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A CHARACTERIZATION OF STRONGLY WEAKLY COMPACTLY GENERATED BANACH SPACES

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ABSTRACT. A Banach space X is strongly weakly compactly generated if and only if there is a reflexive space R and a continuous linear operator $T : R \to X$ such that $T(B_R)$ almost absorbs every weakly compact set in X.

1. Introduction and notation. In 1988, Schlüchtermann and Wheeler introduced a property closely related to the weakly compactly generated property of Banach spaces [**3**, Theorem 2.1].

Definition. We call a Banach space X *strongly weakly compactly generated*, or SWCG, if X contains a weakly compact subset K such that if L is any other weakly compact subset and $\varepsilon > 0$, then there is a positive integer n so that $L \subset nK + \varepsilon B_X$ where $B_X = \{x \in X : \|x\| \leq 1\}$. (When two subsets K and L satisfy this condition, we say that K almost absorbs L.)

In short, then, a Banach space X is SWCG if it contains a weakly compact subset that almost absorbs every other weakly compact set in X. Not surprisingly, the reason for the name is that the SWCG property is a slight strengthening in the conditions of the weakly compactly generated (WCG) property. Every SWCG space is also WCG, but the two properties are distinct: c_0 is an example of a space that is WCG but not SWCG.

This paper provides an alternate characterization of the SWCG property which is analogous to the following well-known WCG result of Davis, Figiel, Johnson and Pełczyński [**1**, Corollary 3].

Proposition 1. *A Banach space X is* WCG *if and only if there is a reflexive space R and a one-to-one operator $T : R \to X$ with $T(B_R)$ dense in X.*

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2. The characterization.

Theorem 1. *A Banach space X is strongly weakly compactly generated if and only if there is a reflexive space R and a continuous linear operator $T : R \to X$ such that $T(B_R)$ almost absorbs every weakly compact set in X.*

Proof. To begin, we observe that the theorem is trivial if X is reflexive, so we assume that X is nonreflexive. Let K be a weakly compact set in X that almost absorbs every weakly compact set in X. Let $W = \overline{\mathrm{conv}\,(K \cup (-K))}$. Then it follows from the Krein-Smulian theorem [**2**, p. 434] that W is weakly compact and W is clearly symmetric and convex. Also, W still almost absorbs every weakly compact set in X since it is a superset of K. Let $U_n = 2^n W + 2^{-n} B_X$ and define a functional on X by

$$\|x\|_n = \inf\{t > 0 : x \in tU_n\}.$$

Then $\|\cdot\|_n$ is a norm on X for all n [**4**]. Define a new space R by

$$R = \left\{ x \in X : |||x||| = \left(\sum_{n=1}^{\infty} \|x\|_n^2 \right)^{1/2} < \infty \right\}.$$

By definition, R can be identified with the subspace of $(\sum_{n=1}^{\infty}(X, \|\cdot\|_n))_{l_2}$ of constant sequences, since we are placing the same element from X in each coordinate. Clearly a subspace of constant sequences in the l_2-sum is closed, and it follows that R is a Banach space.

Let T be the formal identity from $(R, |||\cdot|||)$ to $(X, \|\cdot\|)$. Since $\|\cdot\|_n$ is equivalent to the standard norm on X for all n, T is a bounded linear operator.

If $w \in W$, then $w \in 2^{-n} U_n$ so $\|w\|_n \le 2^{-n}$ for all n. Thus,

$$|||w||| \le \left(\sum_{n=1}^{\infty} (2^{-n})^2 \right)^{1/2} < 1$$

and so $w \in T(B_R)$. Thus $W \subset T(B_R)$ and, since W almost absorbs every weakly compact set in X, it follows that $T(B_R)$ almost absorbs every weakly compact set in X as required. It is shown in [**1**, Lemma 1] that R is reflexive, completing the first half of the proof.

Conversely, if we have a reflexive space R and a linear operator T as in the hypotheses of the theorem, then B_R is weakly compact by the Banach-Alaoglu theorem and so $T(B_R)$ is weakly compact in X. But, by hypothesis, $T(B_R)$ almost absorbs every weakly compact set in X, and so X is strongly weakly compactly generated.

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