ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 34, Number 3, Fall 2004

RELATIONS FOR IMAGINARY PARTS OF ZEROS OF ENTIRE FUNCTIONS

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ABSTRACT. Finite order entire functions are considered. New relations for the imaginary parts of the zeros are derived. They particularly generalize the Cartwright-Levinson theorem. By virtue of these relations, under some restriction, the Hadamard theorem on the convergence exponent of the zeros is improved.

1. The main result. Consider the finite order entire function

$$f(\lambda) = \sum_{k=0}^{\infty} \frac{a_k \lambda^k}{(k!)^{\gamma}}, \quad \lambda \in \mathbf{C}, \quad a_0 = 1, \quad \gamma > 0, \tag{1.1}$$

with complex, in general, coefficients. Assume that

$$w(f) \equiv \sum_{k=1}^{\infty} |a_k|^2 < \infty \tag{1.2}$$

and put

$$\psi_f \equiv \left[|Im\ a_1|^2 + \sum_{k=2}^{\infty} |a_k|^2 \right]^{1/2}.$$

Everywhere below $\{z_k(f)\}_{k=1}^{m}$, $m \le \infty$, is the set of all the zeros of f taken with their multiplicities. In this section it is assumed that the zeros are numerated in the following way

$$\left| Im \frac{1}{z_k(f)} \right| \ge \left| Im \frac{1}{z_{k+1}(f)} \right|, \quad k = 1, \dots, m-1.$$

In the sequel, if $m < \infty$, then we take $|z_k(f)|^{-1} = 0$ for $k > m$.

AMS *Mathematics Subject Classification.* 30D20, 30C15.
Key words and phrases. Entire functions, estimates for zeros.
This research was supported by the Israel Ministry of Science and Technology.
Received by the editors on November 18, 1999.

Since $a_0 = 1$, we have $z_k(f) \neq 0$, $k = 1, 2, \dots$. The aim of the present paper is to prove the following

Theorem 1.1. *Under condition* (1.2), *the following inequalities are valid:*

$$\sum_{k=1}^{j} \left| Im \frac{1}{z_k(f)} \right| \leq \psi_f + \sum_{k=1}^{j} (k+1)^{-\gamma}, \quad j = 1, 2, \dots . \tag{1.3}$$

The proof of this theorem is given in the next section. As is shown below, under some restrictions, Theorem 1.1 with $j = \infty$ particularly generalizes the Cartwright-Levinson theorem, cf. [**5**, p. 126]. Moreover, by virtue of Theorem 1.1 under some restrictions we improve the Hadamard theorem [**5**, p. 18].

Furthermore, put $\omega_I(f) = \psi_f + 2^{-\gamma}$. The well-known Lemma II.3.4 [**4**] and Theorem 1.1 imply

Corollary 1.2. *Let* $\phi(t)$, $0 \leq t < \infty$, *be a convex continuous scalar-valued function, such that* $\phi(0) = 0$. *Then, under condition* (1.2), *the inequalities*

$$\sum_{k=1}^{j} \phi\left(\left| Im \frac{1}{z_k(f)} \right| \right) \leq \phi(\omega_I(f)) + \sum_{k=2}^{j} \phi((k+1)^{-\gamma}), \quad j = 2, 3, \dots ,$$

are valid. In particular, for any real $r \geq 1$,

$$\begin{aligned} \sum_{k=1}^{j} \left| Im \frac{1}{z_k(f)} \right|^r &\leq \omega_I^r(f) + \sum_{k=2}^{j} (k+1)^{-\gamma r} \\ &= (\psi_f + 2^{-\gamma})^r + \sum_{k=2}^{j} (k+1)^{-r\gamma}, \\ &\quad j = 2, 3, \dots . \end{aligned} \tag{1.4}$$

Assume that

$$r\gamma > 1 \quad \text{and} \quad r \geq 1. \tag{1.5}$$

Then the series

$$\sum_{k=2}^{\infty}(k+1)^{-r\gamma} = \zeta(\gamma r) - 1 - 2^{-\gamma r}$$

converges. Here $\zeta(.)$ is the Riemann zeta-function. Now relation (1.4) yields

Corollary 1.3. *Under conditions* (1.2) *and* (1.5), *the inequality*

$$\sum_{k=1}^{\infty}\left|Im\frac{1}{z_k(f)}\right|^r \leq (2^{-\gamma}+\psi_f)^r + \zeta(\gamma r) - 1 - 2^{-\gamma r} \tag{1.6}$$

is valid. In particular, if $\gamma > 1$, *then*

$$\sum_{k=1}^{\infty}\left|Im\frac{1}{z_k(f)}\right| \leq \psi_f + \zeta(\gamma) - 1. \tag{1.7}$$

Consider now a positive scalar-valued function $\Phi(t_1, t_2, \ldots, t_j)$ defined on the domain

$$-\infty < t_j \leq t_{j-1} \leq t_2 \leq t_1 < \infty$$

and satisfying

$$\frac{\partial \Phi}{\partial t_1} > \frac{\partial \Phi}{\partial t_2} > \cdots > \frac{\partial \Phi}{\partial t_j} > 0 \quad \text{for } t_1 > t_2 > \cdots > t_j. \tag{1.8}$$

Then Theorem 1.1 and the well-known Lemma II.3.5 [**4**] yield

Corollary 1.4. *Under conditions* (1.2) *and* (1.8), *for any natural* $j \geq 1$, *the inequality*

$$\Phi\left(\left|Im\tfrac{1}{z_1(f)}\right|, \left|Im\tfrac{1}{z_2(f)}\right|, \ldots, \left|Im\tfrac{1}{z_j(f)}\right|\right) \leq \Phi(\omega_I(f), 3^{-\gamma}, \ldots, (1+j)^{-\gamma})$$

is valid. In particular, let $\{d_k\}_{k=1}^{\infty}$ be a decreasing sequence of positive numbers. Then

$$\sum_{k=1}^{j} d_k \Big| Im \frac{1}{z_k(f)} \Big| \leq d_1 \psi_f + \sum_{k=1}^{j} d_k (k+1)^{-\gamma}, \quad j = 1, 2, \dots .$$

For instance, let $0 < \gamma < 1$. Take $d_k = (k+1)^{-(1+\varepsilon-\gamma)}$ with an arbitrary positive ε. Then

$$\sum_{k=1}^{j} (k+1)^{-(1+\varepsilon-\gamma)} \Big| Im \frac{1}{z_k(f)} \Big| \leq 2^{-(1+\varepsilon-\gamma)} \psi_f + \sum_{k=1}^{j} (k+1)^{-1-\varepsilon},$$
$$j = 1, 2, \dots .$$

Hence,

$$\sum_{k=1}^{\infty} (k+1)^{-(1+\varepsilon-\gamma)} |Im\ z_k^{-1}(f)| \leq 2^{-(1+\varepsilon-\gamma)} \psi_f + \zeta(1+\varepsilon) - 1.$$

Let f belong to the Cartwright class C, i.e., it is of exponential type and satisfies the property

$$\int_{-\infty}^{\infty} \frac{\log^+ |f(t)|}{1+t^2}\, dt < \infty.$$

Recall that if $f \in C$, then the Cartwright-Levinson theorem asserts, among other results, that the series in the lefthand part of (1.7) converges. At the same time Corollary 1.3 under more general assumptions not only asserts the convergence of that series but gives us the estimate for the sum of imaginary parts of the zeros.

2. Proof of Theorem 1.1. To prove Theorem 1.1, for a natural $n \geq 2$, consider the polynomial

$$p_n(\lambda) = \sum_{k=0}^{n} \frac{a_k \lambda^{n-k}}{(k!)^{\gamma}} \tag{2.1}$$

with the zeros $z_k(p_n)$ ordered in the following way

$$|Im\ z_k(p_n)| \geq |Im\ z_{k+1}(p_n)|, \quad k = 1, \ldots, n.$$

Put

$$\psi(p_n) \equiv [|Im\ a_1| + \sum_{k=2}^{n} |a_k|^2]^{1/2}.$$

Lemma 2.1. *The zeros of p_n satisfy the inequalities*

$$\sum_{k=1}^{j} |Im\ z_k(p_n)| \leq \psi(p_n) + \sum_{k=1}^{j} (k+1)^{-\gamma}, \quad j = 1, \ldots, n-1$$

and

$$\sum_{k=1}^{n} |Im\ z_k(p_n)| \leq \psi(p_n) + \sum_{k=1}^{n-1} (k+1)^{-\gamma}.$$

Proof. Introduce the $n \times n$-matrix

$$B_n = \begin{pmatrix} -a_1 & -a_2 & -a_3 & \cdots & -a_{n-1} & -a_n \\ 1/2^{\gamma} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1/3^{\gamma} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1/n^{\gamma} & 0 \end{pmatrix}.$$

The direct calculations show that $p_n(\lambda) = \det(B_n - \lambda I_n)$, $\lambda \in \mathbf{C}$, where I_n is the unit matrix. So

$$z_k(p_n) = \lambda_k(B_n) \tag{2.2}$$

where $\lambda_k(B)$, $k = 1, \ldots, n$, mean the eigenvalues of an $n \times n$ matrix B with their multiplicities. Denote $Im\ B_n = (B_n - B_n^*)/2i$. Here and below the asterisk means the adjointness.

Due to the well-known Lemma II.6.1 [**4**]

$$\sum_{k=1}^{j} |Im\ \lambda_k(B_n)| \leq \sum_{k=1}^{j} s_k(Im\ B_n), \quad j = 1, \ldots, n \tag{2.3}$$

where $s_k(B)$, $k = 1, \dots, n$, denote the singular numbers of an $n \times n$-matrix $B : s_k^2(B) = \lambda_k(BB^*)$ ordered in the decreasing way. Here and below the asterisk means the adjointness. Clearly,

$$B_n - B_n^* = \begin{bmatrix} \bar{a}_1 - a_1 & -a_2 - 1/2^\gamma & -a_3 & \cdots & -a_{n-1} & -a_n \\ 1/2^\gamma + \bar{a}_2 & 0 & -1/3^\gamma & \cdots & 0 & 0 \\ \bar{a}_3 & 1/3^\gamma & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \bar{a}_n & 0 & 0 & \cdots & 1/n^\gamma & 0 \end{bmatrix}.$$

With the notation

$$C = \begin{pmatrix} (-a_1 + \bar{a}_1)/2 & -a_2 & -a_3 & \cdots & -a_n \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

and

$$D = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1/2^\gamma & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1/3^\gamma & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1/n^\gamma & 0 \end{pmatrix}$$

we have $B_n - B_n^* = C - C^* + D - D^*$ and

$$CC^* = \begin{pmatrix} \psi^2(p_n) & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

and

$$DD^* = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1/2^{2\gamma} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1/3^{2\gamma} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1/n^{2\gamma} \end{pmatrix}.$$

Since the diagonal entries of diagonal matrices are the eigenvalues, we can write out

$$s_1(C) = \psi(p_n), s_k(C) = 0, \quad k = 2, \dots, n.$$

In addition,

$$s_k(D) = (k+1)^{-\gamma}, \quad k = 1, \dots, n-1, \quad s_n(D) = 0.$$

Taking into account that

$$\begin{aligned}\sum_{k=1}^{j} s_k(B_n - B_n^*) &= \sum_{k=1}^{j} s_k(D - D^* + C - C^*) \\ &\le 2\sum_{k=1}^{j} s_k(D) + 2\sum_{k=1}^{j} s_k(C),\end{aligned}$$

cf. [**4**, Lemma II.4.2]. So

$$\sum_{k=1}^{j} s_k(Im\ B_n) \le \psi(p_n) + \sum_{k=1}^{j}(k+1)^{-\gamma}, \quad j = 1, \dots, n-1,$$

and

$$\sum_{k=1}^{n} s_k(Im\ B_n) \le \psi(p_n) + \sum_{k=1}^{n-1}(k+1)^{-\gamma}.$$

Now (2.2) and (2.3) yield the required result. □

Proof of Theorem 1.1. Consider the polynomial

$$q_n(\lambda) = 1 + a_1\lambda + \dots + a_n n^{-\gamma}\lambda^n, \quad 2 \le n < \infty,$$

with the zeros $z_k(q_n)$ ordered in the following way

$$\left|Im\frac{1}{z_k(q_n)}\right| \ge \left|Im\frac{1}{z_{k+1}(q_n)}\right|, \quad k = 1, \dots, n-1.$$

With a fixed $k \le n$ and $z = z_k(q_n)$, we get

$$q_n(z) = z^n \sum_{k=0}^{n} a_k k^{-\gamma} z^{k-n} = 0.$$

Hence, $z_k^{-1}(q_n) = z_k(p_n)$ where p_n is defined by (2.1). Now Lemma 2.1 yields the inequalities

$$\begin{aligned}\sum_{k=1}^{j}\left|Im\frac{1}{z_k(q_n)}\right| &\le \psi(p_n) + \sum_{k=1}^{j}(k+1)^{-\gamma}\\ &\le \psi_f + \sum_{k=1}^{j}(k+1)^{-\gamma}, \quad j = 1, \dots, n-1.\end{aligned} \tag{2.4}$$

But the zeros of entire functions continuously depend on its coefficients:

$$\sum_{k=1}^{j}\left|Im\frac{1}{z_k(q_n)}\right| \longrightarrow \sum_{k=1}^{j}\left|Im\frac{1}{z_k(f)}\right|$$

as $n \to \infty$. Now (2.4) implies the required result. □

3. Inequalities for the real parts of the zeros. In this section it is assumed that the zeros of the function f defined by (1.1) are enumerated in the following way

$$\left|Re\frac{1}{z_k(f)}\right| \ge \left|Re\frac{1}{z_{k+1}(f)}\right|, \quad k = 1, 2, \dots .$$

Let us suppose that (1.2) holds, and put

$$\theta_f \equiv \left[|Re\ a_1|^2 + \sum_{k=2}^{\infty}|a_k|^2\right]^{1/2}.$$

Substituting in (1.1) $z = iw$ we get by virtue of Theorem 1.1,

Theorem 3.1. *Under condition* (1.2)*, the inequalities*

$$\sum_{k=1}^{j}\left|Re\frac{1}{z_k(f)}\right| \le \theta_f + \sum_{k=1}^{j}(k+1)^{-\gamma}, \quad j = 1, 2, \dots ,$$

are valid.

Furthermore, denote $\omega_R(f) = \theta_f + 2^{-\gamma}$. The well-known Lemma II.3.4 [**4**] and Theorem 3.1 imply

Corollary 3.2. *Let $\phi(t)$, $0 \le t < \infty$, be a convex continuous scalar-valued function, such that $\phi(0) = 0$. Then, under condition* (1.2)*, the inequalities*

$$\sum_{k=1}^{j} \phi\left(\left|Re \frac{1}{z_k(f)}\right|\right) \le \phi(\omega_R(f)) + \sum_{k=2}^{j} \phi((k+1)^{-\gamma}), \quad j = 2, 3, \dots ,$$

are valid. In particular, for any real $r \ge 1$ and all real $j \ge 2$,

$$\sum_{k=1}^{j} \left|Re \frac{1}{z_k(f)}\right|^r \le \omega_R^r(f) + \sum_{k=2}^{j} (k+1)^{-\gamma r} = (\theta_f + 2^{-r\gamma})^r + \sum_{k=2}^{j} (k+1)^{-r\gamma}.$$

This corollary with $j = \infty$ yields

Corollary 3.3. *Under conditions* (1.2) *and* (1.5)*, the inequality*

$$\sum_{k=1}^{\infty} \left|Re \frac{1}{z_k(f)}\right|^r \le (2^{-\gamma} + \theta_f)^r + \zeta(\gamma r) - 1 - 2^{-\gamma r}$$

is valid. In particular, if $\gamma > 1$, then

$$\sum_{k=1}^{\infty} \left|Re \frac{1}{z_k(f)}\right| \le \theta_f + \zeta(\gamma) - 1.$$

Moreover, Theorem 3.1 and the well-known Lemma II.3.5 [**4**] yield

Corollary 3.4. *Under conditions* (1.2) *and* (1.8)*, for any natural $j \ge 2$, the inequality*

$$\Phi\left(\left|Re \frac{1}{z_1(f)}\right|, \left|Re \frac{1}{z_2(f)}\right|, \dots , \left|Re \frac{1}{z_j(f)}\right|\right) \le \Phi(\omega_R(f), 3^{-\gamma}, \dots , (1+j)^{-\gamma})$$

is valid. In particular, let $\{d_k\}_{k=1}^{\infty}$ *be a decreasing sequence of positive numbers. Then*

$$(3.1)\qquad \sum_{k=1}^{j} d_k \left| Re\frac{1}{z_k(f)} \right| \le d_1\theta_f + \sum_{k=1}^{j} d_k (k+1)^{-\gamma}, \quad j = 1, 2, \dots .$$

For instance, let $0 < \gamma < 1$. Take $d_k = (k+1)^{-(1+\varepsilon-\gamma)}$ with an arbitrary positive ε. Then

$$\sum_{k=1}^{j}(k+1)^{-(1+\varepsilon-\gamma)} \left| Re\frac{1}{z_k(f)} \right| \le 2^{-(1+\varepsilon-\gamma)}\theta_f + \sum_{k=1}^{j}(k+1)^{-1-\varepsilon}, \quad j = 1, 2, \dots .$$

Hence,

$$\sum_{k=1}^{\infty}(k+1)^{-(1+\varepsilon-\gamma)} \left| Re\frac{1}{z_k(f)} \right| \le 2^{-(1+\varepsilon-\gamma)}\theta_f + \zeta(1+\varepsilon) - 1.$$

Furthermore, due to the inequality $a^r + b^r \le (a+b)^r$, $r > 1$, for arbitrary positive a, b, Corollaries 1.3 and 3.3 yield

Corollary 3.5. *Under conditions* (1.2) *and* (1.5)*, the inequality*

$$(3.2)\quad \sum_{k=1}^{\infty} \left| \frac{1}{z_k(f)} \right|^r \le (2^{-\gamma} + \theta_f)^r + (2^{-\gamma} + \psi_f)^r + 2(\zeta(\gamma r) - 1 - 2^{-\gamma r})$$

is valid. In particular, if $\gamma > 1$*, then*

$$\sum_{k=1}^{\infty} \left| \frac{1}{z_k(f)} \right| \le \psi_f + \theta_f + 2(\zeta(\gamma) - 1)$$
$$\le \left[2|a_1|^2 + 4\sum_{k=2}^{\infty}|a_k|^2\right]^{1/2} + 2(\zeta(\gamma) - 1).$$

Under conditions (1.2) and (1.5), this corollary improves the Hadamard theorem, since our result not only asserts the convergence of the series in (3.2) but gives us the estimate for the sum of absolute values of the zeros.

4. The case $\gamma > 1/2$. In this section we are going to make relations (1.6) and (3.2) more sharp under the conditions $r = 1$ and

$$\gamma > 1/2. \tag{4.1}$$

Moreover, under (4.1) we will derive an estimate for the quantity

$$S(f) \equiv \max_{j \neq k} \left| \frac{1}{z_k(f)} - \frac{1}{z_j(f)} \right|.$$

Theorem 4.1. *Under conditions* (1.2) *and* (4.1), *for the zeros of function* (1.1), *the following inequalities are true*:

$$\sum_{k=1}^{\infty} |z_k(f)|^{-2} \leq w(f) + \zeta(2\gamma) - 1 \tag{4.2}$$

and

$$\sum_{k=1}^{\infty} |Im\ z_k(p_n)|^2 \leq |Im\ a_1|^2 + 2^{-\gamma} Re\, a_2 + \frac{1}{2}\Big[\sum_{k=2}^{\infty} |a_k|^2 + \zeta(2\gamma) - 1\Big]. \tag{4.3}$$

To prove Theorem 4.1, again consider polynomial (2.1).

Lemma 4.2. *With the notation*

$$w(p_n) = \sum_{k=1}^{n} |a_k|^2,$$

the zeros of p_n satisfy the inequalities

$$\sum_{k=1}^{n} |z_k(p_n)|^2 \leq w(p_n) + \sum_{k=2}^{n} k^{-2\gamma} \tag{4.4}$$

and

$$\sum_{k=1}^{n} |Im\ z_k(p_n)|^2 \le |Im\ a_1|^2 + 2^{-\gamma} Re\, a_2 + \frac{1}{2}\sum_{k=2}^{n}(|a_k|^2 + k^{-2\gamma}). \tag{4.5}$$

Proof. Due to the Schur inequality [**6**, Section III.1.4],

$$\sum_{k=1}^{n} |\lambda_k(B_n)|^2 \le N^2(B_n) \tag{4.6}$$

where $N(B)$ is the Frobenius (Hilbert-Schmidt) norm of a matrix $B : N^2(B) = \mathrm{Trace}\, BB^*$. Obviously,

$$N^2(B_n) = w(p_n) + \sum_{k=2}^{n} k^{-2\gamma}. \tag{4.7}$$

Now (2.2) and (4.6) imply inequality (4.4).

Furthermore, let us use the inequality

$$\sum_{k=1}^{n} |Im\ \lambda_k(B_n)|^2 \le N^2(Im\ B_n), \tag{4.8}$$

cf. [**6**, Section III.1]. Obviously,

$$2N^2(Im\ B_n) = 2|Im\ a_1|^2 + |1/2^\gamma + a_2|^2 + \sum_{k=3}^{n}(|a_k|^2 + k^{-2\gamma}).$$

Simple calculations show that

$$|1/2^\gamma + a_2|^2 = 1/4^\gamma + |a_2|^2 + 2^{1-\gamma} Re\ a_2.$$

So

$$N^2(Im\ B_n) = |Im\ a_1|^2 + 2^{-\gamma}\, Re\, a_2 + \frac{1}{2}\sum_{k=2}^{n}(|a_k|^2 + k^{-2\gamma}).$$

Now (2.2) and (4.8) imply the required inequality (4.5). $\square$

Proof of Theorem 4.1. As was shown in the proof of Theorem 1.1, $z_k^{-1}(q_n) = z_k(p_n)$. Now inequality (4.4) yields

$$\sum_{k=1}^{n} |z_k(q_n)|^{-2} \le \theta(p_n) + \sum_{k=2}^{n} k^{-2\gamma}.$$

Hence

$$\sum_{k=1}^{j} |z_k(q_n)|^{-2} \le w(f) + \sum_{k=2}^{\infty} k^{-2\gamma} = w(f) + \zeta(2\gamma) - 1 \tag{4.9}$$

for any $j \le n$. Letting $n \to \infty$ in (4.9), we get

$$\sum_{k=1}^{j} |z_k(f)|^{-2} \le w(f) + \zeta(2\gamma) - 1$$

for any natural j. This implies inequality (4.2). Similarly, inequality (4.5) yields (4.3). □

Theorem 4.3. *Let f be defined by* (1.1) *under conditions* (1.2) *and* (4.1). *Then the following inequality is true*:

$$S^2(f) \le 2(w(f) + \zeta(2\gamma) - 1). \tag{4.10}$$

To prove this theorem, again consider polynomial (2.1).

Lemma 4.4. *The zeros of p_n satisfy the inequality*

$$\max_{j \ne k} |z_k(p_n) - z_j(p_n)|^2 \le 2\Big(w(p_n) + \sum_{k=2}^{n} k^{-2\gamma} - \frac{|a_1|^2}{n}\Big).$$

Proof. Due to the well-known inequality III.4.2.1 [**6**], the inequality

$$\max_{j \ne k} |\lambda_k(B_n) - \lambda_j(B_n)| \le \left(2N^2(B_n) - \frac{2}{n}|\mathrm{Trace}\, B_n|^2\right)^{1/2}$$

is valid, where $N(.)$ is again the Frobenius norm. Obviously, Trace $B_n = a_1$. So, according to (4.7),

$$\max_{j\neq k} |\lambda_k(B_n) - \lambda_j(B_n)|^2 \leq 2\biggl(w(p_n) + \sum_{k=2}^{n} k^{-2\gamma} - \frac{|a_1|^2}{n}\biggr).$$

Now (2.2) proves the statement of the lemma. $\square$

Proof of Theorem 4.3. As was shown in the proof of Theorem 1.1, $z_k^{-1}(q_n) = z_k(p_n)$. Now the previous lemma yields

$$\max_{j\neq k} \left|\frac{1}{z_k(q_n)} - \frac{1}{z_j(q_n)}\right|^2 \leq 2(w(p_n) + \zeta(2\gamma) - 1) \leq 2(w(f) + \zeta(2\gamma) - 1).$$

Furthermore, the zeros of entire functions continuously depend on its coefficients. So, letting in the latter inequality $n \to \infty$, we get inequality (4.10). $\square$

5. Equalities for zeros of second order entire functions. Rewrite (1.1) in the form

$$f(\lambda) = 1 + b_1\lambda + b_2\lambda^2 + b_3\lambda^3 + \cdots (\lambda \in \mathbf{C}). \tag{5.1}$$

That is,

$$b_k = a_k k^{-\gamma}. \tag{5.2}$$

Denote by $\rho(f)$ the order of the growth of f. In this section we are going to establish equalities for the zeros which supplement relations (1.6) and (3.2) in the case $r = 2$ and

$$\rho(f) \leq 2. \tag{5.3}$$

Theorem 5.1. *Let the set of the zeros of function f defined by* (5.1) *be non-empty and condition* (5.3) *hold. Then the equality*

$$\sum_{k=1}^{\infty}[|z_k(f)|^{-2} - 2(Re\ z_k^{-1}(f))^2] = J(f)$$

is valid with the notation

$$J(f) \equiv Re\ b_1^2 - 2Re\ b_2 = Re\ a_1^2 - 2^{1-\gamma} Re\ a_2.$$

To prove Theorem 5.1, we need the following

Lemma 5.2. *The zeros* $z_k(p_n)$, $k = 1, 2, \ldots, n$, *of polynomial* (2.1) *and* (5.2) *satisfy the equality*

$$\sum_{k=1}^{n} |z_k(p_n)|^2 - 2(Im\ z_k(p_n))^2 = J(f).$$

Proof. Consider the $n \times n$-matrix

$$E_n = \begin{pmatrix} -b_1 & -b_2 & -b_3 & \cdots & -b_{n-1} & -b_n \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdot & \cdot & 1 & 0 \end{pmatrix}.$$

Clearly $p_n(\lambda) = \det(E_n - I_n \lambda)$. So $z_k(p_n) = \lambda_k(E_n)$, $k = 1, \ldots, n$. Due to Corollary 1.3.7 [**3**, p. 19] (see also [**2**]), we can write out

$$N^2(E_n) - \sum_{k=1}^{n} |\lambda_k(E_n)|^2 = N^2(E_n - E_n^*)/2 - 2\sum_{k=1}^{n} |Im\ \lambda_k(E_n)|^2. \tag{5.4}$$

Simple calculations show that $N^2(E_n) = |b_1|^2 + \cdots + |b_n|^2 + n - 1$ and

$$N^2(E_n - E_n^*)/2 = 2(Im\ b_1)^2 + |b_2 + 1|^2 + |b_3|^2 + |b_4|^2 + \cdots + |b_n|^2 + n - 2.$$

Now (5.4) yields

$$\sum_{k=1}^{n} |\lambda_k(E_n)|^2 - 2\sum_{k=1}^{n} |Im\ \lambda_k(E_n)|^2 = |b_1|^2 + |b_2|^2 + 1 - 2(Im\ b_1)^2 - |b_2 + 1|^2 = J(f).$$

This proves the required result. $\square$

Proof of Theorem 5.1. As was shown in the proof of Theorem 1.1, $z_k^{-1}(q_n) = z_k(p_n)$. Now Lemma 5.2 yields

$$\sum_{k=1}^{n} |z_k(q_n)|^{-2} - 2(Im\, z_k^{-1}(q_n))^2 = J(f). \tag{5.5}$$

Since the zeros of entire functions continuously depend on its coefficients, for any natural $j > 2$, we have

$$\sum_{k=1}^{j} |z_k(q_n)|^{-2} - 2(Im\, z_k^{-1}(q_n))^2 \longrightarrow \sum_{k=1}^{j} |z_k(f)|^2 - 2(Im\, z_k^{-1}(f))^2$$

as $n \to \infty$. Now (5.5) implies the required result. $\square$

Since $|z|^2 = (Re\ z)^2 + (Im\ z)^2$, $z \in \mathbf{C}$, Theorem 5.1 yields

Corollary 5.3. *Under condition* (5.3), *we have*

$$\sum_{k=1}^{\infty} (Re\ z_k^{-1}(f))^2 - (Im\ z_k^{-1}(f))^2 = J(f).$$

Finally, note that Lemmata 2.2, 4.2, 4.4 and 5.2 supplement the well-known results on zeros of polynomials [**1**].

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