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# EXTREME POINTS AND THE DIAMETER NORM

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ABSTRACT. Let $A$ be a point-separating closed linear subspace of $C(X)$, $X$ compact, which contains the constants. We characterize the extreme points of the closed unit ball of $A$ when endowed with the diameter (semi) norm.

**1. Introduction.** Let $X$ be a (infinite) compact Hausdorff space. As usual, $C(X)$ stands for the space of all continuous real-valued functions on $X$ endowed with the supremum norm $\|\cdot\|_\infty$. The diameter (semi) norm of any function $f$ in $C(X)$ is defined to be the diameter of the range of $f$. This norm was first studied by Győry and Molnár in [**9**] when dealing with linear bijections of $C(X)$ ($X$ compact and first countable) which leave the diameter of the range of every function invariant, which is to say, diameter preserving mappings. Since then, considerable attention has been given to these mappings and to the diameter norm. Namely, González and Uspenskij ([**7**]) and, independently, Cabello ([**2**]) removed the hypothesis of first countability on $X$ and characterized the extreme points of the closed unit ball of the dual of $C(X)$ endowed with the diameter norm (see also [**6**] for similar results on certain subspaces of continuous functions). Rao and Roy [**10**] have obtained analogue characterizations in the context of spaces of affine functions and vector-valued continuous functions. Recently, in [**3**], the authors have proved that the diameter norm is *maximal* on $C_0(X)$, where $X$ is a connected non-compact manifold.

Let $A$ be a point-separating closed linear subspace of $(C(X), \|\cdot\|_\infty)$ which contains the constant functions. Let $\mathcal{C}$ denote the constant functions on $X$. In this paper we study the extreme points of the closed unit ball of the dual of the quotient $A/\mathcal{C}$ endowed with the diameter norm.

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**2. Preliminaries.** Let $x, y \in X$. By $\mathcal{E}_x$ we shall denote the evaluation functional; that is, $\mathcal{E}_x(f) = f(x)$ for every $f \in C(X)$. The restriction of $\mathcal{E}_x$ to any linear subspace of $C(X)$ will be denoted by the same $\mathcal{E}_x$. In any case, the domain of $\mathcal{E}_x$ will be clearly stated by the context.

Let $A$ be a point-separating closed linear subspace of $(C(X), \|\cdot\|_\infty)$ which contains $\mathcal{C}$ (the constant functions). We shall write $Ch(A)$ to denote the Choquet boundary for $A$, which is to say, the subspace of $X$ consisting of the points $x$ for which $\varepsilon_x$ is an extreme point of the closed unit ball of the dual, $A^\star$, of $A$. By the Hahn-Banach theorem, we can assume that the elements of $A^\star$ are regular Borel measures on $X$ whose norm, $\|\cdot\|$, equals their total variation.

Let $A_d$ denote the quotient space $A/\mathcal{C}$ endowed with the following norm:

$$\|\pi(f)\|_d := \operatorname{diam}(f),$$

where $\operatorname{diam}(f)$ stands for the diameter of the range of the function $f \in A$.

Since $A_d$ is isomorphic to a quotient of $A$, then $(A_d)^\star$ is isomorphic to a subspace of $A^\star$. Indeed, see Lemma 1 below,

$$(A_d)^\star = \{\mu \in A^\star : \mu(X) = 0\}.$$

Furthermore $A_d$ is, up to a constant factor 2, isometric to the quotient of $(A, \|\cdot\|_\infty)$ by $\mathcal{C}$ since

$$\operatorname{diam}(f) = 2 \cdot \inf\{\|f - \alpha \cdot 1_X\|_\infty : \alpha \in \mathbf{R}\}$$

for all $f \in A$. Thus, if we denote by $\|\cdot\|_{d^\star}$ the original norm in $(A_d)^\star$, then it turns out that

$$2\|\mu\|_{d^\star} = \|\mu\|$$

for every $\mu \in (A_d)^\star$.

## 3. The extreme points of the unit ball of $(A_d)^\star$.

**Lemma 1.** *Let $A$ be a point-separating closed linear subspace of $C(X)$ which contains $\mathcal{C}$. Then*

$$(A_d)^\star = \{\mu \in A^\star : \mu(X) = 0\}.$$

*Proof.* By [**4**, Theorem V.2.2], the annihilator $\mathcal{C}^\perp$ of $\mathcal{C}$ (which coincides with the measures of mass zero) is isometric to $(A/\mathcal{C})^\star$. So the result follows from the fact that the set $(A/\mathcal{C})^\star$ is the same as $(A_d)^\star$. □

**Theorem 1.** *Let $A$ be a point-separating closed linear subspace of $C(X)$ with the constants. Then the set of extreme points of the closed unit ball of $(A_d)^\star$ is included in $\{\varepsilon_x - \varepsilon_y : x, y \in Ch(A), x \neq y\}$.*

*Proof.* Let us define the set

$$A \times A := \{h \in C(X \times X) : h(x,y) = h_1(x) + h_2(y), h_1, h_2 \in A\}$$

endowed with the supremum norm and consider the linear operator

$$L : A_d \longrightarrow A \times A$$

defined as

$$L(\pi(f))(x,y) := f(x) - f(y).$$

Since $\|\pi(f)\|_d = \|L(f)\|_\infty$, we have that $L$ is a linear into isometry. As a consequence, the adjoint map $L^\star : (A \times A)^\star \to (A_d)^\star$ sends the closed unit ball of $(A \times A)^\star$ onto the closed unit ball of $(A_d)^\star$. Indeed, given an extreme point $\mu$ of the closed unit ball of $(A_d)^\star$, we can find an extreme point $\eta$ of the closed unit ball of $(A \times A)^\star$ such that $L^\star(\eta) = \mu$. Hence, since $\eta = \varepsilon_{(x,y)}$ for some $(x,y) \in Ch(A \times A)$, we have

$$\mu = L^\star(\eta) = L^\star(\varepsilon_{(x,y)}) = \varepsilon_x - \varepsilon_y.$$

Finally, by Theorem 2 of [**7**], we know that $Ch(A \times A) = Ch(A) \times Ch(A)$, which implies that $x, y \in Ch(A)$. □

The following example, borrowed from [**10**], shows that the inclusion obtained in Theorem 1 may be proper.

**Example 1.** Let us consider the square $S$ in the plane with vertices at the points $x_1 = (1,1)$, $x_2 = (-1,1)$, $x_3 = (-1,-1)$ and $x_4 = (1,-1)$. Let $A$ be the linear subspace of $C(S)$ consisting of the functions of the form $f(x,y) = ax + by + c$ ($a, b, c \in \mathbf{R}$). It is apparent that $x_1$, $x_2$, $x_3$

and $x_4$ belong to $Ch(A)$. However, $\varepsilon_{x_1} - \varepsilon_{x_2}$ is not an extreme point of the closed unit ball of $(A_d)^\star$. Indeed,

$$\varepsilon_{x_1} - \varepsilon_{x_2} = \frac{1}{2}(\varepsilon_{x_4} - \varepsilon_{x_2}) + \frac{1}{2}(\varepsilon_{x_1} - \varepsilon_{x_3}).$$

In a similar way, we can check that $\varepsilon_{x_2} - \varepsilon_{x_3}$, $\varepsilon_{x_3} - \varepsilon_{x_4}$ and $\varepsilon_{x_4} - \varepsilon_{x_1}$ are not extreme points either.

It is now clear that an additional condition on $A$ is needed in order to characterize the extreme points of the closed unit ball of $(A_d)^\star$ completely.

**Definition 1.** Let $A$ be a point-separating closed linear subspace of $C(X)$ with the constants. We shall say that $A$ satisfy the *unique decomposition property* if given $x, y \in Ch(A)$ and two positive measures $\mu_1$ and $\mu_2$ such that

$$\varepsilon_x - \varepsilon_y = \mu_1 - \mu_2$$

and

$$\|\varepsilon_x - \varepsilon_y\| = \|\mu_1\| + \|\mu_2\|,$$

then $\mu_1 = \varepsilon_z$ and $\mu_2 = \varepsilon_t$ for some $z, t \in Ch(A)$.

*Remark* 6. Let us note that the subspace in the above example does not satisfy the unique decomposition property since

$$\varepsilon_{x_1} - \varepsilon_{x_2} = \varepsilon_{(1,0)} - \varepsilon_{(-1,0)}$$

and

$$\|\varepsilon_{x_1} - \varepsilon_{x_2}\| = \|\varepsilon_{(1,0)}\| + \|\varepsilon_{(-1,0)}\|,$$

but neither $(1,0)$ nor $(-1,0)$ is in $Ch(A)$.

On the other hand, *simplicial function spaces*, i.e., point-separating closed linear subspaces of $C(X)$ ($X$ compact) which contains the constants and whose state space is a Choquet simplex, satisfy the unique decomposition property (indeed, $z = x$ and $t = y$ in Definition 1 for such subspaces) since their dual is a lattice, see, e.g., [**2**, p. 69 and Theorem 4.9, p. 14]. For example,

$$\left\{ f \in C([-1,1]) : f(0) = \frac{f(-1) + f(1)}{2} \right\}$$

is a simplicial function space [**2**, p. 118, Example 3.2].

In order to provide more examples, which need not be necessarily simplicial, let us recall that, given a compact convex set $K$, the space of all continuous affine functions on $K$, $A(K)$, is a point-separating closed linear subspace of $C(K)$ which contains the constants. Then $A(K)$ satisfies the unique decomposition property if either every norm-exposed face of $K$ is projective [**2**, Theorem 6.4, p. 246] or if $K$ satisfies an intersection property [**1**].

**Theorem 2.** *Let $A$ be a point-separating closed linear subspace of $C(X)$ with the constants. If $A$ satisfies the unique decomposition property, then the set of extreme points of the closed unit ball of $(A_d)^\star$ consists exactly of $\{\varepsilon_x - \varepsilon_y : x, y \in Ch(A), x \neq y\}$.*

*Proof.* From Theorem 1, it suffices to check that every point of $\{\varepsilon_x - \varepsilon_y : x, y \in Ch(A), x \neq y\}$ is an extreme point of the closed unit ball of $(A_d)^\star$.

To this end, let $x, y$, $x \neq y$, be two elements of the Choquet boundary for $A$ and suppose that there exist $\mu$ and $\nu$ in $(A_d)^\star$ such that $\varepsilon_x - \varepsilon_y = \mu + \nu$ and $\|\varepsilon_x - \varepsilon_y\|_{d^\star} = \|\mu\|_{d^\star} + \|\nu\|_{d^\star}$. To show that $\varepsilon_x - \varepsilon_y$ is an extreme point of the closed unit ball of $(A_d)^\star$, it suffices to check that $\mu = k_1(\varepsilon_x - \varepsilon_y)$ and $\nu = k_2(\varepsilon_x - \varepsilon_y)$, for some $k_1, k_2 \in \mathbf{R}$. To this end, it is clear that

$$\begin{aligned} \|\varepsilon_x - \varepsilon_y\| &= \|\mu\| + \|\nu\| = \|\mu^+\| + \|\mu^-\| + \|\nu^+\| + \|\nu^-\| \\ &= \|\mu^+ + \nu^+\| + \|\mu^- + \nu^-\|, \end{aligned}$$

where $\mu^+$ and $\mu^-$ denote the positive and the negative parts of $\mu$, respectively. Furthermore, since $\varepsilon_x - \varepsilon_y = \mu^+ - \mu^- + \nu^+ - \nu^-$ and $A$

satisfies the unique decomposition property, we have that $\varepsilon_z = \mu^+ + \nu^+$ and $\varepsilon_t = \mu^- + \nu^-$ for some $z, t \in Ch(A)$. Consequently,

$$\begin{aligned}\|\varepsilon_z\| &= \|\mu^+\| + \|\nu^+\|\\ \|\varepsilon_t\| &= \|\mu^-\| + \|\nu^-\|\end{aligned}$$

and, since $\varepsilon_z$ and $\varepsilon_t$ are, by hypothesis, extreme points of the closed unit ball of $(A, \|\cdot\|_\infty)^\star$, we infer that

$$\begin{aligned}\mu^+ &= k_1 \cdot \varepsilon_z\\ \nu^+ &= k_2 \cdot \varepsilon_z\\ \mu^- &= k_3 \cdot \varepsilon_t\\ \nu^- &= k_4 \cdot \varepsilon_t,\end{aligned}$$

where, clearly, $k_1 = \mu^+(X)$, $k_3 = \mu^-(X)$, $k_2 = \nu^+(X)$ and $k_4 = \nu^-(X)$. Now, since $\mu(X) = 0$ and $\nu(X) = 0$ according to Lemma 1, we infer that $\mu^+(X) = \mu^-(X)$ and $\nu^+(X) = \nu^-(X)$ and, clearly, $\mu = \mu^+(X)(\varepsilon_z - \varepsilon_t) = \mu^+(X)(\varepsilon_x - \varepsilon_y)$ and $\nu = \nu^+(X)(\varepsilon_z - \varepsilon_t) = \nu^+(X)(\varepsilon_x - \varepsilon_y)$, as was to be proved.

**Question 1.** Is the unique decomposition property a necessary condition in Theorem 2?

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