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A SPECTRAL TRANSFORM FOR THE MATRIX HILL'S EQUATION

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ABSTRACT. The mapping from the $K \times K$ coefficient $Q(x)$ of a matrix Hill's equation

$$-Y'' + Q(x)Y = \lambda Y, \quad Q(x+1) = Q(x),$$
$$Y(x) \in \mathbf{C}^K, \quad Q(x) \in M_K,$$

to the Floquet matrix may be considered a type of spectral transform. For matrix functions Q with square integrable components the range of the transform is described with the aid of the Paley-Wiener Hilbert spaces of entire functions. Local diffeomorphism results describe the auxiliary data that may be added to the spectrum to provide local 'spectral' coordinates for the space of coefficients. Some applications to conventional inverse spectral problems are provided.

1. Introduction. We consider the matrix Hill's equation,

$$(1) \qquad \begin{gathered} -Y'' + Q(x)Y = \lambda Y, \qquad Q(x+1) = Q(x), \\ Y(x) \in \mathbf{C}^K, \qquad Q(x) \in M_K. \end{gathered}$$

The function $Q(x)$ is $K \times K$ matrix-valued, with square integrable complex entries $Q_{jk}(x) \in L^2[0,1]$. Initially no symmetry assumptions are made on $Q(x)$.

For each $\lambda \in \mathbf{C}$ Hill's equation (1) has a $2K$-dimensional space of solutions. Translation by the period 1 acts linearly on this space. The standard basis of solutions is given by the columns of the $K \times K$ matrices $C(x,\lambda)$, $S(x,\lambda)$, which are matrix solutions of (1) satisfying the initial conditions

$$(2) \qquad \begin{gathered} C(0,\lambda) = I_K, \qquad S(0,\lambda) = 0_K, \\ C'(0,\lambda) = 0_K, \qquad S'(0,\lambda) = I_K. \end{gathered}$$

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The Floquet matrix

$$\Psi_1(\lambda) = \begin{pmatrix} C(1,\lambda) & S(1,\lambda) \\ C'(1,\lambda) & S'(1,\lambda) \end{pmatrix}$$

represents translation by 1 with respect to the standard basis. We define the Floquet transform to be the mapping $F(Q)$ from $Q(x)$ to the entire function $\Psi_1(\lambda)$. A major goal of this work is to describe the range and mapping properties of the Floquet transform for $Q \in L^2$.

The Floquet matrix carries considerable spectral data, including the periodic, Dirichlet and Neumann eigenvalues. Using techniques of Levinson the author has recently shown [**6**] that $Q(x) \in L^1$ is uniquely determined by the matrix functions $C(1,\lambda)$ and $S(1,\lambda)$ so that the Floquet matrix contains enough 'auxiliary data' for uniquely solvable inverse spectral problems. Inverse problems for the Floquet transform are also related to spectrum preserving evolution equations such as the matrix Kd V equation

$$Q_t = -(1/4)Q''' + (3/4)QQ' + (3/4)Q'Q,$$

which may be expressed in the Lax form $dL/dt = [L, A]$. The paper [**31**] treats this and other equations of Lax type with a matrix operator $L = -D^2 + Q$ via scattering theory.

There is a vast literature on the spectral theory of second order ordinary differential operators with scalar-valued coefficients, and inverse problems for the classical Hill's equation have been thoroughly analyzed [**11, 13, 20–22, 30**]. A recent review with many additional references is [**18**]. Inverse spectral problems are treated systematically from the Hilbert space viewpoint in [**24**]. Extensions of the earlier Hill's equation work to scalar potentials in $L^2[0,1]$ were considered in [**12, 16, 25**].

The literature is much thinner in the case of matrix coefficients. An inverse scattering problem for the matrix Schrodinger equation was treated long ago [**1**]; a more current discussion is in [**7**, p. 370]. Inverse Sturm-Liouville problems are attacked in [**15**]. Very little of the inverse spectral theory for the matrix Hill's equation has been developed. Spectral theoretic characterizations of the very restricted classes of operators have been considered in [**2, 10, 14**]. The author has treated compactness of isospectral sets [**5**] and trace formulas [**3,**

4]. The basic problem of describing the set of matrix potentials with the same set of Floquet multipliers is wide open.

With its novel emphasis on the mapping properties of the Floquet transform $F(Q)$, this work presents results which appear new even in the scalar case. Section 2 reviews basic estimates for solutions of (1) and begins to treat the analyticity of $F(Q)$ as a function of $Q \in L^2([0,1], M_K)$. The gross behavior of $\Psi_1(\lambda, Q)$ is determined by $Q_0 = \int_0^1 Q(x)\, dx$, and the most interesting mapping properties of the Floquet transform arise when the differences $\Psi_1(\lambda, Q) - \Psi_1(\lambda, Q_0)$ are considered.

Section 3 examines the sampled differences

$$\Psi_1(n^2\pi^2, Q) - \Psi_1(n^2\pi^2, Q_0), \quad n = 1, 2, 3, \dots .$$

Write $Q(x) = Q_0 + P(x)$. After fixing Q_0 and performing some rescalings, these sampled differences lead to an analytic map from $P \in L^2([0,1], M_K)$ to a matrix-valued sequence space $l^2(M_{2K})$. Focusing on the upper $K \times K$ blocks of Ψ_1, the normalized sequences of Floquet matrix samples

$$n\pi[C(1, n^2\pi^2, Q_0 + P) - C(1, n^2\pi^2, Q_0)],$$
$$n^2\pi^2[S(1, n^2\pi^2, Q_0 + P) - S(1, n^2\pi^2, Q_0)],$$

provide a local diffeomorphism from the potentials $Q_0 + P$ near 0_K and sequences near 0 in $l^2(M_{K \times 2K})$. This result establishes that certain inverse problems for Floquet matrices are locally well posed.

Section 4 enriches the results considerably by recognizing that the samples discussed above come from functions in one of the Paley-Wiener Hilbert spaces of entire functions. The local diffeomorphism results established for sample sequences extend to local diffeomorphism results with range in the Paley-Wiener space. Applications of the Paley-Wiener space techniques are provided in Section 5. Several results show that only a small set of matrix Hill's equations can have the spectral data of a scalar equation.

2. Preliminaries. Some notational conventions are considered first. For $\lambda \in \mathbf{C}$ let $\omega = \sqrt{\lambda}$, where the square root is chosen continuously for $-\pi < \arg(\lambda) \le \pi$ and positive for $\lambda > 0$ unless otherwise noted.

Denote by $\Im\omega$ the imaginary part of ω. A vector $Y \in \mathbf{C}^K$ is given by the Euclidean norm

$$|Y| = \left[\sum_{k=1}^{K} |y_k|^2\right]^{1/2}, \quad Y = \begin{pmatrix} y_1 \\ \vdots \\ y_K \end{pmatrix},$$

while a $K \times K$ matrix Q is given by the operator norm

$$\|Q\| = \sup_{|Y|=1} |QY|.$$

M_K denotes the set of $K \times K$ matrices with complex entries, and the $K \times K$ identity and zero matrices are I_K and 0_K respectively.

2.1 Basic estimates. Here is a version of Gronwall's inequality suited to our needs.

Lemma 2.1. *Assume that $v(t)$, $u(t)$, and $c(t)$ are real valued non-negative functions on an interval $[a, b]$. Suppose that $v(t)$ is continuous, $u(t)$ is integrable, and $c(t)$ is nondecreasing. If*

$$v(t) \le c(t) + \int_a^t u(s)v(s)\,ds, \quad a \le t \le b,$$

then

$$v(t) \le c(t) \exp\left(\int_a^t u(s)\,ds\right), \quad a \le t \le b.$$

Proof. The proof is standard [**8**, p. 241] if $u(t)$ is continuous and $c(t)$ is constant. The standard result may be extended easily to the case when $u(t)$ is merely integrable by approximating u in L^1 by nonnegative continuous functions. Finally, if $a \le t_0 \le b$, then since $c(t)$ is nondecreasing,

$$v(t) \le c(t_0) + \int_a^t u(s)v(s)\,ds, \quad a \le t \le t_0.$$

Apply the standard inequality at $t = t_0$ to get

$$v(t_0) \le c(t_0) \exp\left(\int_a^{t_0} u(s)\,ds\right). \qquad \square$$

Estimates for solutions of (1) may be established using standard techniques. The model equation $-Y'' = \lambda Y$ has a basis of $2K$ solutions which are the columns of the $K \times K$ diagonal matrix-valued functions $\cos(\omega x)I_K$, $\omega^{-1}\sin(\omega x)I_K$. Using the variation of parameters formula, a solution of (1) satisfying $Y(0,\lambda) = \alpha$, $Y'(0,\lambda) = \beta$, with $\alpha, \beta \in \mathbf{C}^K$, will satisfy the integral equation

$$\begin{aligned} Y(x,\lambda) &= \cos(\omega x)\alpha + \omega^{-1}\sin(\omega x)\beta \\ &\quad + \omega^{-1}\int_0^x \sin(\omega[x-t])Q(t)Y(t,\lambda)\,dt. \end{aligned} \tag{3}$$

Differentiation with respect to x gives

$$Y'(x,\lambda) = -\omega\sin(\omega x)\alpha + \cos(\omega x)\beta + \int_0^x \cos(\omega[x-t])Q(t)Y(t,\lambda)\,dt. \tag{4}$$

Start with the elementary estimates

$$\begin{gathered} |\sin(\omega x)| \le e^{|\Im\omega|x}, \quad |\cos(\omega x)| \le e^{|\Im\omega|x}, \\ |\omega^{-1}\sin(\omega x)| = \left|\int_0^x \cos(\omega t)\,dt\right| \le x e^{|\Im\omega|x}. \end{gathered}$$

In case $\beta = 0$ and $0 \le x \le 1$, the integral equation (3) gives

$$|e^{-|\Im\omega|x}Y(x,\lambda)| \le |\alpha| + \int_0^x \|Q(t)\||e^{-|\Im\omega|t}Y(t,\lambda)|\,dt.$$

By Gronwall's inequality

$$|e^{-|\Im\omega|x}Y(x,\lambda)| \le |\alpha| \exp\left(\int_0^x \|Q(t)\|\,dt\right).$$

Thus (3) implies that

$$|Y(x,\lambda)-\cos(\omega x)\alpha| \le |\alpha||\omega^{-1}|e^{|\Im\omega|x}\int_0^x \|Q(t)\| \exp\left(\int_0^t \|Q(s)\|\,ds\right)dt$$
$$= |\alpha||\omega^{-1}|e^{|\Im\omega|x}\left[\exp\left(\int_0^x \|Q(t)\|\,dt\right)-1\right].$$

A similar inequality holds when $\alpha = 0$ and (4) leads to inequalities for $|Y'|$. These estimates are summarized in the first lemma [**24**, p. 13].

Lemma 2.2. *Suppose that $Y(x,\lambda)$ satisfies* (1) *for $0 \le x \le 1$. Let*

$$C_1(Q,x) = \exp\left(\int_0^x \|Q(t)\|\,dt\right)-1.$$

If $Y(0,\lambda)=\alpha$, $Y'(0,\lambda)=0$, then

$$|Y(x,\lambda)-\cos(\omega x)\alpha| \le |\alpha||\omega^{-1}|e^{|\Im\omega|x}C_1(Q,x),$$
$$|Y'(x,\lambda)+\omega\sin(\omega x)\alpha| \le |\alpha|e^{|\Im\omega|x}C_1(Q,x).$$

Similarly, if $Y(0,\lambda)=0$, $Y'(0,\lambda)=\beta$, then

$$|Y(x,\lambda)-\omega^{-1}\sin(\omega x)\beta| \le |\beta||\omega^{-2}|e^{|\Im\omega|x}C_1(Q,x),$$
$$|Y'(x,\lambda)-\cos(\omega x)\beta| \le |\beta||\omega^{-1}|e^{|\Im\omega|x}C_1(Q,x).$$

The following result [**24**, p. 13] expresses these inequalities for the matrix functions $C(x,\lambda)$ and $S(x,\lambda)$.

Lemma 2.3. *Let*

$$C_2(Q,x) = K^{1/2}\left[\exp\left(\int_0^x \|Q(t)\|\,dt\right)-1\right].$$

For $0 \le x \le 1$, the $K \times K$ matrix solutions $C(x,\lambda)$ and $S(x,\lambda)$ satisfy

$$\|C(x,\lambda)-\cos(\omega x)I_k\| \le |\omega^{-1}|e^{|\Im\omega|x}C_2(Q,x),$$
$$\|C'(x,\lambda)+\omega\sin(\omega x)I_K\| \le e^{|\Im\omega|x}C_2(Q,x),$$
$$\|S(x,\lambda)-\omega^{-1}\sin(\omega x)I_K\| \le |\omega^{-2}|e^{|\Im\omega|x}C_2(Q,x),$$
$$\|S'(x,\lambda)-\cos(\omega x)I_K\| \le |\omega^{-1}|e^{|\Im\omega|x}C_2(Q,x).$$

2.2 Analytic dependence on the potential. Introduce the matrix

$$\Psi_1(x,\lambda,Q) = \begin{pmatrix} C(x,\lambda) & S(x,\lambda) \\ C'(x,\lambda) & S'(x,\lambda) \end{pmatrix}.$$

The conversion of (1) to a first order system gives

$$\mathbf{Y}' = A(x,\lambda)\mathbf{Y}, \quad A(x,\lambda) = \begin{pmatrix} 0_K & I_K \\ Q(x) - \lambda I_K & 0_K \end{pmatrix},$$

and $\Psi_1(x,\lambda,Q)$ is the $2K \times 2K$ matrix solution satisfying $\mathbf{Y}(0,\lambda) = I_{2K}$.

Estimates for the matrix function $\Psi_1^{-1}(x,\lambda)$ are also needed. A straightforward computation [**9**, p. 70] shows that

$$(\Psi_1^{*-1})' = -A^*\Psi_1^{*-1} = \begin{pmatrix} 0_K & \bar{\lambda} I_K - Q^* \\ -I_K & 0_K \end{pmatrix} \Psi_1^{*-1}.$$

The system $\mathbf{Z}' = -A^*\mathbf{Z}$ is satisfied by

$$\mathbf{Z} = \begin{pmatrix} Y' \\ -Y \end{pmatrix}$$

if

$$-Y'' + Q^*Y = \bar{\lambda} Y.$$

Since $\Psi_1^{*-1}(0,\lambda) = I_{2K}$,

$$\Psi_1^{-1}(x,\lambda) = \begin{pmatrix} (S^*)'(x,\bar{\lambda},Q^*) & -S^*(x,\bar{\lambda},Q^*) \\ -(C^*)'(x,\bar{\lambda},Q^*) & C^*(x,\bar{\lambda},Q^*) \end{pmatrix}. \tag{5}$$

Estimates for Ψ_1^{-1} may now be obtained directly from Lemma 2.3. The form of Ψ_1^{-1} may also be developed using the matrix Wronskian identities.

Lemma 2.4. *The following estimate holds*

$$\begin{aligned} &\|\Psi_1^{-1}(x,\lambda,Q)[\Psi_1(x,\lambda,P) - \Psi_1(x,\lambda,Q)]\| \\ &\le K(x)\exp\Big(\int_0^x \|\Psi_1^{-1}(t,\lambda,Q)\| \Big\|\begin{pmatrix} 0_K & 0_K \\ P(t) - Q(t) & 0_K \end{pmatrix}\Big\| \|\Psi_1(t,\lambda,Q)]\|\, dt\Big), \end{aligned}$$

where

$$K(x) = C_1 \int_0^x e^{2|\Im\omega|t} \|P(t) - Q(t)\| \, dt.$$

The value of C_1 *is bounded on sets of potentials* P, Q *such that*

$$\int_0^x \|Q(t)\| \, dt, \quad \int_0^x \|P(t)\| \, dt,$$

are uniformly bounded.

Proof. To relate $\Psi_1(x, \lambda, P)$ to $\Psi_1(x, \lambda, Q)$, write the first order system corresponding to (1) with potential $P(x)$ as

$$\mathbf{Y}' = \begin{pmatrix} 0_K & I_K \\ Q(x) - \lambda & 0_K \end{pmatrix} Y + \begin{pmatrix} 0_K & 0_K \\ P(x) - Q(x) & 0_K \end{pmatrix} \mathbf{Y},$$
$$\mathbf{Y}(0, \lambda) = I_{2K}.$$

Treating this equation as a nonhomogeneous version of the system satisfied by $\Psi_1(x, \lambda, Q)$, the variation of parameters formula [**9**, p. 74], gives

$$\Psi_1(x, \lambda, P) = \Psi_1(x, \lambda, Q) + \Psi_1(x, \lambda, Q) \times \int_0^x \Psi_1^{-1}(t, \lambda, Q) \begin{pmatrix} 0_K & 0_K \\ P(t) - Q(t) & 0_K \end{pmatrix} \Psi_1(t, \lambda, P) \, dt. \tag{6}$$

This may be rewritten as

$$\begin{aligned} \Psi_1(x, \lambda, P) - \Psi_1(x, \lambda, Q) = {} & \Psi_1(x, \lambda, Q) \int_0^x \Psi_1^{-1}(t, \lambda, Q) \begin{pmatrix} 0_K & 0_K \\ P(t) - Q(t) & 0_K \end{pmatrix} \\ & \times \Psi_1(t, \lambda, Q) \, dt + \Psi_1(x, \lambda, Q) \\ & \times \int_0^x \Psi_1^{-1}(t, \lambda, Q) \begin{pmatrix} 0_K & 0_K \\ P(t) - Q(t) & 0_K \end{pmatrix} [\Psi_1(t, \lambda, P) - \Psi_1(t, \lambda, Q)] \, dt. \end{aligned} \tag{7}$$

After some simple algebra, take norms to get

$$\begin{aligned} & \|\Psi_1^{-1}(x, \lambda, Q)[\Psi_1(x, \lambda, P) - \Psi_1(x, \lambda, Q)]\| \\ & \leq K_1(x) + \int_0^x \|\Psi_1^{-1}(t, \lambda, Q)\| \left\| \begin{pmatrix} 0_K & 0_K \\ P(t) - Q(t) & 0_K \end{pmatrix} \right\| \\ & \times \|\Psi_1(t, \lambda, Q)\| \|\Psi_1^{-1}(t, \lambda, Q)[\Psi_1(t, \lambda, P) - \Psi_1(t, \lambda, Q)]\| \, dt, \end{aligned} \tag{8}$$

where

$$K_1(x) = \Big\| \int_0^x \Psi_1^{-1}(t,\lambda,Q) \begin{pmatrix} 0_K & 0_K \\ P(t)-Q(t) & 0_K \end{pmatrix} \Psi_1(t,\lambda,Q)\,dt \Big\|.$$

Now (5) and Lemma 2.3 imply that there is a C_1 which is bounded if

$$\int_0^x \|Q(t)\|\,dt, \quad \int_0^x \|P(t)\|\,dt,$$

are uniformly bounded such that

$$K_1(x) \le K(x) = C_1 \int_0^x e^{2|\Im\omega|t} \|P(t)-Q(t)\|\,dt.$$

Replace $K_1(x)$ with $K(x)$ in (8). Since $K(x)$ is nonnegative and increasing, Gronwall's inequality gives the desired estimate.

The next result concerns the analyticity of the function taking matrix potentials $Q(x)$ to the solution matrix $\Psi_1(x,\lambda,Q)$. Background material on the extension of calculus and analyticity to functions on a Banach space is discussed in [**24**] and [**28**]. Denote by $L^2([0,1],M_K)$ the complex Hilbert space of square integrable $K \times K$ matrix valued functions on the unit interval, for which the norm is given by

$$\|Q\|_2 = \Big[\sum_{j,k=1}^K \int_0^1 |Q_{jk}(x)|^2 \Big]^{1/2}.$$

Suppose that $U \subset \mathbf{C}$ is compact. For each $Q \in L^2([0,1],M_K)$ the function $\Psi_1(x,\lambda,Q)$ is in $C([0,1]\times U, M_{2K})$, the continuous $2K \times 2K$ matrix valued functions on $[0,1]\times U$ with the norm

$$\|\Psi_1(x,\lambda)\|_U = \sup_{(x,\lambda)\in[0,1]\times U} \|\Psi_1(x,\lambda)\|.$$

Theorem 2.5. *Suppose that $U \subset \mathbf{C}$ is compact. The function $Q \to \Psi_1(x,\lambda,Q)$ is analytic from $L^2([0,1],M_K)$ to $C([0,1]\times U, M_{2K})$. The derivative at Q is the bounded linear map taking $H \in L^2([0,1],M_K)$ to*

$$(\partial_Q \Psi_1)H = \Psi_1(x,\lambda,Q) \int_0^x \Psi_1^{-1}(t,\lambda,Q) \begin{pmatrix} 0_K & 0_K \\ H & 0_K \end{pmatrix} \Psi_1(t,\lambda,Q)\,dt.$$

Proof. We have to show that $\Psi_1(x,\lambda,Q)$ is continuously differentiable with respect to Q. The main computation is (7), and the remaining point to check is the estimate for the last term. Lemma 2.4 and the Cauchy-Schwarz inequality show that

$$\|\Psi_1(x,\lambda,P) - \Psi_1(x,\lambda,Q)\| = O(\|P-Q\|_2),$$

with estimates holding uniformly for $(x,\lambda) \in [0,1] \times U$ and P, Q in an L^2 bounded set. This shows that the last term in (7) is $(\|P-Q\|_2^2)$, giving the differentiability and the formula for the derivative, which is continuous in Q by the estimates of Lemma 2.4. □

3. A local diffeomorphism of Hilbert spaces. In this section we begin to consider the recovery of $Q(x)$ from its Floquet matrix The essential idea is that data from the Floquet matrix can be used to compute mild perturbations of the Fourier coefficents for Q. The inverse function theorem is applied to recover the Fourier coefficients themselves.

It will be advantageous to rewrite (1) in the form

$$(9) \qquad -Y'' + Q_0 Y + P(x)Y = \lambda Y, \quad \int_0^1 P(x)\,dx = 0,$$

and to assume that

$$Q_0 = \int_0^1 Q(x)\,dx$$

is a diagonal matrix $Q_0 = \operatorname{diag}[q_1, \ldots, q_k]$. In case Q_0 is similar to a diagonal matrix a simple change of basis achieves this form.

If $\Omega(\lambda, Q_0)$ denotes the $K \times K$ diagonal matrix

$$\Omega = \operatorname{diag}\left[\sqrt{\lambda - q_1}, \ldots, \sqrt{\lambda - q_K}\right],$$

then

$$\cos(\Omega t) = \operatorname{diag}\left[\cos(\sqrt{\lambda - q_1}\,t), \ldots, \cos(\sqrt{\lambda - q_K}\,t)\right],$$
$$\sin(\Omega t) = \operatorname{diag}\left[\sin(\sqrt{\lambda - q_1}\,t), \ldots, \sin(\sqrt{\lambda - q_K}\,t)\right].$$

Define the matrix function

$$\Psi_0(t,\lambda) = \Psi_1(t,\lambda,Q_0) = \begin{pmatrix} \cos(\Omega t) & \Omega^{-1}\sin(\Omega t) \\ -\Omega\sin(\Omega t) & \cos(\Omega t) \end{pmatrix}.$$

This is an analytic function of λ and Q_0.

Lemma 2.3 provides estimates for the Floquet matrix

$$\Psi_1(1,\lambda,Q_0+P) = \begin{pmatrix} C(1,\lambda) & S(1,\lambda) \\ C'(1,\lambda) & S'(1,\lambda) \end{pmatrix}.$$

These estimates may be refined if λ is near $n^2\pi^2$.

Lemma 3.1. *Suppose that for positive integers n we have $\lambda = n^2\pi^2 + O(1)$. The following estimates hold uniformly if $\int_0^1 \|Q(t)\|\,dt$ is bounded.*

(10)

$$\begin{aligned}
C(1,\lambda) &= \cos(\Omega) - (-1)^n 2^{-1} n^{-1} \int_0^1 \sin(2n\pi t)Q(t)\,dt + O(n^{-2}),\\
C'(1,\lambda) &= -\Omega\sin(\Omega) + (-1)^n 2^{-1} \int_0^1 \cos(2n\pi t)Q(t)\,dt + O(n^{-1}),\\
S(1,\lambda) &= \Omega^{-1}\sin(\Omega) + (-1)^n 2^{-1} n^{-2} \int_0^1 \cos(2n\pi t)Q(t)\,dt + O(n^{-3}),\\
S'(1,\lambda) &= \cos(\Omega) + (-1)^n 2^{-1} n^{-1} \int_0^1 \sin(2n\pi t)Q(t)\,dt + O(n^{-2}).
\end{aligned}$$

Proof. The argument for $C(1,\lambda)$ is typical. Using (3) we obtain

$$\begin{aligned}
C(1,\lambda) &= \cos(\omega)I_K + \omega^{-1}\int_0^1 \sin(\omega[1-t])Q(t)\cos(\omega t)\,dt \\
&\quad + \omega^{-1}\int_0^1 \sin(\omega[1-t])Q(t)[C(t,\lambda) - \cos(\omega t)I_K]\,dt.
\end{aligned} \tag{11}$$

The estimates of Lemma 2.3 give

$$\left\|\omega^{-1}\int_0^1 \sin(\omega[1-t])Q(t)[C(t,\lambda)-\cos(\omega t)I_K]\,dt\right\| \le C_2|\omega^{-2}|e^{|\Im\omega|}\int_0^1 \|Q(t)\|\,dt.$$

Let
$$\|Q\|_1 = \int_0^1 \|Q(t)\|\,dt.$$
Since
$$e^x - 1 = \int_0^x e^t\,dt,$$
there is a constant C_3 such that
$$C_2 \le K^{1/2} \int_0^{\|Q\|_1} e^t\,dt \le C_3\|Q\|_1$$
as long as $\|Q\|_1$ is bounded.

Letting $\varepsilon_n = \lambda - n^2\pi^2$, Taylor expansions give
$$\omega = n\pi\left[1 + \frac{\varepsilon_n}{n^2\pi^2}\right]^{1/2} = n\pi + O(n^{-1}),$$
and
$$\sqrt{\lambda - q_k} = \omega[1 - q_k/\lambda]^{1/2} = \omega + O(\omega^{-1}).$$
We have
$$\sin(\omega[1-t])\cos(\omega t) = 2^{-1}[\sin(\omega[1-2t]) + \sin(\omega)],$$
and $\sin(\omega) = O(n^{-1})$, so that
$$\begin{aligned}C(1,\lambda) = \cos(\omega) + \omega^{-1}2^{-1}\int_0^1 \sin(\omega[1-2t])Q(t)\,dt \\ + Q_0 O(n^{-2}) + O(\|Q\|_1^2\omega^{-2}e^{|\Im\omega|}).\end{aligned} \tag{12}$$
Note that Q_0 is bounded by a constant times $\|Q\|_1$.

The remaining simplifications are also elementary. The trigonometric identities
$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b),$$
$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b),$$

give

$$\sin(\omega[1-2t]) = \sin([n\pi + O(n^{-1})][1-2t])$$
$$= (-1)^{n+1}\sin(2n\pi t) + O(n^{-1}).$$

Since

$$\sin(\sqrt{\lambda - q_k})O(n^{-1}) = \sin(n\pi + O(n^{-1}))O(n^{-1}) = O(n^{-2}),$$

we also get

$$\cos(\omega) = \cos(\sqrt{\lambda - q_k}) + O(n^{-2}),$$

achieving the desired result. □

Writing $\Psi_1(n^2\pi^2, Q_0 + P)$ for $\Psi_1(1, n^2\pi^2, Q_0 + P)$, we define a sequence of $2K \times 2K$ complex matrices for nonnegative integers n

$$F_1(n, Q) = \begin{pmatrix} n\pi I_K & 0_K \\ 0_K & I_K \end{pmatrix} [\Psi_1(n^2\pi^2, Q_0 + P) - \Psi_0(n^2\pi^2)] \times \begin{pmatrix} I_K & 0_K \\ 0_K & n\pi I_K \end{pmatrix}.$$

Denote by $l^2(M_{2K})$ the Hilbert space of sequences of $2K \times 2K$ complex matrices A_n with inner product

$$\langle \{A_n\}, \{B_n\} \rangle = \sum_{n=0}^{\infty} \operatorname{tr}(B_n^*, A_n).$$

Also introduce the subspace $L_0^2([0,1], M_K)$ of $L^2([0,1], M_K)$ consisting of those functions $Q(x)$ satisfying $\int_0^1 Q = 0$.

Theorem 3.2. *Suppose $Q_0 = \operatorname{diag}[q_1, \dots, q_K]$ is fixed. The map*

$$F_1 : P \longrightarrow \{F_1(n, Q_0 + P)\}, \quad n \geq 0,$$

takes $L_0^2([0,1], M_K)$ to $l^2(M_{2K})$ and is analytic. For $H \in L_0^2([0,1], M_K)$ the derivative with respect to P has the form

$$(\partial_P F_1)H = \frac{1}{2}\int_0^1 \begin{pmatrix} -\sin(2n\pi t)H(t) & \cos(2n\pi t)H(t) \\ \cos(2n\pi t)H(t) & \sin(2n\pi t)H(t) \end{pmatrix} dt + K(Q_0 + P)H,$$

where $K(Q_0+P)$ is a Hilbert-Schmidt operator. The operator $K(Q_0)$ is an analytic function of Q_0 and $K(0)=0$.

Proof. Lemma 3.1 shows that the sequence is in $l^2(M_{2K})$ since the blocks of $F_1(n,Q)$ are the sum of $\pm 1/2$ times the Fourier sine or cosine coefficients of Q plus terms from a square summable sequence.

Theorem 2.5 shows that the component maps $F_1(n,Q)$ are analytic from $P \in L^2$ to M_{2K}. By [**24**, p. 138] the map F_1 is analytic if it is locally bounded, but this follows from Lemma 3.1.

Since $\Psi_0(n^2\pi^2)$ depends only on Q_0, the component calculations from Theorem 2.5 are simply restricted to $H \in L_0^2([0,1], M_K)$. These calculations give

$$\partial_P F_1(n,Q)H = \begin{pmatrix} n\pi I_K & 0_K \\ 0_K & I_K \end{pmatrix} \Psi_1(1,n^2\pi^2,Q) \times \int_0^1 \Psi_1^{-1}(t,n^2\pi^2,Q) \begin{pmatrix} 0_K & 0_K \\ H & 0_K \end{pmatrix} \times \Psi_1(t,n^2\pi^2,Q)\,dt \begin{pmatrix} I_K & 0_K \\ 0_K & n\pi I_K \end{pmatrix}.$$

Lemma 2.3 gives

$$\Psi_1(t,n^2\pi^2,Q) = \begin{pmatrix} \cos(n\pi t)I_K + O(n^{-1}) & (n\pi)^{-1}\sin(n\pi t)I_K + O(n^{-2}) \\ -n\pi\sin(n\pi t)I_K + O(1) & \cos(n\pi t)I_K + O(n^{-1}) \end{pmatrix}$$

and (5) gives

$$\Psi_1^{-1}(t,n^2\pi^2,Q) = \begin{pmatrix} \cos(\omega t)I_K + O(n^{-1}) & -(n\pi)^{-1}\sin(n\pi t)I_K + O(n^{-2}) \\ n\pi\sin(n\pi t)I_K + O(1) & \cos(n\pi t)I_K + O(n^{-1}) \end{pmatrix}.$$

Thus, for $n \ge 1$,

$$\partial_P F_1(n,Q)H = \begin{pmatrix} n\pi I_K & 0_K \\ 0_K & I_K \end{pmatrix} \times \int_0^1 \begin{pmatrix} -(n\pi)^{-1}\sin(n\pi t)\cos(n\pi t)H - (n\pi)^{-2}\sin^2(n\pi t)H \\ \cos^2(n\pi t)H (n\pi)^{-1}\sin(n\pi t)\cos(n\pi t)H \end{pmatrix} dt$$

$$\times \begin{pmatrix} I_K & 0_K \\ 0_K & n\pi I_K \end{pmatrix} + O(n^{-1})$$
$$= \frac{1}{2}\int_0^1 \begin{pmatrix} -\sin(2n\pi t)H & \cos(2n\pi t)H \\ \cos(2n\pi t)H & \sin(2n\pi t)H \end{pmatrix} dt + O(n^{-1}).$$

The upshot is that the derivative of F_1 takes H to the sequence of (matrix) inner products with functions which are (up to sign) square summable perturbations of the orthonormal trig function basis

$$\cos(2n\pi t)I_k, \quad \sin(2n\pi t)I_k.$$

By [**24**, p. 163] this map $\partial_P F_1$ is a Hilbert-Schmidt perturbation $K(Q_0 + P)$ of the map to Fourier coefficients, as claimed. One checks easily that $K(0) = 0$, while the analyticity of $K(Q_0)$ follows by recomputing $\partial_P F_1(n, Q_0)H$ and then using elementary estimates. □

It is proven in [**6**] that Q is uniquely determined by the matrix functions $C(1, \lambda, Q)$ and $S(1, \lambda, Q)$. Let $l^2(M_{K\times 2K})$ denote the square summable sequences of $K \times 2K$ complex matrices $[A_n, B_n]$ for $n = 1, 2, 3, \dots$. We have an immediate corollary of Theorem 3.2 and the inverse function theorem [**24**, p. 142].

Corollary 3.3. *Suppose* $Q_0 = \operatorname{diag}[q_1, \dots, q_K]$ *is fixed and sufficiently close to* 0_K. *Then the map taking* $P \in L_0^2([0,1], M_K)$ *to* $l^2(M_{K\times 2K})$ *defined by*

$$A_n = n\pi[C(1, n^2\pi^2, Q_0 + P) - C(1, n^2\pi^2, Q_0)],$$
$$B_n = n^2\pi^2[S(1, n^2\pi^2, Q_0 + P) - S(1, n^2\pi^2, Q_0)],$$

is an analytic diffeomorphism between the potentials P *near* 0 *and sequences near* 0.

One may show that the map of Corollary 3.3 is generically a local diffeomorphism by using the analytic Fredholm theorem [**26**, p. 201].

4. The Paley-Wiener space. Although Corollary 3.3 provides some information about the well-posedness of recovering Q from its

Floquet matrix, it sheds little direct light on which spectra are realizable for various boundary conditions, how to supplement discrete spectral data to (locally) uniquely determine Q, or how to characterize the matrix functions $\Psi_1(\lambda)$ which are Floquet matrices for Hill's operators. These questions can be better addressed by understanding more about the functions $\Psi_1(\lambda)$ and in particular their relationship with the Paley-Wiener Hilbert spaces of entire functions.

First recall Paley and Wiener's well known [**23**, p. 13], [**27**, p. 370] characterization of the range of the Fourier transform on $L^2[-A,A]$.

A function $G(z)$ is the analytic continuation of the Fourier transform of a function $g \in L^2[-A,A]$ for $A > 0$,

$$G(z) = \int_{-A}^{A} g(t) e^{it(x+iy)}\, dt, \quad z = x + iy,$$

if and only if $G(z)$ is an entire function satisfying

$$\int_{-\infty}^{\infty} |G(x)|^2\, dx < \infty,$$

and

$$|G(z)| \le C e^{A|z|}.$$

This theorem will extend to the matrix valued entire functions; it will be convenient to interpret $|G(z)|$ as the matrix norm

$$|G(z)|^2 = \operatorname{tr}(G^*(z)G(z))$$

in the last inequality, in which case we will say that the matrix function has exponential type A. For our purposes the Fourier transforms of $K \times K$ matrix functions $g \in L^2([-1,1], M_K)$ are most important. Denote by $\mathcal{PW}$ this Hilbert space of functions with

$$\|G(\omega)\|_P^2 = \frac{1}{\pi} \int_{-\infty}^{\infty} |G(\omega)|^2\, d\omega.$$

Since $\Psi_1(1,\lambda,Q)$ is an entire function of λ, $\Psi_1(1,\omega^2,Q)$ is an entire function of ω. Again abbreviating $\Psi_1(1,\omega^2,Q)$ by $\Psi_1(\omega^2,Q)$, define the entire function

$$F_0(\omega,Q) = \Psi_1(\omega^2,Q) - \Psi_0(\omega^2).$$

We will suppress the Q and write $F_0(\omega)$ for $F_0(\omega, Q)$ when confusion is unlikely. Since the matrix norms satisfy $c_1|G(z)| \le \|G(z)\| \le c_2|G(z)|$ for some positive constants c_1, c_2, Lemma 2.3 gives blockwise estimates

$$\Psi_1(\omega^2, Q) - \Psi_0(\omega^2) = \begin{pmatrix} O(\omega^{-1}e^{|\Im\omega|}) & O(\omega^{-2}e^{|\Im\omega|}) \\ O(e^{|\Im\omega|}) & O(\omega^{-1}e^{|\Im\omega|}) \end{pmatrix}$$

with respect to either matrix norm. Break the $2K \times 2K$ function $F_0(\omega)$ into $K \times K$ blocks

$$F_0(\omega) = \begin{pmatrix} f_{11}(\omega) & f_{12}(\omega) \\ f_{21}(\omega) & f_{22}(\omega) \end{pmatrix},$$

and it follows that $F_0(\omega)$ has exponential type 1 as a function of ω. The entire matrix functions $f_{11}(\omega)$, $f_{22}(\omega)$ and $\omega f_{12}(\omega)$ are square integrable over the real axis. A modification

$$\tilde{f}_{21}(\omega) = \left[f_{21}(\omega) - f_{21}(0)\,\frac{\sin(\omega)}{\omega} \right] / \, \omega$$

of $f_{21}(\omega)$ will also be entire of type 1 and square integrable over the real axis.

Functions $G(\omega) \in \mathcal{PW}$ have an important characterization in terms of sampling [**19**, p. 150].

Each entire function $G(\omega) \in \mathcal{PW}$ admits the representation

$$G(\omega) = \sum_{j=-\infty}^{\infty} A_j \frac{\sin(\omega)}{\omega - j\pi}, \tag{13}$$

where

$$A_j = (-1)^j G(j\pi),$$

and

$$\frac{1}{\pi} \int_{-\infty}^{\infty} |G(\omega)|^2 \, d\omega = \sum_{j=-\infty}^{\infty} |G(j\pi)|^2.$$

The functions $\{\sin(\omega)/(\omega - j\pi)\}$ form an orthonormal basis for $\mathcal{PW}$, so (13) *describes a diffeomorphism of $\mathcal{PW}$ and $l^2(M_K)$.*

Theorem 3.2 already establishes that the samples $f_{11}(n\pi)$ grow more slowly than samples of the general element of $\mathcal{PW}$. Let $\mathcal{PW}_1$ denote the set of functions $G(\omega) \in \mathcal{PW}$ satisfying

$$\sum_{j=-\infty}^{\infty} (j\pi)^2 |G(j\pi)|^2 < \infty.$$

We can make $\mathcal{PW}_1$ into a Hilbert space with the inner product

$$\langle G(\omega), H(\omega) \rangle = \operatorname{trace} H^*(0)G(0) + \sum_{j=-\infty}^{\infty} (j\pi)^2 H^*(j\pi)G(j\pi).$$

Let $\mathcal{PW}_1^0$ be the subspace of $\mathcal{PW}_1$ consisting of functions vanishing at 0.

Theorem 4.1. *Suppose that Q_0 is fixed. The map*

$$P \longrightarrow f_{11}(\omega) + i\omega f_{12}(\omega)$$

takes $L_0^2([0,1], M_K)$ into the Paley-Wiener space $\mathcal{PW}_1$. If Q_0 is sufficiently close to 0_K, the map

$$P \longrightarrow f_{11}(\omega) - f_{11}(0)\frac{\sin(\omega)}{\omega} + i\omega f_{12}(\omega)$$

is an analytic diffeomorphism from a neighborhood of $0 \in L_0^2([0,1], M_K)$ to a neighborhood of 0 in $\mathcal{PW}_1^0$.

Proof. The function $f_{11}(\omega) + i\omega f_{12}(\omega)$ is in $\mathcal{PW}_1$ by the estimates of Lemma 2.3 and Lemma 3.1. Let

$$g(\omega) = f_{11}(\omega) - f_{11}(0)\frac{\sin(\omega)}{\omega} + i\omega f_{12}(\omega).$$

Subtraction of $f_{11}(0)\sin(\omega)/\omega$ projects $f_{11}(\omega) + i\omega f_{12}(\omega)$ into $\mathcal{PW}_1^0$.

Since the evaluations $n\pi g(n\pi)$ are generalized Fourier coefficients for $g(\omega)$ with respect to the orthonormal basis

$$\frac{(-1)^n \sin(\omega)}{n\pi(\omega - n\pi)}, \quad n = 1, 2, 3, \ldots$$

for $\mathcal{PW}_1^0$, the arguments of Theorem 3.2 apply to show that the map is analytic and the derivative at $Q = 0$ is the linear map

$$H \in L_0^2 \longrightarrow \sum_{n=1}^{\infty} \bigg[-\frac{1}{2} \int_0^1 H(x) \sin(2\pi n x)\, dx + \frac{i}{2} \int_0^1 H(x) \cos(2\pi n x)\, dx \bigg] \frac{(-1)^n \sin(\omega)}{n\pi(\omega - n\pi)}.$$

This map is a linear diffeomorphism from $L_0^2([0,1], M_K)$ to $\mathcal{PW}_1^0$, so applying the inverse function theorem completes the proof. $\square$

The functions

$$f_{11}(\omega) = C(1,\lambda) - \cos(\Omega), \quad f_{12}(\omega) = S(1,\lambda) - \Omega^{-1}\sin(\Omega)$$

are entire functions of $\lambda = \omega^2$, so are even functions of ω. It follows that $\omega f_{12}(\omega)$ is an odd function of ω. We consider the cases Q even, when the periodic extension of Q to the line satisfies $Q(x) = Q(-x)$ and Q odd, when $Q(x) = -Q(-x)$. In terms of their values on the unit interval, a periodic function Q is even if $Q(x) = Q(1-x)$ and odd if $-Q(x) = Q(1-x)$. The symmetry just noted above for $f_{11}(\omega)$ and $\omega f_{12}(\omega)$ means that we may consider the maps from even functions Q to odd functions $\omega f_{12}(\omega)$, or from odd functions Q to even functions $f_{11}(\omega)$ in $\mathcal{PW}_1$. The argument of the previous theorem now extends to give the following result.

Theorem 4.2. *Fix Q_0. The map*

$$P \longrightarrow \omega f_{12}(\omega)$$

takes even $P(x) \in L_0^2([0,1], M_K)$ to odd functions in $\mathcal{PW}_1$. If Q_0 is sufficiently close to 0_K, this map is an analytic diffeomorphism from a neighborhood of 0 in the even functions in $L_0^2([0,1], M_K)$ to a neighborhood of 0 in the odd functions in $\mathcal{PW}_1^0$.

The map

$$P \longrightarrow f_{11}(\omega)$$

takes odd $P(x) \in L^2_0([0,1], M_K)$ *to even functions in* $\mathcal{PW}_1$*. If* Q_0 *is sufficiently close to* 0_K*, the image of* $\partial_P f_{11}$ *acting on odd functions* P *has codimension* K^2 *in the space of even functions, and the map*

$$P \longrightarrow f_{11}(\omega) - f_{11}(0)\frac{\sin(\omega)}{\omega}$$

is an analytic diffeomorphism from a neighborhood of 0 *in the odd functions in* $L^2_0([0,1], M_K)$ *to a neighborhood of* 0 *in the even functions in* $\mathcal{PW}_1^0$.

5. Applications. With a few notable exceptions [**2, 5, 10, 14, 29**], it appears that very little is known about the set of $K \times K$ matrix potentials $Q(x)$ with a given set of spectral data when $K > 1$. Here we apply the results of the previous section to show that isospectral sets are small if the spectral data is 'similar to' the data of a scalar problem and the potential is near 0_K. Conditions on the Floquet multipliers are considered first, followed by results on Dirichlet eigenvalues for even matrix potentials.

5.1 Periodic problems. The complex number λ is a periodic or anti-periodic eigenvalue for (1), or equivalently there are Floquet multipliers ± 1 at $\lambda \in \mathbf{C}$ if and only if

$$\det[\Psi_1(\lambda) \mp I_{2K}] = 0.$$

In the simple case $Q = 0$ the periodic eigenvalues are $\lambda = 0$ with multiplicity K and $\lambda = (2n\pi)^2$ with multiplicity $2K$ for $n = 1, 2, 3, \dots$, while the anti-periodic eigenvalues are $\lambda = ([2n-1]\pi)^2$ with multiplicity $2K$. The estimates of Lemma 3.1 may be used to show that, if $\|Q\|_2$ is sufficiently small, then the roots of the entire functions $\det[\Psi_1(\lambda) \mp I_{2K}]$, counted with multiplicity, will remain inside disks D_n of radius 1 centered at $n^2\pi^2$. Moreover, the dimensions of the generalized periodic and anti-periodic eigenspaces within the disks D_n will not change. (See [**4**, pp. 221–222] for a related argument, and particularly [**17**, p. 379].)

We single out a class of potentials Q with highly degenerate periodic and anti-periodic eigenvalues. Say that Q satisfies condition $\mathcal{C}_g$ if the Floquet matrix is $(-1)^n I_{2K}$ at $\lambda_n \in D_n$ for $n > g$. In the scalar case when $q(x)$ is a real-valued function, the condition $\mathcal{C}_g$ implies that $q(x)$

is a finite gap potential. Moreover, inside any ball $\|q\|_2 < \varepsilon$ there are scalar potentials satisfying C_g with exactly g gaps [**13**], which lie in $D_1, \ldots, D_g$.

Suppose that $p_1(x), \ldots, p_K(x)$ are g-gap scalar potentials with the same spectrum, small norm, and gaps in $D_1, \ldots, D_g$. Then the operator $-D^2 + \mathrm{diag}\,[p_1, \ldots, p_K]$ will satisfy the condition $\mathcal{C}_g$. Other examples with the same spectral properties are obtained by conjugating with a constant invertible matrix.

Theorem 5.1. *Suppose that Q_0 is fixed and for $n > g$ the sequence $\lambda_n = n^2\pi^2 + \mu_n$ satisfies $\sum_{n>g} |\mu_n|^2 < \varepsilon$ for a sufficiently small $\varepsilon > 0$. Then the set of potentials $Q(x) = Q_0 + P(x)$ with $\|Q\|_2$ sufficiently small, and with Floquet matrix $(-1)^n I_{2K}$ at λ_n for $n > g$, is contained in a complex manifold of dimension at most $2gK^2$.*

Proof. Define $\mu_n = 0$ for $n = 1, \ldots, g$. The condition that $\sum_{n>g} |\mu_n|^2$ is sufficiently small means that the sequence $n^2\pi^2 + \mu_n$ is the Dirichlet spectrum for a scalar potential $q(x) \in L^2[0,1]$ with $\|q\|_2$ small. This is explicitly proven for real sequences μ_n and real potentials q in [**24**, p. 53], but the same inverse function theorem arguments extend to complex potentials in $L^2[0,1]$ and complex sequences $\{\mu_n\}$ near 0 in l^2.

The Dirichlet eigenvalues $n^2\pi^2 + \mu_n$ are the roots of the function $S(1, \lambda, q)$. Define $s(\omega) = S(1, \omega^2, q)$. Let $\omega_0 = 0$ and, for $n = 1, 2, 3, \ldots$, let $\omega_{\pm n} = \pm\sqrt{n^2\pi^2 + \mu_n}$. In the terminology of [**19**, p. 163], the function $\omega s(\omega)$ is a sine-type function, with indicator diagram of width 2. This implies [**19**, p. 165] that evaluation of functions in $\mathcal{PW}$ at the sequence $\{\omega_j\}$ is an isomorphism of $\mathcal{PW}$ to $l^2(M_K)$.

The Floquet matrix of Q is $(-1)^n I_{2K}$ at λ_n for $n > g$. We thus know

$$F_0(\omega_j, Q) = \Psi_1(\omega_j^2, Q) - \Psi_0(\omega_j^2), \quad |j| > g,$$

and in particular the values of the even function $f_{11}(\omega)$ and the odd function $\omega f_{12}(\omega)$ at ω_j for $|j| > g$. Since these functions are in $\mathcal{PW}$ they are determined by the additional values $f_{11}(\omega_j)$ for $0 \le j \le g$ and $\omega_j f_{12}(\omega_j)$ for $1 \le j \le g$. This data is a set of $2g + 1$ additional $K \times K$ complex matrices.

By Theorem 4.1 the potential P is actually determined by the orthogonal projection of the function $f_{11}(\omega) + i\omega f_{12}(\omega)$ onto $\mathcal{PW}_0$, so the $K \times K$ matrix $f_{11}(0)$ is superfluous. Thus the matrices $f_{11}(j\pi) + i\omega f_{12}(j\pi)$ for $j = 1, \dots, g$ provide coordinates for a set which includes any of the functions $P(x)$. $\square$

5.2 Even potentials and Dirichlet eigenvalues. Here we consider the implications of the identity $Q(x) = Q(1-x)$. First we make some simple observations about symmetry. If a potential is even and $Y(x,\lambda)$ solves (1), then so does $Y(1-x,\lambda)$. In particular, the matrix-valued functions $C(1-x,\lambda)$ and $S(1-x,\lambda)$ will satisfy (1). The uniqueness theorem and evaluation at $x = 0$ give

$$\begin{aligned} C(1-x,\lambda) &= C(x,\lambda)C(1,\lambda) - S(x,\lambda)C'(1,\lambda), \\ S(1-x,\lambda) &= C(x,\lambda)S(1,\lambda) - S(x,\lambda)S'(1,\lambda). \end{aligned}$$

Evaluation at $x = 1$ gives

$$\begin{aligned} I_K &= C^2(1,\lambda) - S(1,\lambda)C'(1,\lambda), \\ 0_K &= C(1,\lambda)S(1,\lambda) - S(1,\lambda)S'(1,\lambda). \end{aligned} \tag{14}$$

Evaluation of the derivative at $x = 1$ gives

$$\begin{aligned} 0_K &= C'(1,\lambda)C(1,\lambda) - S'(1,\lambda)C'(1,\lambda), \\ -I_K &= C'(1,\lambda)S(1,\lambda) - S'(1,\lambda)S'(1,\lambda). \end{aligned} \tag{15}$$

These identities lead to the following matrix generalization of a theorem which is well known in the scalar case.

Theorem 5.2. *If $Q(x) = Q(1-x)$, then λ is a Dirichlet or Neumann eigenvalue if and only if it is a periodic or anti-periodic eigenvalue.*

Proof. A computation gives

$$\begin{aligned} &\Psi_1^2 \\ &= \begin{pmatrix} C^2(1,\lambda)+S(1,\lambda)C'(1,\lambda) & C(1,\lambda)S(1,\lambda)+S(1,\lambda)S'(1,\lambda) \\ C'(1,\lambda)C(1,\lambda)+S'(1,\lambda)C'(1,\lambda) & C'(1,\lambda)S(1,\lambda)+(S')^2(1,\lambda) \end{pmatrix} \\ &= I_{2K} + 2\begin{pmatrix} S(1,\lambda)C'(1,\lambda) & C(1,\lambda)S(1,\lambda) \\ S'(1,\lambda)C'(1,\lambda) & C'(1,\lambda)S(1,\lambda) \end{pmatrix} \\ &= I_{2K} + 2\begin{pmatrix} S(1,\lambda) & C(1,\lambda) \\ S'(1,\lambda) & C'(1,\lambda) \end{pmatrix}\begin{pmatrix} C'(1,\lambda) & 0 \\ 0 & S(1,\lambda) \end{pmatrix}. \end{aligned}$$

Having a periodic or anti-periodic eigenvalue at λ means that the matrix $\Psi_1(\lambda)$ has ± 1 as an eigenvalue. If

$$\begin{pmatrix} V \\ W \end{pmatrix}$$

is an eigenvector, the above identity for Ψ^2 leads to

$$\begin{pmatrix} S(1,\lambda) & C(1,\lambda) \\ S'(1,\lambda) & C'(1,\lambda) \end{pmatrix} \begin{pmatrix} C'(1,\lambda) & 0 \\ 0 & S(1,\lambda) \end{pmatrix} \begin{pmatrix} V \\ W \end{pmatrix} = 0$$

and, since

$$\begin{pmatrix} S(1,\lambda) & C(1,\lambda) \\ S'(1,\lambda) & C'(1,\lambda) \end{pmatrix}$$

is invertible, either $C'(1,\lambda)V = 0$ or $S(1,\lambda)W = 0$. But these are the conditions for a Dirichlet or Neumann eigenvalue. Conversely, if $C'(1,\lambda)V = 0$ for $V \neq 0$, then

$$\begin{pmatrix} V \\ 0 \end{pmatrix}$$

is an eigenvector for $\Psi_1^2(\lambda)$ with eigenvalue 1, and similarly for the case $S(1,\lambda)W = 0$. $\square$

In the real scalar case, we know that even potentials are uniquely determined by the Dirichlet spectrum. Here are two versions for the matrix case: the first does not require that Q be self-adjoint. A similar question was considered in [**29**]. Rather than requiring Q to be near 0, the main hypotheses in [**29**] are that Q is even, real symmetric and $Q_{ij}(x) = 0$ if $|i-j| \geq 2$.

Theorem 5.3. *Suppose that $\{\lambda_n\}$ is a sequence of complex numbers of the form $\lambda_n = n^2\pi^2 + \mu_n$ with $\sum |\mu_n|^2 < \varepsilon$ for a sufficiently small $\varepsilon > 0$. Then there is an open neighborhood U_ε of 0_K in $L^2([0,1], M_K)$ in which there exists a unique even $K \times K$ matrix potential $Q = Q_0 + P \in L^2([0,1], M_K)$ whose Dirichlet eigenvalues are precisely $\{\lambda_n\}$, each eigenvalue having geometric multiplicity K. The matrix potential has the form $Q(x) = q(x)I_K$, where $q \in L^2[0,1]$.*

Proof. As we noted in Theorem 5.1, there is an even function $q \in L^2[0,1]$ with $qI_K \in U_\varepsilon$ which has $\{\lambda_n\}$ as its Dirichlet spectrum. Suppose that $Q = Q_0 + P$ is another such matrix potential. Then the estimates for $S(1,\lambda)$ in Lemma 3.1 imply that the diagonal matrix Q_0 is 0_K.

For $n = 1,2,3,\dots$, let $\omega_{\pm n} = \pm\sqrt{\lambda_n}$ and let $\omega_0 = 0$. If the Dirichlet eigenvalues of (1) have geometric multiplicity K, then

$$\omega_j S(1,\omega_j^2, qI_K) = \omega_j S(1,\omega_j^2, Q) = 0_K, \quad j = 0, \pm 1, \pm 2, \dots .$$

If $s(\omega) = S(1,\omega^2,q)$, the function $\omega s(\omega)$ is again a sine-type function. Since the functions

$$\omega f_{12}(\omega, qI_K) = \omega S(1,\omega^2, qI_K) - \sin(\omega) I_K \in \mathcal{PW}$$

and

$$\omega f_{12}(\omega, Q) = \omega S(1,\omega^2, Q) - \sin(\omega) I_K \in \mathcal{PW}$$

agree at the roots of $\omega s(\omega)$, they are the same.

Finally, Theorem 4.2 shows that $Q = qI_K$. $\square$

If $Q(x) = Q^*(x)$, additional information is available with the aid of a result from [**6**]. Recall that the Dirichlet spectra of real even functions $q \in L^2[0,1]$ are completely characterized in [**24**, p. 115]. Notice that if $S(1,\lambda_n,Q) = 0_K$, then (14) gives $C^2(1,\lambda_n,Q) = I_K$.

Theorem 5.4. *Suppose that the real sequence $\lambda_1 < \lambda_2 < \cdots$ is the set of Dirichlet eigenvalues for* (i) *a real even function $q \in L^2[0,1]$ and* (ii) *an even potential $Q \in L^2([0,1], M_K)$ satisfying $Q(x) = Q^*(x)$ with each eigenvalue λ_n having geometric multiplicity K. Then $Q(x) = q(x)I_K$ if and only if $C(1,\lambda_n,Q) = (-1)^n I_K$ and $C(1,0,Q) = C(1,0,q)I_K$.*

Proof. Since each eigenvalue λ_n has geometric multiplicity K, $S(1,\lambda_n,Q) = 0_K$. Taking λ real, (5) implies the matrix Wronskian identity

$$(S^*)'(1,\lambda,Q)C(1,\lambda,Q) - S^*(1,\lambda,Q)C'(1,\lambda,Q) = I_K.$$

Evaluation at λ_n gives

$$(S^*)'(1,\lambda_n,Q)C(1,\lambda_n,Q) = I_K.$$

In addition, (14) gives

$$C^2(1,\lambda_n,Q) = I_K,$$

so that

$$(S^*)'(1,\lambda_n,Q) = C(1,\lambda_n,Q).$$

Suppose that $Q(x) = q(x)I_K$. One trivially finds $C(1,0,Q) = C(1,0,q)I_K$. In the scalar case it is known [**24**, p. 41] that $S(x,\lambda_n,q)$ has exactly $n+1$ simple roots in $[0,1]$. This implies that $S'(1,\lambda_n,q) \neq 0$ and has the same sign as $(-1)^n$. Since q is even and real, $S'(1,\lambda_n,q)$ is real and (15) gives $S'(1,\lambda_n,q)^2 = 1$, so

$$C(1,\lambda_n,Q) = C(1,\lambda_n,q)I_K = S'(1,\lambda_n,q)I_K = (-1)^n I_K.$$

Now suppose that $C(1,\lambda_n,Q) = (-1)^n I_K$ and $C(1,0,Q) = C(1,0,q)I_K$.

Define $s(\omega) = S(1,\omega^2,q)$ and let $\bar{q} = \int_0^1 q$. Since the Dirichlet spectrum of Q agrees with that of $q(x)I_K$, the estimates for $S(1,\lambda,Q)$ in Lemma 3.1 imply that the diagonal matrix Q_0 is $\bar{q}I_K$. The functions

$$\omega S(1,\omega^2,qI_K) - \omega S(1,\omega^2,\bar{q}I_K) \in \mathcal{PW}$$

and

$$\omega S(1,\omega^2,Q) - \omega S(1,\omega^2,\bar{q}I_K) \in \mathcal{PW}$$

are the same since they agree at the roots ω_j of the sine-type function $\omega s(\omega)$. Thus $S(1,\omega^2,qI_K) = S(1,\omega^2,Q)$.

Since $\omega_j^2 = \lambda_{|j|}$ for $|j| \geq 1$ and $C(1,0,Q) = C(1,0,q)I_K$, the functions

$$C(1,\omega^2,qI_K) - C(1,\omega^2,\bar{q}I_K) \in \mathcal{PW},$$

and

$$C(1,\omega^2,Q) - C(1,\omega^2,\bar{q}I_K) \in \mathcal{PW},$$

agree at each of the roots ω_j of $\omega s(\omega)$. Thus, $C(1,\omega^2,qI_K) = C(1,\omega^2,Q)$.

Finally, the main result of [**6**] says that Q is uniquely determined by $C(1,\lambda)$ and $S(1,\lambda)$. $\square$

The analysis of Q near 0_K suggests that the condition $C(1,0,Q) = C(1,0,q)I_K$ may be superfluous.

REFERENCES

1. Z. Agranovich and V. Marchenko, *The inverse problem of scattering theory*, Gordon and Breach, New York, 1963.

2. E. Belokolos, F. Gesztesy, K. Makarov and L. Sakhnovich, *Matrix-valued generalizations of the theorems of Borg and Hochstadt*, in *Evolution Equations* (G. Ruiz Goldstein, R. Nagel and S. Romanelli, eds.), Lecture Notes in Pure and Appl. Math., vol. 234, Marcel Dekker, New York, 2003, pp. 1–34.

3. R. Carlson, *Large eigenvalues and trace formulas for matrix Sturm-Liouville problems*, SIAM J. Math. Anal. **30** (2000), 949–962.

4. ———, *Eigenvalue estimates and trace formulas for the matrix Hill's equation*, J. Differential Equations **167** (2000), 211–244.

5. ———, *Compactness of Floquet isospectral sets for the matrix Hill's equation*, Proc. Amer. Math. Soc. **128** (2000), 2933–2941.

6. ———, *An inverse problem for the matrix Schrödinger equation*, J. Math. Anal. Appl. **267** (2002), 564–574.

7. K. Chadan and P.C. Sabatier, *Inverse problems in quantum scattering theory*, Springer-Verlag, New York, 1989.

8. E.A. Coddingtion and R. Carlson, *Linear ordinary differential equations*, SIAM, Philadelphia, 1997.

9. E.A. Coddingtion and N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill, New York, 1955.

10. B. Després, *The Borg theorem for the vectorial Hill's equation*, Inverse Problems **11** (1995), 97–121.

11. B. Dubrovin, V. Matveev and S. Novikov, *Nonlinear equations of Korteweg-de Vries type, finite zoned linear operators, and Abelian varieties*, Russian Math. Surveys **31** (1976), 59–146.

12. A. Finkel, E. Isaacson and E. Trubowitz, *An explicit solution of the inverse periodic problem for Hill's equation*, SIAM J. Math. Anal. **18** (1987), 46–53.

13. J. Garnett and E. Trubowitz, *Gaps and bands of one dimensional periodic Schrödinger operators*, Comment. Math. Helv. **59** (1984), 258–321.

14. F. Gesztesy and L. Sakhnovich, *A class of matrix-valued Schrödinger operators with prescribed finite-band spectra*, in *Reproducing Kernel Hilbert Spaces, Positivity, System Theory and Related Topics* (D. Alpay, ed.), Oper. Theory Adv. Appl., vol. 143, Birkhaüser, Basel, 2003, pp. 213–253.

15. M. Jodeit and B. Levitan, *Isospectral vector-valued Sturm-Liouville problems*, Lett. Math. Phys. **43** (1998), 117–122.

16. T. Kappeler, *Fibration of the phase space for the Korteweg-De Vries equation*, Ann. Institute Fourier **41** (1991), 539–575.

17. T. Kato, *Perturbation theory for linear operators*, Springer-Verlag, New York, 1995.

18. I. Krichever and S. Novikov, *Periodic and almost-periodic potentials in inverse problems*, Inverse Problems **15** (1999), R117–R144.

19. B. Levin, *Lectures on entire functions*, Amer. Math. Soc., Providence, 1996.

20. B. Levitan and M. Gasymov, *Determination of a differential equation by two of its spectra*, Russian Math. Surveys **19** (1964), 1–63.

21. H.P. McKean and E. Trubowitz, *Hill's operator and hyperelliptic function theory in the presence of infinitely many branch points*, Comm. Pure Appl. Math. **29** (1976), 143–226.

22. H.P. McKean and P. van Moerbeke, *The spectrum of Hill's equation*, Invent. Math. **30** (1975), 217–274.

23. R. Paley and N. Wiener, *Fourier transforms in the complex domain*, Amer. Math. Soc., Providence, 1934.

24. J. Pöschel and E. Trubowitz, *Inverse spectral theory*, Academic Press, Orlando, 1987.

25. J. Ralston and E. Trubowitz, *Isospectral sets for boundary value problems on the unit interval*, Ergodic Theory Dynamical Systems **8** (1988), 301–358.

26. M. Reed and B. Simon, *Methods of modern mathematical physics*, Vol. 1, Academic Press, New York, 1972.

27. W. Rudin, *Real and complex analysis*, McGraw-Hill, New York, 1966.

28. J. Schwartz, *Non-linear functional analysis*, Gordon and Breach, New York, 1969.

29. C. Shen and C. Shieh, *Two inverse eigenvalue problems for vectorial Sturm-Liouville equations*, Inverse Problems **14** (1998), 1331–1343.

30. E. Trubowitz, *The inverse problem for periodic potentials*, Comm. Pure Appl. Math. **30** (1977), 321–337.

31. M. Wadati and T. Kamijo, *On the extension of inverse scattering method*, Progress Theoret. Physics **52** (1974), 397–414.

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