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REAL GENUS 12

BEATA MOCKIEWICZ

ABSTRACT. The *real genus* of a finite group G is the minimum algebraic genus of any compact bordered Klein surface on which G acts as a group of automorphisms. This notion was introduced by May at the beginning of the 1990s. For infinitely many integers p, the number of groups of the real genus p is positive. There is no group of the real genus 2 and so far the number 2 has been the only known integer with such property. In this note we show that additionally there is no group of the real genus 12.

1. Introduction. Let G be a finite group. There is a bordered Klein surface X on which the group G acts as a group of automorphisms [**1**]. The *real genus* $\rho(G)$ of G is the minimum algebraic genus of any such surface. The systematic study of the real genus was initiated by May in [**8**]. He has computed the real genus of all groups of order smaller than 32 [**9**] and the real genera for some important infinite families of groups [**8**]. All groups of the real genus ranging between 0 and 8 are known at present. The only groups of the real genus 0 are Z_n and D_n, all groups of the real genus 1 are $Z_2 \times D_n$ with even n and $Z_2 \times Z_n$ with even $n \geq 4$ and, surprisingly, there are no groups of the real genus 2 [**8**]. The groups S_4 and A_4 are the only groups of the real genus 3 [**8**], there are four groups of the real genus 4 [**7**] and nine groups of the real genus 5 [**9**]. Recently we have found in [**5**] all groups of the real genera 6, 7 and 8; there are 4, 3 and 2 such groups, respectively. The number of groups of the real genus p for each integer $p \geq 2$ is finite and for infinitely many integers $p > 2$ this number is positive [**8**]. A natural problem, recognized by May as intriguing [**8**], was to decide whether there exists an integer $p \neq 2$ which cannot stand as the real genus of some finite group G. May has proved himself in [**8**] that such an integer cannot be odd, showing that for $n > 3$ the dicyclic group $\langle x, y | x^{2n}, x^n y^{-2}, y^{-1}xyx \rangle$ has the real genus $1 + 2n$. In this paper we

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prove that there are no groups of the real genus 12. Of course it does not mean that there are no groups acting on surfaces of genus 12. What it really does mean is that every such group acts also on a surface of a smaller genus. What made this result still more surprising is the recent paper of May and Zimmerman [**11**], where the authors show that the groups of arbitrary strong symmetric genus exist.

2. Arithmetic restrictions. In our study we will use a standard representation of a finite group G as a quotient Λ/Γ for some NEC groups Λ and Γ [**2, 3, 8**]. The algebraic structure of an NEC group Λ is determined by the signature $\sigma(\Lambda)$, being a sequence of numbers and symbols of the form

$$(1) \quad \sigma = (h; \pm; [m_1, \ldots, m_r]; \{(n_{11}, \ldots, n_{1s_1}), \ldots, (n_{k1}, \ldots, n_{ks_k})\}).$$

A general presentation of Λ with signature (1) can be found in [**3**], for example. We do not give it here for two simple reasons; it is rather complicated in general while we shall deal with rather special signatures. Instead we shall provide the presentations for considered cases. The hyperbolic area of any fundamental region for Λ with signature (1) equals

$$\mu(\Lambda) = 2\pi\left(\alpha h + k - 2 + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right) + \sum_{i=1}^{k}\sum_{j=1}^{s_i}\frac{1}{2}\left(1 - \frac{1}{n_{ij}}\right)\right),$$

where $\alpha = 1$ or 2 according to whether the sign in the signature is $-$ or $+$, respectively. A group Γ normalize a surface X on which G acts. It has signature $(g; \pm; [-]; \{(-), \overset{k'}{\ldots}, (-)\})$, with $k' \geq 1$ and is called a *bordered surface group.* The quotient Λ/Γ is said to be a *smooth factor* of Λ. Important here is the Hurwitz-Riemann formula which says that $[\Lambda : \Gamma] = \mu(\Gamma)/\mu(\Lambda)$.

Assume then that G is a group of the real genus 12. Then $\mu(\Gamma) = 22\pi$. By [**9**], $|G| \geq 32$ and by [**8**], $|G| \leq 132$. The first gives $\mu(\Lambda) \leq 11\pi/16$ and the second $\mu(\Lambda) \geq \pi/6$ by virtue of the Riemann-Hurwitz formula. In particular, we see that we have to deal with NEC groups Λ having a small area of fundamental region and so having rather simple signatures; for example, $k \geq 1$ and $\alpha h + k \leq 2$. However, not all arithmetically admissible signatures are involved. For example the signature of Λ must contain an empty period cycle or a period cycle with two consecutive link periods n_{ij} equal to 2 [**4**]. On the other hand, all quadrilateral groups $(0;+;[-];\{(2,2,m,n)\})$ admit smooth factors, e.g., [**4**], though such a factor must contain dihedral subgroups D_n and D_m which lead to further limitations. For example, for $m=2$, $|G| = 44n/(n-2)$ and therefore $n \in \{3,4,13\}$ since $|G|$ is an integer, and for $n=24$, $G = D_n$ has the real genus 0.

Similar arguments, cf. [**3**, Lemma 4.1.1] leads to the following possible signatures of Λ; useful here is for instance an observation that $|G| \notin \{33,35,37,41,43\}$ since in such cases G is cyclic.

Case	$\sigma(\Lambda)$	$\mu(\Lambda)$	$\lvert G\rvert$
(1)	$(0;+;[-];\{(2,2,2,3)\})$	$\pi/6$	132
(2)	$(0;+;[-];\{(2,2,2,4)\})$	$\pi/4$	88
(3)	$(0;+;[-];\{(2,2,3,3)\})$	$\pi/3$	66
(4)	$(0;+;[2,3];\{(-)\})$	$\pi/3$	66
(5)	$(0;+;[3];\{(2,2)\})$	$\pi/3$	66
(6)	$(0;+;[-];\{(2,2,2,13)\})$	$11\pi/26$	52
(7)	$(0;+;[2,4];\{(-)\})$	$\pi/2$	44
(8)	$(0;+;[2];\{(2,2,2)\})$	$\pi/2$	44
(9)	$(0;+;[4];\{(2,2)\})$	$\pi/2$	44
(10)	$(0;+;[-];\{(2,2,2,2,2)\})$	$\pi/2$	44
(11)	$(0;+;[-];\{(-),(2)\})$	$\pi/2$	44
(12)	$(0;+;[-];\{(2,2,3,7)\})$	$11\pi/21$	42
(13)	$(0;+;[-];\{(2,2,4,5)\})$	$11\pi/20$	40

Smooth factors of quadrilateral group Λ with $\sigma(\Lambda) = (0;+;[-];\{(2,2,m,n)\})$ are known to have simple presentation [**3**, p. 99]. Namely, $G = \Lambda/\Gamma$ for bordered surface group Γ if and only if G can be generated by elements a,b,c of order 2 such that ab and bc have order m and n, respectively. In particular, such a G contains a dihedral subgroup of order $2n$ as we already mentioned.

Similar criteria can be easily established for the remaining signatures appearing above. For instance, in case (4) an NEC group Λ has the presentation $\langle x_1, x_2, c | x_1^2, x_2^3, c^2, (x_1x_2)^{-1}c(x_1x_2)c\rangle$. Clearly $c \in \Gamma$ since Γ contains a reflection and on the other hand all reflections of Λ are conjugated to c. Now x_1, x_2 represent in Λ/Γ generators of order 2 and 3, respectively, since otherwise Γ would have elliptic elements contrary to the assumption that it is a bordered surface group. Therefore, G is a group of order 66 generated by two elements of order 2 and 3. Observe, however, that different signatures may lead to the same type of presentation of G. For example, in case (5), Λ has the presentation $\langle x, c_0, c_1 | x^3, c_0^2, c_1^2, (c_0c_1)^2, (c_1xc_0x^{-1})^2\rangle$. We have $c_0 \in \Gamma$ or $c_1 \in \Gamma$. Suppose that $c_0 \in \Gamma$. Then $c_1 \notin \Gamma$ since otherwise c_0c_1 would be an orientation preserving element of order 2 in Γ, which is impossible. Similarly for $c_1 \in \Gamma$ we have $c_0 \notin \Gamma$. Thus G is generated by the images in Λ/Γ of x and c_i for $i = 0$ or $i = 1$. These generators have orders 3 and 2, respectively, since Γ is a bordered surface group. Therefore G is a group of order 66 generated by two elements of order 2 and 3 as in the case (4). In this way we obtain the following candidates for the smooth factors of the above listed NEC groups.

Cases	Order	The presentation type of G
(1)	132	$\langle a, b, c\rangle, \lvert a\rvert = \lvert b\rvert = \lvert c\rvert = \lvert ab\rvert = 2,\ \lvert bc\rvert = 3$
(2)	88	$\langle a, b, c, \rangle, \lvert a\rvert = \lvert b\rvert = \lvert c\rvert = \lvert ab\rvert = 2,\ \lvert bc\rvert = 4$
(3)	66	$\langle a, b, c\rangle, \lvert a\rvert = \lvert b\rvert = \lvert c\rvert = 2,\ \lvert ab\rvert = \lvert bc\rvert = 3$
(4),(5)	66	$\langle a, b\rangle, \lvert a\rvert = 2,\ \lvert b\rvert = 3$
(6)	52	$\langle a, b, c\rangle, \lvert a\rvert = \lvert b\rvert = \lvert c\rvert = \lvert ab\rvert = 2,\ \lvert bc\rvert = 13$
(7),(9)	44	$\langle a, b\rangle, \lvert a\rvert = 2,\ \lvert b\rvert = 4$
(8),(10)	44	$\langle a, b, c\rangle, \lvert a\rvert = \lvert b\rvert = \lvert c\rvert = \lvert bc\rvert = 2$
(10)	44	$\langle a, b, c, d\rangle, \lvert a\rvert = \lvert b\rvert = \lvert c\rvert = \lvert d\rvert = \lvert ab\rvert = 2$
(11)	44	$\langle a, b\rangle, \lvert a\rvert = 2$
(12)	42	$\langle a, b, c\rangle, \lvert a\rvert = \lvert b\rvert = \lvert c\rvert = 2,\ \lvert ab\rvert = 3,\ \lvert bc\rvert = 7$
(13)	40	$\langle a, b, c\rangle, \lvert a\rvert = \lvert b\rvert = \lvert c\rvert = 2,\ \lvert ab\rvert = 4,\ \lvert bc\rvert = 5$

3. Groups of the real genus 12. Now it can be proven that there are no groups of type (1), (2), (3), (4), (5), (7) and (9) and G is dihedral in cases (6), (8), (10), (11) and (12). Finally there is a unique non-dihedral group G of type (13). But in this group ac represents an element of order 4 and so G acts on a surface of algebraic genus $p = 11$, being a smooth factor of a group with signature $(0; +; [-]; \{(2, 2, 4, 4)\})$. These are straightforward but take space tasks, which require use of Sylow theorems and other standard group theory ideas. So instead of dealing with all cases we restrict ourselves just to consideration of a few samples, some of them kindly suggested by the referee, to illustrate the methodology.

Proposition 1. *There is no group of order* 66 *generated by two elements of order* 2 *and* 3*, respectively.*

Proof. Assume that $G = \langle a, b\rangle$ has order 66 and a, b are elements of order 2 and 3, respectively. Then by Sylow theory G has a normal subgroup H of order 11. Then $G/H = \langle \tilde{a}, \tilde{b}\rangle$ has order 6 and $\tilde{a}, \tilde{b}$ has order 2 and 3, respectively. Assume first that $G/H = D_3$. Then $\tilde{a}\tilde{b}$ has order 2 and $H = \langle (ab)^2\rangle$. Since $H \trianglelefteq G$, $a(ab)^2a = (ab)^{2\alpha}$ for some $1 \le \alpha \le 10$. Then $(ab)^2 = ((ab)^2)^{\alpha^2}$ and so 11 divides $\alpha^2 - 1$. Therefore, $\alpha = 1$ or $\alpha = 10$ and $a(ab)^2a = (ab)^2$ or $a(ab)^2a = (ab)^{-2}$, respectively. In the first case $(ba)^2 = (ab)^2$ and then $(ba)^6 = 1$, which is impossible. In the second one $b^2a = ab$ and $(ab)^2 = 1$, a contradiction. The case $G/H = Z_6$ can be ruled out similarly. □

Proposition 2. *There is no group G of order* 88 *generated by three elements a, b and c of order* 2 *such that ab and bc have order* 2 *and* 4*, respectively.*

Proof. Assume that $G = \langle a, b, c\rangle$ is such a group. There is a normal subgroup H of G of order 11. Therefore, $G/H = \langle \tilde{b}, \tilde{c}\rangle$ since $|G/H| = 8$. We obtain $\tilde{a} = (\tilde{b}\tilde{c})^2$ or $\tilde{a} = (\tilde{b}\tilde{c})^{\alpha}\tilde{b}$ for some $0 \le \alpha \le 3$. In the first case $H = \langle a(bc)^2\rangle$, and $ca(bc)^2c = a(bc)^2$ or $ca(bc)^2c = (a(bc)^2)^{-1}$. Therefore $\langle bc\rangle \trianglelefteq G$, which is impossible. In the second case we obtain $\alpha = 2$ and $H = \langle acbc\rangle$. Moreover $a(acbc)c = acbc$ or $c(acbc)c = cbca$. Then $\langle bc\rangle \trianglelefteq G$, a contradiction again. □

Proposition 3. *The only group G of order 52 generated by three elements a, b and c of order 2 such that ab and bc have orders 2 and 13, respectively, is D_{26}.*

Proof. Assume that G is such a group. Then by Sylow theory it contains a normal subgroup H of order 13. Therefore $H = \langle bc \rangle$ and so $abca = bc$ or $abca = cb$. In the first case $(ac)^2 = 1$ and $(abc)^2 \in H$. Then $|abc| = 26$ because $abc \neq 1$ and abc cannot have order 13. Now $\langle ab, c \rangle = D_{26}$ and $G = D_{26}$. In the second case $(abc)^2 = 1$ and $(ac)^2 \in H$. Therefore we obtain $G = \langle a, c \rangle = D_{26}$. □

Proposition 4. *Let G be a group of order 40 generated by three elements a, b and c of order 2 such that ab and bc have orders 4 and 5, respectively. Then $G = D_{20}$ or $G = \langle a, b, c | a^2, b^2, c^2, (ab)^4, (bc)^5, (ac)^4, abcacb \rangle$.*

Proof. Assume that G is such a group. Let $H = \langle bc \rangle$. Then $H \trianglelefteq G$ by Sylow theory and $|G/H| = 8$. On the other hand $G/H = \langle \tilde{a}, \tilde{b} | \tilde{a}^2, \tilde{b}^2, (\tilde{a}\tilde{b})^4 \rangle$. Since $\tilde{a}\tilde{c} = \tilde{a}\tilde{b}$, $\tilde{a}\tilde{c}$ has order 4. Therefore ac has order 4 or 20. In the second case $G = \langle a, c \rangle = D_{20}$. So let $|ac| = 4$. Since $H \trianglelefteq G$, we have $abca = bc$ or $abca = cb$. In the first case $G \cong G_1$ where $G_1 = \langle a, b, c | a^2, b^2, c^2, (ab)^4, (bc)^5, (ac)^4, abcacb \rangle$, since the last has order 40. Indeed, since $\langle bc \rangle \trianglelefteq G_1$, $|G_1| \leq 8 \cdot 5 = 40$ using software package GAP we obtain that actually $|G_1| = 40$. Now if $abca = cb$ then $c(ab)c = ba$ and so ab generates a normal subgroup of G, which is impossible. □

4. Concluding remarks. It is natural to wonder whether the other positive integers that cannot stand as the real genus exist. The role of $p = 2$ was rather easy to discover. The groups acting on surfaces of genera 0 and 1 exhaust few groups of order not bigger than 12 acting on surfaces of genus 2. The number $p = 12$ was the smallest one for which no groups of the real genus p were known.

Observe that primitivity of $p - 1 = 11$ is not a sufficient condition for other p's to share the property of $p = 12$ because there are groups of the real genera 6, 8 and 14 [**9, 10**]. An example of $p = 6$ shows that also the fact that p lies between twin primes is still not sufficient. So the most apparent features of $p = 12$ do not provide sufficiently empirical material for generalizations, though some results of the paper can be generalized.

For example, recently we have shown in [**12**] that Proposition 2 holds for groups of order $8q$ for arbitrary prime $q > 5$, i.e., there is no group of order $8q$ generated by elements a, b, c of order 2 such that ab and bc have orders 2 and 4, respectively. This together with a similar result of May [**6**] concerning M^*-groups implies

Corollary (cf. [**6**]). *If $q > 5$ is a prime, then a group of the real genus $p = q + 1$ has order smaller than $8q$.*

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Institute of Mathematics, Kazimierz Wielki Academy of Bydgoszcz, Chodkiewicza 30, 85-064 Bydgoszcz, Poland
E-mail address: brmock@ab-byd.edu.pl