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A NOTE ON MOMENTS OF SCALING FUNCTIONS

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ABSTRACT. In this note we give a proof of a reproducing formula for polynomials using natural decay hypothesis. This leads to a new exact formula for computation of moments of even order.

1. Introduction. Let us briefly recall some facts about wavelets and scaling functions.

In the context of wavelets theory, a multi-resolution analysis of $L^2(\mathbf{R})$ is a sequence V_j, $j \in \mathbf{Z}$, of closed subspaces of $L^2(\mathbf{R})$ such that

(i) $V_j \subset V_{j+1}$ for all $j \in \mathbf{Z}$

(ii) $\cap_{j\in\mathbf{Z}} V_j = \{0\}$, $\cup_{j\in\mathbf{Z}} V_j$ is dense in $L^2(\mathbf{R})$

(iii) $f \in V_0 \Leftrightarrow f(.-k) \in V_0$, for all $k \in \mathbf{Z}$ and $f \in V_j \Leftrightarrow f(2x) \in V_{j+1}$, for all $j \in \mathbf{Z}$.

(iv) There is a function φ (called the *scaling function* or the *father wavelet*) such that the family $\{\varphi(.-k) : k \in \mathbf{Z}\}$ is an orthonormal basis for V_0.

A classical procedure (for example, see [**5**]) leads to the construction of wavelets, i.e., an orthonormal basis $\{\psi_{j,k} : j, k \in \mathbf{Z}\}$ of $L^2(\mathbf{R})$ with

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k).$$

The function ψ is called the wavelet or the *mother wavelet*.

If p is a natural number and f is a function defined on $\mathbf{R}$ such that $x \mapsto x^p f(x)$ is integrable, the *moment of order* p of f is defined as

$$\int_{\mathbf{R}} x^p f(x)\, dx.$$

Moments of wavelets and scaling functions are considered in a wide literature, especially in numerical algorithms. Under some very weak

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and natural hypotheses, it is known that for a wavelet ψ, there is a natural number p such that $\int_{\mathbf{R}} x^l \psi(x)\, dx = 0$ for $l = 0, \dots, p$. On the other hand, for the scaling function φ, we always have $\int_{\mathbf{R}} \varphi(x)\, dx \neq 0$, and the other moments are also usually not equal to zero.

In this context, in order to find the coefficients of the representation of a polynomial in the approximation spaces V_j, one has to calculate the moments of the scaling function.

In what follows, we always use the following notations for the moments

$$M_j = \int_{\mathbf{R}} x^j \varphi(x)\, dx.$$

In [**6**], Sweldens and Piessens present the following result

$$M_2 = (M_1)^2$$

in case the scaling function is compactly supported and the wavelet has at least three vanishing moments. Then, considering the shifted moments, they cancel the first and the second error terms in approximations, hence get an interesting quadrature formula. To obtain this result about moments, they use a reproducing formula for polynomials.

In this note, under natural hypotheses and Strang-Fix conditions on a function φ (not necessarily a scaling function), we prove the reproducing formula for polynomials with absolute uniform convergence on compact sets and obtain the unicity of the coefficients. The result was obtained by Meyer in [**5**] but under a stronger regularity hypothesis. The proof we give does not follow the lines of Meyer's proof and only uses a trigonometric Fourier series. Moreover, our result leads to relations showing that moments M_j of even order can be expressed in terms of a linear combination of products of moments of smaller order, with coefficients directly computable. In particular, we obtain $M_2 = (M_1)^2$.

Recurrence relations to compute these moments or approximations of them can be found in [**1, 6**]. These relations involve approximations or computations of auxiliary numbers related to the specific property of scaling functions. Here, we present relations leading to the exact computation of moments of even order using only combinatory coefficients.

In what follows, the set of natural numbers greater than or equal to 0, respectively strictly greater than 0, is denoted $\mathbf{N}$, respectively $\mathbf{N}_0$, and the set of all integers, respectively all integers not equal to 0, is denoted $\mathbf{Z}$, respectively $\mathbf{Z}_0$.

We also use the following notation

$$C_m^n = \frac{m!}{n!(m-n)!}$$

where $m, n \in \mathbf{N}$, $m \geq n$.

2. Results.

Proposition 2.1. *Let φ be a function defined on $\mathbf{R}$ satisfying*

$$|\varphi(x)| \leq \frac{C}{(1+|x|)^{m+1+\varepsilon}}$$

for some $m \in \mathbf{N}_0$, C, $\varepsilon > 0$ and such that the functions $\varphi(x-k)$, $k \in \mathbf{Z}$, satisfy

$$\int_{\mathbf{R}} \varphi(x-k)\varphi(x-j)\,dx = \delta_{kj}, \quad j,k \in \mathbf{Z}.$$

If in addition φ is such that

$$M_0 = \hat{\varphi}(0) = 1$$

and satisfies the Strang-Fix conditions

$$D^j\hat{\varphi}(2k\pi) = 0 \quad \text{for } k \in \mathbf{Z}_0, \quad 1 \leq j \leq m,$$

then for every $j = 0, \ldots, m$, there is a unique sequence $(a_k^{(j)})_{k\in\mathbf{Z}}$ such that

$$x^j = \sum_{k\in\mathbf{Z}} a_k^{(j)}\varphi(x-k)$$

almost everywhere where the series is absolutely and uniformly convergent on every compact subset of $\mathbf{R}$ and where $a_k^{(j)}$ is a polynomial of degree j in the variable k. These coefficients are

$$a_k^{(j)} = \int_{\mathbf{R}} x^j\varphi(x-k)\,dx, \quad j = 0, \ldots, m, \quad k \in \mathbf{Z}.$$

In particular, we have

$$a_0^{(j)} = M_j.$$

Proof. We first prove unicity. Assume that for every $j = 0, \dots, m$, the sequence $(a_k^{(j)})_{k\in\mathbf{Z}}$ is such that

$$x^j = \sum_{k\in\mathbf{Z}} a_k^{(j)} \varphi(x-k)$$

almost everywhere with absolute and uniform convergence on every compact subset of $\mathbf{R}$, and where $a_k^{(j)}$ is a polynomial of degree j in the variable k. To obtain the unicity and the announced form of the coefficients, using integration of series, it suffices to prove that for every $l \in \mathbf{Z}$ and $j = 0, \dots, m$, the series

$$\sum_{k\in\mathbf{Z}} \int_{\mathbf{R}} \left| a_k^{(j)} \varphi(x-k)\varphi(x-l)\right| dx$$

converges.

For $p \in \mathbf{N}$ and $y, x > 0$, we have

$$(x+y)^p \leq 2^{p-1}(x^p + y^p);$$

hence for every $k \in \mathbf{Z}$, $x \in \mathbf{R}$ and $j \in \mathbf{N}$, we have

$$(1+|k|)^j \leq (1+|k-x|+1+|x|)^j \leq 2^{j-1}((1+|k-x|)^j + (1+|x|)^j).$$

It follows that there is $C_{1,j} = C_1 > 0$ such that

$$\begin{aligned} |a_k^{(j)}|\,|\varphi(x-k)| &\leq C_1 \frac{(1+|k|)^j}{(1+|x-k|)^{m+1+\varepsilon}} \\ &\leq C_1 2^{j-1}\left(\frac{1}{(1+|x-k|)^{m-j+1+\varepsilon}} + \frac{(1+|x|)^j}{(1+|x-k|)^{m+1+\varepsilon}}\right). \end{aligned}$$

As for every $s = 0, \dots, m$, the series

$$\sum_{k\in\mathbf{Z}} \frac{1}{(1+|x-k|)^{m-s+1+\varepsilon}}$$

converges uniformly on every compact subset of $\mathbf{R}$ and defines a 1-periodic and continuous function on $\mathbf{R}$; it is bounded. Hence there is $C_{2,j} = C_2 > 0$ such that

$$\sum_{|k|\le K} \int_{\mathbf{R}} \left|a_k^{(j)} \varphi(x-k)\varphi(x-l)\right| dx \le C_2 \int_{\mathbf{R}} (1+|x|)^j |\varphi(x-l)|\, dx,$$
$$\forall\, K \in \mathbf{N}_0.$$

The convergence follows.

Now we prove the existence. For every $j = 0, \dots, m$, using the hypothesis of decay on φ, the series

$$\sum_{k\in\mathbf{Z}} (x-k)^j \varphi(x-k)$$

is absolutely and uniformly convergent on every compact subset of $\mathbf{R}$ and defines a 1-periodic and bounded function. Using trigonometric Fourier series we have, almost everywhere,

$$\begin{aligned}\sum_{k\in\mathbf{Z}} (x-k)^j \varphi(x-k) &= \sum_{l\in\mathbf{Z}} \left(\int_0^1 \sum_{k\in\mathbf{Z}} (t-k)^j \varphi(t-k) e^{-2il\pi t}\, dt \right) e^{2i\pi l x} \\ &= \sum_{l\in\mathbf{Z}} \left(\int_{\mathbf{R}} t^j \varphi(t) e^{-2il\pi t}\, dt \right) e^{2i\pi l x}.\end{aligned}$$

The Strang-Fix conditions give

$$\int_{\mathbf{R}} t^j \varphi(t) e^{-2il\pi t}\, dt = 0, \quad \text{for } l \in \mathbf{Z}_0$$

hence

$$\sum_{k\in\mathbf{Z}} (x-k)^j \varphi(x-k) = \int_{\mathbf{R}} t^j \varphi(t)\, dt = M_j \tag{1}$$

almost everywhere absolutely and uniformly on $\mathbf{R}$. It follows that, for $j \ge 1$,

$$\begin{aligned}\sum_{k\in\mathbf{Z}} k^j \varphi(x-k) &= \sum_{k\in\mathbf{Z}} (k-x+x)^j \varphi(x-k) \\ &= \sum_{l=0}^{j} C_j^l x^{j-l} \sum_{k\in\mathbf{Z}} (k-x)^l \varphi(x-k) \\ &= \sum_{l=0}^{j} C_j^l x^{j-l} (-1)^l M_l\end{aligned}$$

hence

$$x^j = \sum_{k\in\mathbf{Z}} k^j \varphi(x-k) - \sum_{l=1}^{j} C_j^l x^{j-l}(-1)^l M_l. \tag{2}$$

Finally we conclude by a recurrence argument. □

We immediately have

$$a_k^{(j)} = \int_{\mathbf{R}} (x+k)^j \varphi(x)\,dx = \sum_{l=0}^{j} C_j^l k^l M_{j-l}$$

but this relation does not give anything between moments. The next proposition contains another expression of the polynomials $a_k^{(j)}$, which leads to new relations between moments.

We use some definitions and notations: for $j, l \in \mathbf{N}_0$, we define

$$K_l(j) = \Big\{(i_1,\dots,i_l) \in \mathbf{N}_0^l : \sum_{k=1}^{l} i_k = j\Big\}$$

and

$$K(j) = \bigcup_{l=1}^{j} K_l(j).$$

For $(i_1,\dots,i_l) \in K(j)$, we write $i \in K(j)$. For $j \geq i_1 + \cdots + i_l$ we define

$$\begin{aligned} F_j(i_1,\dots,i_l) =& F_j(i) \\ =& (-1)^{i_1+1}\cdots(-1)^{i_l+1} C_j^{i_1} C_{j-i_1}^{i_2} \cdots C_{j-\sum_{k=1}^{l-1} i_k}^{i_l} M_{i_1}\cdots M_{i_l} \end{aligned}$$

where

$$M_j = \int_{\mathbf{R}} x^j \varphi(x)\,dx = a_0^{(j)}.$$

For $j \in \mathbf{N}$, we also set

$$\sum_{i\in K(0)} F_j(i) = 1. \tag{$*$}$$

Proposition 2.2. *Under the same hypothesis as in Proposition* 2.1 *and using the notations introduced above, we have the following relations*

$$a_k^{(j)} = \sum_{l=0}^{j} k^l \sum_{i \in K(j-l)} F_j(i), \quad k \in \mathbf{Z}, \quad j = 1, \dots, m. \tag{3}$$

Proof. From (1) we have

$$M_j = \int t^j \varphi(t)\, dt = \sum_{k \in \mathbf{Z}} (x-k)^j \varphi(x-k) \quad \text{a.e.}, \quad M_0 = 1,$$

where the convergence is absolute and uniform on $\mathbf{R}$.

For $j = 1$ we have $M_1 = \sum_{k \in \mathbf{Z}} (x-k)\varphi(x-k)$, hence

$$M_0 x = x = M_1 + \sum_{k \in \mathbf{Z}} k\varphi(x-k) = \sum_{n \in \mathbf{Z}} (M_1 + k)\varphi(x-k).$$

Using unicity of the coefficients in the reproducing formula, we get

$$a_k^{(1)} = k + M_1.$$

Since $\sum_{i \in K(0)} F_1(i) = 1$ and $\sum_{i \in K(1)} F_1(i) = M_1$, the relation (3) is verified for $j = 1$.

Assume now that the relation is satisfied for $j = 1, \dots, n-1$. We then have

$$x^j = \sum_{k \in \mathbf{Z}} \left(\sum_{l=0}^{j} k^l \sum_{i \in K(j-l)} F_j(i) \right) \varphi(x-k)$$

for every $j = 1, \dots, n-1$ and also for $j = 0$ using the convention $(*)$.

Hence, using (2), we get

$$\begin{aligned}
x^n &= \sum_{k\in\mathbf{Z}} k^n\varphi(x-k) + \sum_{j=1}^{n} C_n^j(-1)^{j+1}x^{n-j}M_j \\
&= \sum_{k\in\mathbf{Z}} k^n\varphi(x-k) \\
&\quad + \sum_{j=1}^{n} C_n^j(-1)^{j+1}M_j \sum_{k\in\mathbf{Z}}\Bigg(\sum_{l=0}^{n-j} k^l \sum_{i\in K(n-j-l)} F_{n-j}(i)\Bigg)\varphi(x-k) \\
&= \sum_{k\in\mathbf{Z}} k^n\varphi(x-k) \\
&\quad + \sum_{k\in\mathbf{Z}}\Bigg(\sum_{j=1}^{n}\sum_{l=0}^{n-j} k^l C_n^j(-1)^{j+1}M_j \sum_{i\in K(n-l-j)} F_{n-j}(i)\Bigg)\varphi(x-k) \\
&= \sum_{k\in\mathbf{Z}} k^n\varphi(x-k) \\
&\quad + \sum_{k\in\mathbf{Z}}\Bigg(\sum_{l=0}^{n-1} k^l \sum_{j=1}^{n-l} C_n^j(-1)^{j+1}M_j \sum_{i\in K(n-l-j)} F_{n-j}(i)\Bigg)\varphi(x-k) \\
&= \sum_{k\in\mathbf{Z}} k^n\varphi(x-k) \\
&\quad + \sum_{k\in\mathbf{Z}}\Bigg(\sum_{l=0}^{n-1} k^l\Bigg(\sum_{j=1}^{n-l-1}\sum_{i\in K(n-l-j)} F_n(j,i) + C_n^{n-l}(-1)^{n-l+1}M_{n-1}\Bigg)\Bigg) \\
&\quad \cdot\varphi(x-k) \\
&= \sum_{k\in\mathbf{Z}} k^n\varphi(x-k) + \sum_{k\in\mathbf{Z}}\Bigg(\sum_{l=0}^{n-1} k^l \sum_{i\in K(n-l)} F_n(i)\Bigg)\varphi(x-k),
\end{aligned}$$

hence

$$a_k^{(n)} = \sum_{l=0}^{n} k^l \sum_{i\in K(n-l)} F_n(i). \qquad \square$$

We can deduce from the previous relations that the moments of even order can be expressed in terms of a linear combination of products of moments of smaller order in which the coefficients are of type C_m^l.

Corollary 2.3. *Under the same hypotheses as in Proposition* 2.1 *and using the same notations, we have*

$$M_j = \sum_{i \in K(j)} F_j(i) = \sum_{i \in \cup_{l=1}^{j} K_l(j)} F_j(i), \quad j = 1, \dots, m.$$

In particular, if j is even, we have

$$2M_j = \sum_{i \in \cup_{l=2}^{j} K_l(j)} F_j(i).$$

Proof. It suffices to take $k = 0$ in the relations (3) giving $a_k^{(j)}$ in the previous proposition.

For j even, we have

$$F_j(j) = (-1)^{1+j} M_j = -M_j,$$

hence the conclusion. □

As an example, we obtain

$$K_2(2) = \{(1,1)\}, \ F_2((1,1)) = 2,$$

hence

$$2M_2 = 2(M_1)^2;$$

in the same way

$$M_4 = -3(M_1)^4 + 4M_1M_3.$$

Let us also recall that the hypotheses we use are very natural in the context of wavelets. Indeed, we have the following result (see for example [**2, 5**]).

Property 2.4. Assume $\psi_{j,k}$, $j, k \in \mathbf{Z}$, is an orthonormal basis of wavelets associated with a multi-resolution analysis as described in the Introduction. If the scaling function φ and the wavelet ψ satisfy

$$|\varphi(x)|, \ |\psi(x)| \leq \frac{C}{(1+|x|)^{m+1+\varepsilon}}$$

for some $\varepsilon > 0$, $C > 0$, $m \in \mathbf{N}_0$, and if $\psi \in C^m(\mathbf{R})$, $D^j\psi \in L_\infty(\mathbf{R})$ for all $j = 0, \ldots, m$, then the filter m_0 associated to the multi-resolution analysis satisfies

$$m_0(\xi) = (1 + e^{-i\xi})^{m+1} L(\xi)$$

where L is 2π-periodic and $C^m(\mathbf{R})$ and we have

$$|\hat{\varphi}(0)| = 1.$$

The factorization of the filter implies also

$$D^j \hat{\varphi}(2k\pi) = 0 \quad \text{for } k \in \mathbf{Z}_0, \quad 1 \le j \le m.$$

REFERENCES

1. G. Beylkin, R. Coifman and V. Rokhlin, *Fast wavelets transforms and numerical algorithms* I, Comm. Pure Appl. Math. **44** (1991), 141–183.

2. I. Daubechies, *Ten lectures on wavelets*, CBMS-NSF Regional Conf. Ser. in Appl. Math. **61**, SIAM, Philadelphia, 1992.

3. ———, *Orthonormal compactly supported wavelets* II. *Variations on a theme*, SIAM J. Math. Anal. **24** (1993), 499–519.

4. G. Fix and G. Strang, *Fourier analysis of the finite element method in Ritz-Galerkin theory*, Stud. Appl. Math. **18** (1969), 265–274.

5. Y. Meyer, *Ondelettes et opérateurs*, I, Hermann, Paris, 1990.

6. W. Sweldens and R. Piessens, *Calculation of the wavelet decomposition using quadrature formulae*, CWI Quarterly **5** (1992), 33–52.

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