □

We now show the following theorem.

Theorem 9. *Let* $U_1, U_2 \in L(E, F)$ *with closed ranges. Then we have*

(1) $d(R(U_1), R(U_2)) \leqq \|U_1 - U_2\| \max[(1/\gamma(U_1)), (1/\gamma(U_2))]$.

(2) $d(\operatorname{Ker}(U_1), \operatorname{Ker}(U_2)) \leqq \|U_1 - U_2\| \max[(1/\gamma(U_1)), (1/\gamma(U_2))]$.

Proof. (1) If either $U_1 = 0$ or $U_2 = 0$, the proof is trivial. So we may assume that $U_i \neq 0$, $i = 1, 2$. Recall that $\gamma(U_i) > 0$, $i = 1, 2$. Let $y \in R(U_1)$, $y \neq 0$, and let $y = U_1(x)$, $x \in E$. For any $z \in \operatorname{Ker}(U_1)$,

$$\begin{aligned} \operatorname{dist}(y, R(U_2)) &\leqq \|y - U_2(x - z)\| \\ &= \|U_1(x - z) - U_2(x - z)\| \\ &\leqq \|U_1 - U_2\|\|x - z\|. \end{aligned}$$

Hence, $\operatorname{dist}(y, R(U_2)) \leqq \|U_1 - U_2\| \operatorname{dist}(x, \operatorname{Ker}(U_1))$. Then it follows that

$$\frac{\operatorname{dist}(y, R(U_2))}{\|y\|} \leqq \|U_1 - U_2\| \frac{\operatorname{dist}(x, \operatorname{Ker}(U_1))}{\|y\|} \leqq \frac{\|U_1 - U_2\|}{\gamma(U_1)}.$$

Thus we obtain

$$\eta(R(U_1), R(U_2)) \leqq \frac{\|U_1 - U_2\|}{\gamma(U_1)}.$$

Similarly,

$$\eta(R(U_2), R(U_1)) \leqq \frac{\|U_1 - U_2\|}{\gamma(U_2)}.$$

Thus the proof of (1) is complete. $\square$

(2) If either $\mathrm{Ker}\,(U_1) = \{0\}$ or $\mathrm{Ker}\,(U_2) = \{0\}$, the proof is trivial. So we may assume that $\mathrm{Ker}\,(U_i) \neq 0$, $i = 1, 2$. For each $x \in \mathrm{Ker}\,(U_1)$, $x \neq 0$,

$$\gamma(U_2)\mathrm{dist}\,(x, \mathrm{Ker}\,(U_2)) \leqq \|U_2(x)\| = \|(U_2 - U_1)(x)\| \leqq \|U_2 - U_1\|\|x\|.$$

Hence

$$\frac{\mathrm{dist}\,(x, \mathrm{Ker}\,(U_2))}{\|x\|} \leqq \frac{\|U_2 - U_1\|}{\gamma(U_2)}.$$

It follows that

$$\eta(\mathrm{Ker}\,(U_1), \mathrm{Ker}\,(U_2)) \leqq \frac{\|U_2 - U_1\|}{\gamma(U_2)}.$$

Similarly,

$$\eta(\mathrm{Ker}\,(U_2), \mathrm{Ker}\,(U_1)) \leqq \frac{\|U_1 - U_2\|}{\gamma(U_1)}.$$

Thus we can complete the proof of (2). $\square$

5. Perturbations of linear operators. In this section we need the following lemma.

Lemma 10. *Let $B \in L(E)$ be such that $\|B\| < 1$. Then $(I_E + B)^{-1}$ exists and belongs to $L(E)$. Furthermore, $\|(I_E + B)^{-1}\| = 1$.*

From now on, given $A \in L(E, F)$, set $\bar{T} = T + A$.

Lemma 11. *Suppose that T has a generalized inverse S such that either $\|SA\| < 1$ or $\|AS\| < 1$ and such that either* $\dim \mathrm{Ker}\,(\bar{T}) = \dim \mathrm{Ker}\,(T) < \infty$ *or $R(\bar{T}) \cap \mathrm{Ker}\,(S) = \{0\}$. Then*

$$(I_F + AS)^{-1}\bar{T}(\mathrm{Ker}\,(T)) \subset R(T).$$

Proof. At first we remark that, by the assumption $\sum_{k=0}^{\infty}(-SA)^k$ and $\sum_{k=0}^{\infty}(-AS)^k$ converge, so $(I_E + SA)^{-1}$ and $(I_F + AS)^{-1}$ exist and belong to $L(E)$ and $L(F)$, respectively. Set $U = (I_E + SA)^{-1}(I_E - ST)$. We now show that $R(U) = \mathrm{Ker}\,(\bar{T})$. Since $U \in L(E)$ is a projection, $\mathrm{Ker}\,(\bar{T}) \subset R(U)$. If $\dim \mathrm{Ker}\,(\bar{T}) = \dim \mathrm{Ker}\,(T) < \infty$, then

$$\dim R(U) = \dim (I_E - ST) = \dim \mathrm{Ker}\,(\bar{T}).$$

So $\mathrm{Ker}\,(\bar{T}) = R(U)$. Further, if $R(\bar{T}) \cap \mathrm{Ker}\,(S) = \{0\}$, then it holds that $TS\bar{T}(I_E + SA)^{-1} = T$. Hence $TS\bar{T}U = T(I_E - ST) = 0$. Since $STS = S$, $S\bar{T}U = 0$. It follows that, for any $x \in E$, $\bar{T}U(x) \in R(\bar{T}) \cap \mathrm{Ker}\,(S)$. Hence, $\bar{T}U(x) = 0$. From this, $R(U) \subset \mathrm{Ker}\,(\bar{T})$. Thus we showed that $R(U) = \mathrm{Ker}\,(\bar{T})$. Since it holds that $S(I_F + AS)^{-1} = (I_E + SA)^{-1}S$ and $(I_F + AS)^{-1}A = A(I_E + SA)^{-1}$, we have

$$(I_F - TS)(I_F + AS)^{-1}\bar{T}(I_E - ST) = 0.$$

It follows that, for any $x \in \mathrm{Ker}\,(T)$,

$$(I_F + AS)^{-1}\bar{T}(x) = TS(I_F + AS)^{-1}\bar{T}(x) \in R(T).$$

This completes the proof. □

Theorem 12. *Suppose that T has a generalized inverse S such that either $\|SA\| < 1$ or $\|AS\| < 1$ and such that either $\dim \mathrm{Ker}\,(\bar{T}) = \dim \mathrm{Ker}\,(T) < \infty$ or $R(\bar{T}) \cap \mathrm{Ker}\,(S) = \{0\}$. Set $\bar{S} = (I_E + SA)^{-1}S (= S(I_F + AS)^{-1})$. Then $\bar{S}$ is a generalized inverse of $\bar{T}$ such that $\|\bar{S}\| = \|S\|$, $\|\bar{S}A\| = \|SA\|$ and $\|A\bar{S}\| = \|AS\|$.*

Proof. At first we obtain

$$\begin{aligned}
\bar{T} - \bar{T}\bar{S}\bar{T} &= (T + A) - (T + A)\bar{S}(T + A) \\
&= \{(I_F + AS) - (T + A)S\}(I_F + AS)^{-1}(T + A) \\
&= (I_F - TS)(I_F + AS)^{-1}(T + A) \\
&= (I_F - TS)(I_F + AS)^{-1}\{(I_F + AS)T + A(I_E - ST)\} \\
&= (I_F - TS)T + (I_F - TS)(I_F + AS)^{-1}A(I_E - ST) \\
&= 0 \qquad \text{(by Lemma 11)}.
\end{aligned}$$

Next,

$$\begin{aligned}
\bar{S}\bar{T}\bar{S} - \bar{S} &= S(I_F + AS)^{-1}\{(T + A)S(I_F + AS)^{-1} - I_F\} \\
&= S(I_F + AS)^{-1}[\{(TS - I_F) + (I_F + AS)\}(I_F + AS)^{-1} - I_F] \\
&= S(I_F + AS)^{-1}(TS - I_F)(I_F + AS)^{-1} \\
&= 0
\end{aligned}$$

for $\mathrm{Ker}(\bar{S}) = \mathrm{Ker}(S)$ and $(TS - I_F)(I_F + AS)^{-1}(F) \subset \mathrm{Ker}(S)$. Thus we showed that $\bar{S}$ is a generalized inverse of $\bar{T}$. Furthermore, we have

$$\|\bar{S}\| = \|(I_E + SA)^{-1}S\| = \left\| \left(\sum_{k=0}^{\infty} (-SA)^k \right) S \right\| = \|S\|,$$

or

$$\|\bar{S}\| = \|S(I_F + AS)^{-1}\| = \left\| S \sum_{k=0}^{\infty} (-AS)^k \right\| = \|S\|.$$

Similarly, we can show that $\|\bar{S}A\| = \|SA\|$ and $\|A\bar{S}\| = \|AS\|$. □

Lemma 13. *Suppose that T has a generalized inverse S such that either $\|SA\| < 1$ or $\|AS\| < 1$ and such that either* $\dim \mathrm{Ker}(\bar{T}) = \dim \mathrm{Ker}(T) < \infty$ or $R(\bar{T}) \cap \mathrm{Ker}(S) = \{0\}$. *If $A \neq 0$, then $\bar{T} \neq 0$ and $\bar{S} \neq 0$.*

Proof. Suppose that $\bar{T} = 0$. Then $\|A\| = \|ASA\|$. Since $A \neq 0$, it follows that $1 \leqq \|SA\|$ and $1 \leqq \|AS\|$. This contradicts the assumption. □

Lemma 14. (1) *Suppose that T has a pseudoinverse S with $\|SA\| < 1$. If* $\mathrm{Ker}(T) = \{0\}$, *then* $\mathrm{Ker}(\bar{T}) = \{0\}$.

(2) *Suppose that T has a generalized inverse S such that $\|SA\| < 1$ and $R(\bar{T}) \cap \mathrm{Ker}(S) = \{0\}$. Then $\mathrm{Ker}(T) = \{0\}$ if and only if $\mathrm{Ker}(\bar{T}) = \{0\}$.*

Proof. (1) Suppose that $\mathrm{Ker}(\bar{T}) \neq \{0\}$ and let $u \in \mathrm{Ker}(\bar{T})$ with $u \neq 0$. Since $u - ST(u) \in \mathrm{Ker}(T)$, by the assumption it follows that $u + SA(u) = 0$. Hence we have $\|SA\| \geqq 1$. This contradicts $\|SA\| < 1$.

(2) By Theorem 12, $\bar{S}$ is a generalized inverse of $\bar{T}$. Suppose that $\mathrm{Ker}(\bar{T}) = \{0\}$, and let $v \in \mathrm{Ker}(T)$. Then $v - \bar{S}A(v) = 0$. If $v \neq 0$, then $1 \leqq \|\bar{S}A\| = \|SA\|$. This contradicts $\|SA\| < 1$ and the proof is complete. □

Corollary 15. *Suppose that T has a generalized inverse S such that either* $\dim \mathrm{Ker}(\bar{T}) = \dim \mathrm{Ker}(T) < \infty$ *or* $R(\bar{T}) \cap \mathrm{Ker}(S) = \{0\}$.

(1) *If either* $\|SA\| < 1$ *or* $\|AS\| < 1$, *then* $d(R(T), \overline{R(\bar{T})}) \leqq \|S\|\|A\|$.

(2) *If* $\|SA\| < 1$, *then* $d(\mathrm{Ker}\,(T), \mathrm{Ker}\,(\bar{T})) \leqq \|SA\|$.

Proof. We may assume that $A \neq 0$. From Theorem 12 and Lemma 13, we note that $T, S, \bar{T}$ and $\bar{S}$ are not 0.

(1) Let $w \in \overline{R(\bar{T})}$, $w \neq 0$, $w = \lim w_n$ and $w_n = \bar{T}(z_n)$, $z_n \in E$, $n = 1, 2, \dots$. For all $z \in \mathrm{Ker}\,(\bar{T})$, it holds

$$\begin{aligned}\mathrm{dist}\,(w, R(T)) &\leqq \|w - T(z_n - z)\| \leqq \max(\|w - w_n\|, \|w_n - T(z_n - z)\|) \\ &\leqq \max(\|w - w_n\|, \|A\|\|z_n - z\|), \quad n = 1, 2, \dots .\end{aligned}$$

It follows that

$$\mathrm{dist}\,(w, R(T)) \leqq \max(\|w - w_n\|, \|A\|\mathrm{dist}\,(z_n, \mathrm{Ker}\,(\bar{T}))), \quad n = 1, 2, \dots .$$

For a sufficiently large number n, we obtain

$$\begin{aligned}\mathrm{dist}\,(z_n, \mathrm{Ker}\,(\bar{T})) &\leqq \|z_n - (I_E - \bar{S}\bar{T})(z_n)\| \\ &= \|\bar{S}\bar{T}(z_n)\| \leqq \|\bar{S}\|\bar{T}(z_n)\| = \|\bar{S}\|\|w\|\end{aligned}$$

and

$$\|w - w_n\| \leqq \|A\|\|\bar{S}\|\|w\|.$$

Hence we have

$$\mathrm{dist}\,(w, R(T)) \leqq \|A\|\|\bar{S}\|\|w\|.$$

Thus we have that

$$\eta(\overline{R(\bar{T})}, R(T)) \leqq \|A\|\|\bar{S}\| = \|A\|\|S\|.$$

Next let $y \in R(T)$, $y \neq 0$. For an $x \in E$ with $y = T(x)$, we have that

$$\mathrm{dist}\,(x, \mathrm{Ker}\,(T)) \leqq \|x - (I_E - ST)(x)\| \leqq \|ST(x)\| \leqq \|S\|\|y\|.$$

Since, for all $z \in \mathrm{Ker}\,(T)$,

$$\mathrm{dist}\,(y, \overline{R(\bar{T})}) \leqq \|y - \bar{T}(x - z)\| = \|A(x - z)\| \leqq \|A\|\|x - z\|.$$

Hence

$$\mathrm{dist}\,(y, \overline{R(\bar{T})}) \leqq \|A\|\mathrm{dist}\,(x, \mathrm{Ker}\,(T)) \leqq \|A\|\|S\|\|y\|.$$

Therefore we obtain that

$$\eta(R(T), \overline{R(\bar{T})}) \leqq \|A\|\|S\|.$$

Thus we can conclude that

$$d(R(T), \overline{R(\bar{T})}) \leqq \|A\|\|S\|.$$

(2) By Lemma 14 we may assume that $\operatorname{Ker}(T) \neq \{0\}$ and $\operatorname{Ker}(\bar{T}) \neq \{0\}$. At first, we show that $\eta(\operatorname{Ker}(\bar{T}), \operatorname{Ker}(T)) \leqq \|SA\|$. Let $u \in \operatorname{Ker}(\bar{T})$, $u \neq 0$. Then we have

$$\operatorname{dist}(u, \operatorname{Ker}(T)) \leqq \|u-(I_E-ST)(u)\| = \|ST(u)\| = \|SA(u)\| \leqq \|SA\|\|u\|.$$

It follows from this that

$$\eta(\operatorname{Ker}(\bar{T}), \operatorname{Ker}(T)) \leqq \|SA\|.$$

In a similar fashion, we can obtain that

$$\eta(\operatorname{Ker}(T), \operatorname{Ker}(\bar{T})) \leqq \|\bar{S}A\| = \|SA\|.$$

Thus we can conclude that

$$d(\operatorname{Ker}(T), \operatorname{Ker}(\bar{T})) \leqq \|SA\|. \qquad \square$$

From Lemma 8 and Corollary 15, we have the following corollary.

Corollary 16. *Suppose that T has a generalized inverse S such that $\|S\|\|A\| < 1$ and such that either* $\dim \operatorname{Ker}(\bar{T}) = \dim \operatorname{Ker}(T) < \infty$ or $R(\bar{T}) \cap \operatorname{Ker}(S) = \{0\}$. *Then T is surjective if and only if $R(\bar{T})$ is dense in F. In addition, if $R(\bar{T})$ is closed, then T is bijective if and only if $\bar{T}$ is bijective.*

Remark. The following example indicates that in the conclusion of Corollary 15, $\|S\|\|A\|$ cannot be replaced by $\|SA\|$.

Let $E = F = K^2$, and let $\alpha \in K$ with $0 < |\alpha| < 1$. Also, let $T(x_1, x_2) = (0, \alpha(x_1 + x_2))$, $S(x_1, x_2) = (0, \alpha x_1 + \alpha^{-1} x_2)$ and

$A(x_1, x_2) = (\alpha^2(x_1+x_2), 0)$. Then T, S, A and $\bar{T}$ satisfy the conditions of Corollary 15. We also have

$$\eta(R(T), R(\bar{T})) = \eta(R(\bar{T}), R(T)) = |\alpha|,$$

so

$$d(R(T), R(\bar{T})) = |\alpha|.$$

Further, we have $\|SA\| \leqq |\alpha|^3$.

6. Perturbations for $Tx = b$. In this section let b and $\bar{b}$, $b \neq 0$, $\bar{b} \neq 0$, be fixed elements in $R(T)$ and $R(\bar{T})$, respectively. We consider the gap of the sets of solutions of operator equations $T(x) = b$ and $\bar{T}(x) = \bar{b}$. Set

$$X(T, b) = \{x \in E : T(x) = b\},$$
$$X(\bar{T}, \bar{b}) = \{x \in E : \bar{T}(x) = \bar{b}\}.$$

It is clear that $X(T, b), X(\bar{T}, \bar{b}) \in C(E)$.

Proposition 17. *Suppose that T has a pseudoinverse S such that either $\|SA\| < 1$ or $\|AS\| < 1$. Then, for every $\bar{x} \in X(\bar{T}, \bar{b})$, there exists an $x \in X(T, b)$ such that*

$$\frac{\|\bar{x} - x\|}{\|x\|} \leqq \|S\|\|T\| \max\left(\frac{\|\bar{b} - b\|}{\|b\|}, \frac{\|A\|}{\|T\|}\right).$$

Proof. Let $W = ST(E)$. Since $\mathrm{Ker}\,(T) = (I_E - ST)(E)$, $E = \mathrm{Ker}\,(T) \oplus W$. Set $x = S(b) + (I_E - ST)(\bar{x} - S(b))$. Then it holds $T(x) = TS(b) = b$ and $ST(\bar{x} - x) = \bar{x} - x$. Since $(T + A)(\bar{x} - x) = \bar{b} - b - A(x)$, it follows that $(I_E + SA)(\bar{x} - x) = S(\bar{b} - b - A(x))$ and $\bar{x} - x = (I_E + SA)^{-1} S(\bar{b} - b - A(x))$. Thus we have

$$\frac{\|\bar{x} - x\|}{\|x\|} \leqq \|S\|\|T\| \max\left(\frac{\|\bar{b} - b\|}{\|b\|}, \frac{\|A\|}{\|T\|}\right). \qquad \square$$

Corollary 18. *Suppose that T has a pseudoinverse S with $\|S\|\|A\| < 1$ and let $\|b\| = \|\bar{b}\|$. If $\|\bar{b} - b\|/\|b\| \leqq \|A\|/\|T\|$, then*

$$\eta(X(\bar{T}, \bar{b}), X(T, b)) \leqq \|S\|\|A\|.$$

Proof. By Proposition 17, there exists an $x \in X(T,b)$ such that $\|\bar{x}-x\|/\|x\| \leqq \|S\|\|A\| < 1$. It follows from this that $\|\bar{x}\| = \|x\|$, and the proof is complete. □

Remark. Since $T \neq 0$ and S is a pseudoinverse of T, we have $1 \leqq \|ST\| \leqq \|S\|\|T\|$. From the condition $\|S\|\|A\| < 1$, it follows that $\|A\|/\|T\| < 1$. Hence in Corollary 18 we need the condition $\|b\| = \|\bar{b}\|$.

Corollary 19. *Suppose that T has a pseudoinverse S with $\|SA\| < 1$. If $\bar{b} - b \in \operatorname{Ker}(S)$, then*

$$\eta(X(\bar{T},\bar{b}), X(T,b)) \leqq \|SA\|.$$

Proof. Since $\bar{b} - b \in \operatorname{Ker}(S)$, from the proof of Proposition 17 it follows that

$$\|\bar{x}-x\| \leq \|SA(x)\| \leqq \|SA\|\|x\|.$$

Hence $\|\bar{x}\| = \|x\|$, for $\|SA\| < 1$. This implies that

$$\eta(X(\bar{T},\bar{b}), X(T,b)) \leqq \|SA\|. \qquad \square$$

Theorem 20. *Suppose that T has a generalized inverse S such that $\|S\|\|A\| < 1$ and such that either $\dim \operatorname{Ker}(\bar{T}) = \dim \operatorname{Ker}(T) < \infty$ or $R(\bar{T}) \cap \operatorname{Ker}(S) = \{0\}$. Assume that $\|b\| = \|\bar{b}\|$. If $\|\bar{b}-b\|/\|b\| \leqq \|A\|/\|T\|$, then $d(X(T,b), X(\bar{T},\bar{b})) \leqq \|S\|\|A\|$.*

Proof. By Theorem 12 there exists a generalized inverse $\bar{S}$ of $\bar{T}$ such that $\|S\| = \|\bar{S}\|$. Aslo, from the above remark, it follows that $\|A\| < \|T\|$. Therefore, $\|T\| = \|\bar{T}\|$. Thus, from the assumption, we have $\|b-\bar{b}\|/\|\bar{b}\| \leqq \|A\|/\|\bar{T}\|$ and $\|\bar{S}\|\|A\| < 1$. Hence by Corollary 18, it holds that $\eta(X(T,b), X(\bar{T},\bar{b})) \leqq \|\bar{S}\|\|A\| = \|S\|\|A\|$. By combining Corollary 18, we conclude that $d(X(T,b), X(\bar{T},\bar{b})) \leqq \|S\|\|A\|$. □

Theorem 21. *Suppose that T has a generalized inverse S such that $\|SA\| < 1$ and such that either $\dim \operatorname{Ker}(\bar{T}) = \dim \operatorname{Ker}(T) < \infty$ or $R(\bar{T}) \cap \operatorname{Ker}(S) = \{0\}$. If $\bar{b} - b \in \operatorname{Ker}(S)$, then*

$$d(X(T,b), X(\bar{T},\bar{b})) \leqq \|SA\|.$$

Proof. By Theorem 12, there exists a generalized inverse $\bar{S}$ of $\bar{T}$ such that $\|S\| = \|\bar{S}\|$, $\|SA\| = \|\bar{S}A\|$ and $\operatorname{Ker}(S) = \operatorname{Ker}(\bar{S})$. Hence, by Corollary 19, it holds that $\eta(X(T,b), X(\bar{T},\bar{b})) \leqq \|\bar{S}A\| = \|SA\|$. By combining Corollary 19, we can complete the proof. □

Example. Let α, E, F, T and A be as in the remark in Section 5.

(1) Let $b = (0, \alpha + \alpha^2)$, $\bar{b} = (\alpha^2, \alpha)$ and $S(x_1, x_2) = (\alpha x_1 + \alpha^{-1} x_2, 0)$. Then $b, \bar{b}$ and S satisfy the hypotheses of Theorem 20. We can also see that $d(X(T,b), X(\bar{T},\bar{b})) = \|S\|\|A\| = |\alpha|$. Further, we can see that $d(X(T,b), X(\bar{T},\bar{b})) \geq \|SA\| = |\alpha|^3$.

(2) Let $b = (0, \alpha + 1)$, $\bar{b} = (\alpha, 1)$ and $S(x_1, x_2) = (\alpha^{-1}(x_1 + x_2), 0)$. Then $b, \bar{b}$ and S satisfy the hypotheses of Theorem 21. Also we see that $d(X(T,b), X(\bar{T},\bar{b})) = \|SA\| = |\alpha|$.

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