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A NOTE ON WEIGHTED ESTIMATES FOR CERTAIN CLASSES OF PSEUDO-DIFFERENTIAL OPERATORS

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ABSTRACT. We consider certain classes of pseudo-differential operators and prove $L^2_w - L^2_w$, $L^1_w - L^{1,\infty}_w$ and $H^1_w - L^1_w$ estimates.

1. Introduction. For a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$, let $(\partial\xi)^\alpha$ denote the differential operator

$$(\partial/\partial\xi_1)^{\alpha_1} \dots (\partial/\partial\xi_n)^{\alpha_n}.$$

Put $|\alpha| = \alpha_1 + \dots + \alpha_n$. Let $\omega : [0,\infty) \times [0,\infty) \to [0,\infty)$ be such that

(1) for each fixed s, $\omega(s,t)$ is continuous, increasing and concave with respect to t and $\omega(s,0) = 0$;

(2) if $s/2 \le s' \le 2s$, $\omega(s',t) \le C\omega(s,t)$ for some constant C;

(3) $$\sum_{j=0}^{\infty} \omega(2^j, 2^{-j})^2 < \infty.$$

A function ω satisfying these conditions is called a modulus of continuity. Let $\sigma(x,\xi)$ be a continuous, bounded function on $\mathbf{R}^n \times \mathbf{R}^n$. Let L, M be nonnegative integers. We consider the following conditions:

$$|(\partial\xi)^\alpha \sigma(x,\xi)| \le C_\alpha (1+|\xi|)^{-|\alpha|} \quad \text{for all } |\alpha| \le L, \tag{1.1}$$

$$\begin{aligned} &|(\partial\xi)^\alpha \sigma(x+y,\xi) - (\partial\xi)^\alpha \sigma(x,\xi)| \\ &\qquad \le C_\alpha (1+|\xi|)^{-|\alpha|} \omega(1+|\xi|, |y|) \quad \text{for all } |\alpha| \le M. \end{aligned} \tag{1.2}$$

We say that $\sigma \in \Sigma(\omega, L, M)$ if $\sigma(x,\xi)$ satisfies (1.1) and (1.2).

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Let $\sigma(x, D)$ denote the pseudo-differential operator defined by

$$\sigma(x, D)f(x) = \int_{\mathbf{R}^n} \sigma(x, \xi)\hat{f}(\xi)e^{2\pi i\langle x,\xi\rangle}\, d\xi,$$

where $\langle x, \xi\rangle$ denotes the inner product in $\mathbf{R}^n$ and $\hat{f}$, $f \in \mathcal{S}(\mathbf{R}^n)$ (the Schwartz space), is the Fourier transform; we also write $\hat{f} = \mathcal{F}(f)$.

Now we define some function spaces. Let $\omega \in A_1$ where A_p denotes the weight class of Muckenhoupt. A nonnegative, locally integrable function w is of class A_1, by definition, if there exists a constant $c \geq 0$ such that $\mathcal{M}(w)(x) \leq cw(x)$ almost everywhere, where $\mathcal{M}$ denotes the Hardy-Littlewood maximal operator. Let $\varphi \in C_0^\infty(\mathbf{R}^n)$ be nonnegative, radial and such that $\operatorname{supp}(\varphi) \subset \{|x| \leq 1\}$, $\varphi(0) = 1$, $\int \varphi = 1$. Let f be a tempered distribution on $\mathbf{R}^n$. We say $f \in H_w^1(\mathbf{R}^n)$ if

$$\|f\|_{H_w^1} = \int_{\mathbf{R}^n} \sup_{t>0} |f * \varphi_t(x)|w(x)\, dx < \infty,$$

where $\varphi_t(x) = t^{-n}\varphi(t^{-1}x)$. We denote by $L_w^{1,\infty}$ the weak L_w^1 space of all those measurable functions f which satisfy

$$\|f\|_{L_w^{1,\infty}} = \sup_{\lambda>0} \lambda w(\{x \in \mathbf{R}^n : |f(x)| > \lambda\}) < \infty,$$

where $w(E) = \int_E w(x)\, dx$. Finally, for a weight v, L_v^p denotes the weighted Lebesgue space with norm $\|f\|_{L_v^p} = (\int |f(x)|^p v(x)\, dx)^{1/p}$.

In this note we shall prove the following.

Theorem 1. *Let $w \in A_1$. If $\sigma(x, \xi) \in \Sigma(\omega, [n/2]+1, [n/2]+1)$, then the pseudo-differential operator $\sigma(x, D)$ extends to a bounded operator on L_w^2 where $[a]$ denotes the integer such that $a - 1 < [a] \leq a$.*

Theorem 2. *Let $w \in A_1$. If $\sigma(x, \xi) \in \Sigma(\omega, n + 1, [n/2] + 1)$, then $\sigma(x, D)$ extends to a bounded operator from L_w^1 to $L_w^{1,\infty}$ and from H_w^1 to L_w^1.*

When $\omega(s, t) = \omega_0(t)$ and w is a constant function, these mapping properties of the pseudo-differential operators were proved by Coifman-Meyer under stronger assumptions on $\sigma(x, \xi)$, see [**3**, Theorem 9].

Weighted estimates were studied in detail by Yabuta [**9**]. (See also Muramatu-Nagase [**6**], Miyachi-Yabuta [**5**], Carbery-Seeger [**2**] and Yamazaki [**10**].) Theorems 1 and 2 improve results of [**9**].

Taking $\omega(s,t) = s^\delta t$, $0 < \delta < 1$, in Theorems 1 and 2 we have the following two corollaries.

Corollary 1. *Let* $w \in A_1$. *If* $\sigma(x,\xi)$ *satisfies* (1.1) *with* $L = [n/2]+1$ *and*

$$|(\partial x)^\beta (\partial \xi)^\alpha \sigma(x,\xi)| \le C_{\alpha,\beta}(1+|\xi|)^{\delta|\beta|-|\alpha|} \tag{1.3}$$

for all $|\alpha| \le [n/2]+1$ *and* $|\beta| = 1$ *with* $0 < \delta < 1$, *then* $\sigma(x,D)$ *is bounded on* L^2_w.

Corollary 2. *Let* $w \in A_1$. *If* $\sigma(x,\xi)$ *satisfies* (1.1) *with* $L = n+1$ *and* (1.3), *then* $\sigma(x,D)$ *is bounded from* L^1_w *to* $L^{1,\infty}_w$ *and from* H^1_w *to* L^1_w.

Since $\omega(s,t) = s^\delta t$ satisfies (2.1) and (2.2) of [**9**] (see (1.8) and (1.9) below), Corollary 1 follows from Theorem 2.1 of Yabuta [**9**] and Corollary 2 from [**9**, Section 7]. See also Journé [**4**].

Remark 1. Let

$$\sigma_a(x,\xi) = e^{-2\pi i\langle x,\xi\rangle} e^{-|x|^2}(1+|\xi|^2)^{-n/a}, \quad a \ge 2.$$

When w is a constant function and n is odd in Theorem 1, the optimality of $[n/2]+1$ in $\Sigma(\omega,[n/2]+1,[n/2]+1)$ can be seen by taking the symbol $\sigma_4(x,\xi)$. When w is a constant function and $n \ge 3$ in Theorem 2, the optimality of $L = n+1$ in $\Sigma(\omega, n+1, [n/2]+1)$ for the weak (1,1) boundedness can be seen by checking the symbol $\sigma_2(x,\xi)$. See Coifman-Meyer [**3**, p. 12] and Yabuta [**8**, Section 6].

Remark 2. Let $\eta \in C_0^\infty(\mathbf{R})$ be such that $\eta(\xi) = 1$ for $\xi \in [3/4, 5/4]$, $\operatorname{supp}(\eta) \subset [2/3, 4/3]$. Then the optimality of the exponent 2 in the condition $\sum_j \omega(2^j, 2^{-j})^2 < \infty$ can be seen by checking a symbol of the form

$$\sigma(x,\xi) = \sum_{j=0}^{\infty} \omega_j \eta(2\pi 2^{-j}\xi)\exp(-2\pi i 2^j x)$$

with $\sum_j \omega_j^2 = \infty$. See Coifman-Meyer [**3**, pp. 39–40].

In fact, we can refine Theorems 1 and 2 as follows (Theorems 3 and 4). Let $\sigma(x,\xi)$ be continuous and bounded on $\mathbf{R}^n \times \mathbf{R}^n$. Let L and M be nonnegative integers and $0 < a,\, b \le 1$. Let $\omega(s,t)$ be a modulus of continuity. We consider the following conditions

$$|(\partial\xi)^\alpha \sigma(x,\xi)| \le C_\alpha (1+|\xi|)^{-|\alpha|} \quad \text{for } |\alpha| \le L, \tag{1.4}$$

$$\begin{aligned}&|(\partial\xi)^\alpha \sigma(x,\xi+\eta) - (\partial\xi)^\alpha \sigma(x,\xi)| \\ &\quad \le C_\alpha (1+|\xi|)^{-|\alpha|-a}|\eta|^a \quad \text{for } |\eta| < (1+|\xi|)/2 \text{ and } |\alpha| = L,\end{aligned} \tag{1.5}$$

$$\begin{aligned}&|(\partial\xi)^\alpha \sigma(x+y,\xi) - (\partial\xi)^\alpha \sigma(x,\xi)| \\ &\qquad\qquad \le C_\alpha (1+|\xi|)^{-|\alpha|}\omega(1+|\xi|,|y|) \quad \text{for } |\alpha| \le M,\end{aligned} \tag{1.6}$$

$$\begin{aligned}&|(\partial\xi)^\alpha \sigma(x+y,\xi+\eta) - (\partial\xi)^\alpha \sigma(x,\xi+\eta) \\ &\qquad - (\partial\xi)^\alpha \sigma(x+y,\xi) + (\partial\xi)^\alpha \sigma(x,\xi)| \\ &\qquad \le C_\alpha (1+|\xi|)^{-|\alpha|-b}|\eta|^b \omega(1+|\xi|,|y|) \\ &\qquad\qquad \text{for } |\eta| < (1+|\xi|)/2 \text{ and } |\alpha| = M.\end{aligned} \tag{1.7}$$

Theorem 3. *Suppose $\sigma(x,\xi)$ satisfies* (1.4)–(1.7) *with $L = M = [n/2]$ and $a = b$, $0 < a \le 1$, $[n/2] + a > n/2$. Then $\sigma(x,D)$ is bounded on L^2_w for all $w \in A_1$.*

Theorem 4. *Suppose $\sigma(x,\xi)$ satisfies the conditions* (1.4), (1.5) *with $L = n$, $0 < a \le 1$ and the conditions* (1.6), (1.7) *with $M = [n/2]$ and b such that $[n/2] + b > n/2$, $0 < b \le 1$. Then $\sigma(x,D)$ is bounded from L^1_w to $L^{1,\infty}_w$ and from H^1_w to L^1_w for all $w \in A_1$.*

We easily see that Theorems 1 and 2 immediately follow from Theorems 3 and 4, respectively. In Theorem 4, the assumption on M in (1.6) and (1.7) is less restrictive than that of [**9**, Theorem 2.3], see also [**9**, Section 7]. Also we note that Theorem 3 was proved in [**9**] with the additional, superfluous assumptions on ω ((2.1) and (2.2) of [**9**])

$$\int_0^1 \omega(1/t, t^\delta)^2 \, dt/t < \infty \quad \text{for some } 0 < \delta < 1; \tag{1.8}$$

$$\sum_{1\le 2^j\le 1/R} \omega(2^j, R) \le B \quad \text{for all } 0 < R \le 1 \text{ with some } B > 0. \tag{1.9}$$

We can remove these assumptions in Theorem 3.

Remark 3. Let ω_1 be a modulus of continuity such that $\omega_1(s,t) = \log(2+s)[\log(2+1/t)]^{-3/2-\alpha}$, $\omega_1(s,0) = 0$ for $0 \le s$, $0 < t \le 1$, where $0 < \alpha < 1/2$. It is easy to see that ω_1 does not satisfy the condition (1.9). Let $\tilde\omega_2(s,t) = s^{1/2}t^{1/2}[\log(2+1/t)]^{-1/2-\beta}$, $\beta > 0$, $\tilde\omega_2(s,0) = 0$ for $0 \le s$, $0 < t \le 1$. If β is small enough, $\tilde\omega_2(s,t)$ is concave on $[0,1]$ with respect to t and so we can find a modulus of continuity ω_2 such that $\omega_2(s,t) = \tilde\omega_2(s,t)$ for $0 \le s$, $0 \le t \le 1$. We can easily see that ω_2 does not satisfy the condition (1.8). If we define a modulus of continuity ω by $\omega = \omega_1 + \omega_2$, then ω does not satisfy either (1.8) or (1.9).

Theorems 3 and 4 are consequences of more general results (Theorems 5 and 6). Let ρ be a nonnegative function such that $\rho^{-1} \in L^1(\mathbf{R}^n)$. Define

$$\|f\|_{B_\rho} = \left(\int_{\mathbf{R}^n} |\hat f(x)|^2 \rho(x)\,dx\right)^{1/2}.$$

Let $\Psi \in C^\infty(\mathbf{R}^n)$ be a radial function supported in $\{1/2 \le |\xi| \le 2\}$ such that

$$\sum_{j\in\mathbf{Z}} \Psi(2^{-j}\xi) = 1 \quad \text{for } \xi \ne 0,$$

where $\mathbf{Z}$ denotes the set of all integers. Define $\Phi \in C_0^\infty(\mathbf{R}^n)$ by $\Phi(\xi) = 1 - \sum_{j\ge1} \Psi(2^{-j}\xi)$. Then we have the following

Theorem 5. *Let $\sigma(x,\xi)$ be continuous and bounded on $\mathbf{R}^n \times \mathbf{R}^n$. Let $w \in A_1$. Suppose that*

$$\sup_{t>0} \theta_t * w(x) \le Cw(x) \quad \text{a.e. where } \theta(x) = \rho(x)^{-1} \tag{1.10}$$

and that

$$\sup_{j\ge1} \sup_{x\in\mathbf{R}^n} \|\sigma(x, 2^j\cdot)\Psi(\cdot)\|_{B_\rho} < \infty, \tag{1.11}$$

$$\sup_{x\in\mathbf{R}^n} \|\sigma(x+y, 2^j\cdot)\Psi(\cdot) - \sigma(x, 2^j\cdot)\Psi(\cdot)\|_{B^\rho} \le C\omega(2^j, |y|) \tag{1.12}$$
$$j \ge 1,$$

$$\sup_{x\in\mathbf{R}^n} \|\sigma(x,\cdot)\Phi(\cdot)\|_{B_\rho} < \infty. \tag{1.13}$$

Then $\sigma(x,D)$ is bounded on L^2_w.

Let β be a nonnegative function on $[0,\infty)$ such that $\beta(t)>0$ for $t>0$ and

(1) $\beta(s) \le C\beta(t)$ if $t/2 \le s \le 2t$,

(2) $\beta(t) \le C(1+t)$,

(3) $\beta(s) \le C\beta(t)$ for $0 \le s \le t$,

(4) $\sum_{k\ge 1} k\beta(2^k)^{-1} < \infty$.

We assume that functions $w \in A_1$ and ρ satisfy the following condition for some β as above

$$\sup_{t>0} t^{-n} \int_{\mathbf{R}^n} \theta(y/t)(1+\beta(|y|/t))w(x-y)\,dy \le Cw(x) \tag{1.14}$$

almost everywhere, where $\theta(x)$ is as in (1.10). We also assume that $|\eta| * \theta(x) \le C_\eta \theta(x)$ for all $\eta \in \mathcal{S}(\mathbf{R}^n)$. Under these assumptions on ρ and $w \in A_1$, we have the following

Theorem 6. *Let $\sigma(x,\xi)$ be continuous and bounded on $\mathbf{R}^n \times \mathbf{R}^n$. Put*

$$A_j(x,k) = \int_{\mathbf{R}^n} \sigma(x,2^j\xi)\Psi(\xi)\exp(-2\pi i\langle k,\xi\rangle)\,d\xi, \quad j\ge 1,$$
$$B(x,k) = \int_{\mathbf{R}^n} \sigma(x,\xi)\Phi(\xi)\exp(-2\pi i\langle k,\xi\rangle)\,d\xi.$$

Suppose $\sigma(x,D)$ is bounded on L^2_w and

$$|A_j(x,k)| \le C\rho(k)^{-1}, \quad j\ge 1, \qquad |B(x,k)| \le C\rho(k)^{-1}.$$

Then $\sigma(x,D)$ is bounded from L^1_w to $L^{1,\infty}_w$ and from H^1_w to L^1_w.

Examples. Let

(1) $\rho(x) = (1+|x|^2)^{s/2}$, $s > n$;

(2) $\rho(x) = (1+|x|^2)^{n/2}[\log(2+|x|^2)]^3[\log(2+\log(2+|x|^2))]^{2+\varepsilon}$, $\varepsilon > 0$.

Then we can see that these functions ρ satisfy all the requirements assumed in Theorem 6 for all $w \in A_1$ by taking $\beta(t) = t^\tau$ with $0 < \tau < \min(1, s-n)$ and $\beta(t) = [\log(2+t)]^2[\log(2+\log(2+t))]^{1+\varepsilon/2}$, respectively.

As an application of the weighted estimates of Theorem 5 and the extrapolation theorem of Rubio de Francia [**7**], we have the following

Corollary 3. *Let ρ be a nonnegative function such that $\rho^{-1} \in L^1(\mathbf{R}^n)$. Suppose that the condition* (1.10) *holds for all $w \in A_1$. Suppose that σ satisfies the conditions* (1.11)–(1.13). *Let $2 < p < \infty$. Then $\sigma(x, D)$ is bounded on L^p_w for all $w \in A_{p/2}$.*

In particular, we have the conclusion of Corollary 3 under the hypotheses of Theorem 3.

We shall prove Theorem 5 in Section 2. To prove the weighted estimates, Yabuta [**9**] used the sharp function of Fefferman-Stein, which requires the superfluous assumptions on ω stated above ((1.8), (1.9)). Instead of using the sharp function, basically we apply the method of Coifman-Meyer [**3**], the principal part of which is the decomposition of a symbol into the reduced symbols. However, to get the improved results, we need to refine the method. We shall prove Theorem 6 in Section 3 by applying a weighted version of a result of Carbery [**1**]. In Section 4 we shall prove Theorems 3 and 4 by applying Theorems 5 and 6.

In this note C is used to denote nonnegative constants which may be different in different occurrences.

2. Proof of Theorem 5.

Take a radial function $\psi \in C_0^\infty(\mathbf{R}^n)$ such that $\operatorname{supp}(\psi) \subset \{1/4 < |\xi| < 4\}$ and $\psi(\xi) = 1$ if $1/2 \le |\xi| \le 2$.

Decompose

$$\begin{aligned}\sigma(x,\xi) &= \sigma(x,\xi)\Phi(\xi) + \sum_{j\geq 1}\sigma(x,\xi)\Psi(2^{-j}\xi)\\ &= \sigma(x,\xi)\Phi(\xi) + \sum_{j\geq 1}\sigma(x,\xi)\Psi(2^{-j}\xi)\psi(2^{-j}\xi)^2\\ &= \int_{\mathbf{R}^n} B(x,k)e^{2\pi i\langle k,\xi\rangle}\,dk\\ &\quad + \sum_{j\geq 1}\int_{\mathbf{R}^n} A_j(x,k)\exp(2\pi i\langle 2^{-j}k,\xi\rangle)\,dk\psi(2^{-j}\xi)^2,\end{aligned}$$

where $A_j(x,k)$ and $B(x,k)$ are as in Theorem 6.

Lemma 1. *Suppose that the conditions* (1.11) *and* (1.12) *hold. Then we can decompose* $A_j(x,k) = A_j^{(1)}(x,k) + A_j^{(2)}(x,k)$, *where*

$$|A_j^{(i)}(x,k)| = \rho(k)^{-1/2}q^{(i)}(x,k,j)$$

with nonnegative functions $q^{(i)}(x,k,j)$ *satisfying*

$$\sup_{x\in\mathbf{R}^n}\sum_{j\geq 1}\int_{\mathbf{R}^n} q^{(1)}(x,k,j)^2\,dk < \infty, \tag{2.1}$$

$$\sup_{x\in\mathbf{R}^n}\sup_{j\geq 1}\int_{\mathbf{R}^n} q^{(2)}(x,k,j)^2\,dk < \infty. \tag{2.2}$$

Furthermore, the Fourier transform of $A_j^{(2)}(x,k)$ *in the* x*-variable is supported in* $\{|\xi| \leq 2^{j-10}\}$ *uniformly in* k.

Lemma 2. *Suppose that the condition* (1.13) *holds. Then the function*

$$r(x,k) = \rho(k)^{1/2}|B(x,k)|$$

satisfies

$$\sup_{x\in\mathbf{R}^n}\int_{\mathbf{R}^n} r(x,k)^2\,dk < \infty.$$

Now we prove Lemma 1. Put

$$A_j^{(2)}(x,k) = \int_{\mathbf{R}^n} [\hat{\varphi}_{2^{-j+10}} * \sigma(\cdot, 2^j\xi)](x)\Psi(\xi)\exp(-2\pi i\langle k,\xi\rangle)\,d\xi,$$

where $[\hat{\varphi}_{2^{-j+10}} * \sigma(\cdot, 2^j\xi)](x) = \int \hat{\varphi}_{2^{-j+10}}(y)\sigma(x-y, 2^j\xi)\,dy$ and φ is as in the definition of H^1_w in Section 1. Define $A_j^{(1)} = A_j - A_j^{(2)}$. Then we see that

$$\begin{aligned}\int |A_j^{(2)}(x,k)|^2\rho(k)\,dk &\le C\int |\hat{\varphi}_{2^{-j+10}}(y)|\|\sigma(x+y, 2^j\cdot)\Psi(\cdot)\|_{B_\rho}^2\,dy\\ &\le C\sup_{x\in\mathbf{R}^n}\|\sigma(x,2^j\cdot)\Psi(\cdot)\|_{B_\rho}^2.\end{aligned}$$

Therefore, by (1.11) we get (2.2). The support condition for the Fourier transform of $A_j^{(2)}$ is easily seen.

Next, since $\int\hat{\varphi} = 1$, by (1.12) we have

$$\begin{aligned}&\sum_{j\ge1}\int |A_j^{(1)}(x,k)|^2\rho(k)\,dk\\ &\quad\le \sum_{j\ge1} C\int |\hat{\varphi}_{2^{-j+10}}(y)|\|\sigma(x+y,2^j\cdot)\Psi(\cdot) - \sigma(x,2^j\cdot)\Psi(\cdot)\|_{B_\rho}^2\,dy\\ &\quad\le \sum_{j\ge1} C\int |\hat{\varphi}_{2^{-j+10}}(y)|\omega(2^j,|y|)^2\,dy\\ &\quad\le \sum_{j\ge1} C\int |\hat{\varphi}(y)|\omega(2^j, 2^{-j+10}|y|)^2\,dy\\ &\quad\le \sum_{j\ge1} C\omega(2^j,2^{-j})^2\int |\hat{\varphi}(y)|(1+|y|)^2\,dy\\ &\quad\le \sum_{j\ge1} C\omega(2^j,2^{-j})^2 < \infty,\end{aligned}$$

where we have used the inequality $\omega(2^j, a2^{-j}) \le C(1+a)\omega(2^j,2^{-j})$, $a>0$, which holds since $\omega(s,t)$ is increasing and concave in t. This proves (2.1). We have completed the proof of Lemma 1. $\square$

We easily see that the condition (1.13) implies Lemma 2.

Now we turn to the proof of Theorem 5. Put

$$\begin{aligned} E_j(f)(x,k) &= \int_{\mathbf{R}^n} \exp(2\pi i\langle 2^{-j}k,\xi\rangle)\psi(2^{-j}\xi)^2\hat{f}(\xi)\exp(2\pi i\langle x,\xi\rangle)\,d\xi \\ &= (\tau_{-k}\mathcal{F}^{-1}(\psi))_{2^{-j}} * \Delta_j(f)(x), \end{aligned}$$

where $\tau_k f(x) = f(x-k)$ and

$$\Delta_j(f)(x) = \int_{\mathbf{R}^n} \psi(2^{-j}\xi)\hat{f}(\xi)\exp(2\pi i\langle x,\xi\rangle)\,d\xi.$$

Then by (2.1) and the Schwarz inequality we have

$$\begin{aligned} &\left|\sum_{j=1}^{\infty}\int A_j^{(1)}(x,k)E_j(f)(x,k)\,dk\right|^2 \\ &\qquad \le \int \sum_{j\ge 1} q^{(1)}(x,k,j)^2\,dk \sum_{j\ge 1}\int \rho(k)^{-1}|E_j(f)(x,k)|^2\,dk \\ &\qquad \le C\sum_{j\ge 1}\int \rho(k)^{-1}|E_j(f)(x,k)|^2\,dk. \end{aligned}$$

Thus, integrating with respect to $w(x)\,dx$ by (1.10) and the weighted Littlewood-Paley inequality we have

$$\begin{aligned} &\int\left|\sum_{j=1}^{\infty}\int A_j^{(1)}(x,k)E_j(f)(x,k)\,dk\right|^2 w(x)\,dx \\ &\quad \le C\sum_{j\ge 1}\int \rho(k)^{-1}\left(\int |E_j(f)(x,k)|^2 w(x)\,dx\right)dk \\ &\quad \le C\sum_{j\ge 1}\int\left(\int \rho(k)^{-1}\int 2^{jn}|\mathcal{F}^{-1}(\psi)(2^j(x-y)+k)|w(x)\,dx\,dk\right) \\ &\qquad\qquad \cdot |\Delta_j(f)(y)|^2\,dy \\ &\quad \le C\sum_{j\ge 1}\int\left(\int \rho(k)^{-1}w(y-2^{-j}k)\,dk\right)|\Delta_j(f)(y)|^2\,dy \\ &\quad \le C\sum_{j\ge 1}\int w(y)|\Delta_j(f)(y)|^2\,dy \\ &\quad \le C\|f\|_{L^2(w)}^2. \end{aligned}$$

Observing that the Fourier transform of $\int A_j^{(2)}(x,k)E_j(f)(x,k)\,dk$ is supported in an annulus of the form $\{c_1 2^j < |\xi| < c_2 2^j\}$, $c_1, c_2 > 0$, we apply the weighted Littlewood-Paley inequality. Then by the Schwarz inequality and (2.2) we have

$$\begin{aligned}
&\int \left| \sum_{j=1}^{\infty} \int A_j^{(2)}(x,k)E_j(f)(x,k)\,dk \right|^2 w(x)\,dx \\
&\leq C \int \sum_{j=1}^{\infty} \left| \int A_j^{(2)}(x,k)E_j(f)(x,k)\,dk \right|^2 w(x)\,dx \\
&\leq C \sum_{j\geq 1} \int \left(\int q^{(2)}(x,k,j)^2\,dk \right) \left(\int \rho(k)^{-1} |E_j(f)(x,k)|^2\,dk \right) w(x)\,dx \\
&\leq C \sum_{j\geq 1} \iint \rho(k)^{-1} |E_j(f)(x,k)|^2\,dk\,w(x)\,dx \\
&\leq C\|f\|_{L^2(w)}^2,
\end{aligned}$$

where we can have the last inequality as in the previous paragraph. Collecting the results, we see that $\tilde{\sigma}(x,D)$ is bounded on L_w^2 where $\tilde{\sigma}(x,\xi) = \sigma(x,\xi) - \sigma(x,\xi)\Phi(\xi)$.

The operator $\tau(x,D)$ where $\tau(x,\xi) = \sigma(x,\xi)\Phi(\xi)$ can be treated by using Lemma 2 as follows: by Schwarz's inequality, we see that

$$\begin{aligned}
\left| \int \tau(x,\xi)\hat{f}(\xi)e^{2\pi i\langle x,\xi\rangle}\,d\xi \right|^2 &= \left| \int B(x,k)f(x+k)\,dk \right|^2 \\
&\leq \int r(x,k)^2\,dk \int \rho(k)^{-1}|f(x+k)|^2\,dk \\
&\leq C \int \rho(k-x)^{-1}|f(k)|^2\,dk.
\end{aligned}$$

Integrating with respect to $w(x)\,dx$, we get the L_w^2 boundedness. This completes the proof of Theorem 5. □

3. Proof of Theorem 6. The following is a weighted version of Theorem 2 of Carbery [**1**].

Proposition 1. *Let α be a nonnegative function on $\mathbf{Z}$ such that*

$$\sum_{k\le 0}|k|\alpha(k)<\infty.$$

Let $\sigma(x,\xi)$ be continuous and bounded on $\mathbf{R}^n\times\mathbf{R}^n$. Let $w\in A_1$ and suppose that $\sigma(x,D)$ is bounded on L^2_w. Put $\sigma_i(x,\xi)=\sigma(x,\xi)\Psi(2^i\xi)$, $i\in\mathbf{Z}$, where $\Psi\in C_0^\infty(\mathbf{R}^n)$ is as in Section 1. Suppose that

$$|\sigma_i*(\hat\Psi)_{2^{-j}}|_{L^1_w}\le\alpha(i-j)\quad \textit{for all } i,j\in\mathbf{Z} \textit{ with } i\le j,$$

where the convolution is taken in the ξ-variable and $|\sigma|_{L^1_w}$ denotes the L^1_w–L^1_w operator norm of $\sigma(x,D)$. Then $\sigma(x,D)$ is bounded from L^1_w to $L^{1,\infty}_w$ and from H^1_w to L^1_w.

The proof is similar to the one given in [**1**] for the unweighted case. Let T be a singular integral operator with kernel $K(x,y)$. Put $K_j(x,y)=K(x,y)\Psi(2^{-j}(x-y))$ and $T_jf(x)=\int K_j(x,y)f(y)\,dy$. Let φ be as in the proof of Lemma 1 and $P_jf(x)=\varphi_{2^j}*f(x)$. Suppose T is bounded on L^2_w, $w\in A_1$. Then the L^1_w–$L^{1,\infty}_w$ boundedness of T follows from the weighted version of the Hörmander condition

$$\sup_{j\in\mathbf{Z}}\left|\sum_{l\ge 0}T_{j+l}(I-P_j)\right|_{L^1_w}<\infty,$$

where I denotes the identity operator. We can use this result to prove the L^1_w–$L^{1,\infty}_w$ boundedness of Proposition 1. To prove the H^1_w–L^1_w boundedness, we use the atomic decomposition for H^1_w.

To apply Proposition 1 for the proof of Theorem 6, we need the following

Lemma 3. *Let $w\in A_1$, ρ and β be as in Theorem 6. Suppose that*

$$|A_j(x,k)|\le C\rho(k)^{-1}.$$

Then

$$|\tilde\sigma_m*(\hat\Psi)_{2^{-j}}|_{L^1_w}\le C\beta(2^{-m+j})^{-1}\quad \textit{for all } m,\ j\in\mathbf{Z} \textit{ with } m\le j,$$

where $\tilde{\sigma}(x,\xi) = \sigma(x,\xi) - \sigma(x,\xi)\Phi(\xi)$*, as before.*

We also need the following, which can be easily seen.

Lemma 4. *Let* $w \in A_1$*. Suppose that* $\theta * w(x) \le Cw(x)$ *almost everywhere, where* θ *is as in* (1.10)*, and that*

$$|B(x,k)| \le C\rho(k)^{-1}.$$

Then $\tau(x,D)$ *is bounded on* L^1_w *and* L^2_w*, where* $\tau(x,\xi) = \sigma(x,\xi)\Phi(\xi)$*, as before.*

We first prove Lemma 3. Put

$$\begin{aligned} b_j(x,\xi) &= \sigma(x,\xi)\Psi(2^{-j}\xi) \\ &= \int_{\mathbf{R}^n} A_j(x,k)\exp(2\pi i\langle 2^{-j}k,\xi\rangle)\,dk\psi(2^{-j}\xi)^2, \\ K_{j,l,m}(x,y) &= \mathcal{F}^{-1}[(b_l)_m(x,\cdot) * (\hat{\Psi})_{2^{-j}}](y), \end{aligned}$$

where the inverse Fourier transform is taken with respect to the ξ-variable. Then, writing $u(x) = \int |\widehat{\psi^2}(x+k)|\rho(k)^{-1}\,dk$, we have for $m-2 \le -l \le m+2$, $l \ge 1$,

$$\begin{aligned} &\int |K_{j,l,m}(x,x-y)|w(x)\,dx \\ &\quad\le C\int \rho(k)^{-1}\int 2^{(l+m)n}\int |\widehat{\psi^2}(2^{(l+m)}(x-z)+k)||\hat{\Psi}(z)|\,dz \\ &\qquad\qquad \cdot |\Psi(2^{m-j}x)|w(2^m x+y)\,dx\,dk \\ &\quad= C\int \rho(k)^{-1}\iint |\widehat{\psi^2}(x+k)||\Psi(2^{-j-l}x+2^{m-j}z)| \\ &\qquad\qquad \cdot w(2^m z+2^{-l}x+y)\,dx|\hat{\Psi}(z)|\,dz\,dk \\ &\quad= C\iint u(x)|\Psi(2^{-j-l}x+2^{m-j}z)|w(2^m z+2^{-l}x+y)\,dx|\hat{\Psi}(z)|\,dz. \end{aligned}$$

Since Ψ is supported in $\{1/2 \le |x| \le 2\}$, by the properties (1) and (3) of β we see that

$$\begin{aligned} |\Psi(2^{-j-l}x+2^{m-j}z)| &\le C\beta(2^{-m+j})^{-1}\beta(|2^{-m-l}x+z|) \\ &\le C\beta(2^{-m+j})^{-1}[\beta(|x|)+\beta(|z|)]. \end{aligned}$$

Since $u(x) \le C\rho(-x)^{-1}$ by our assumption, by (1.14) we have

$$
\begin{aligned}
&\int |K_{j,l,m}(x, x-y)| w(x)\, dx \\
&\quad \le C\beta(2^{-m+j})^{-1} \iint \rho(-x)^{-1}[\beta(|x|) + \beta(|z|)] \\
&\qquad\qquad\qquad \cdot w(2^m z + 2^{-l}x + y)\, dx |\hat{\Psi}(z)|\, dz \\
&\quad \le C\beta(2^{-m+j})^{-1} \int (1 + \beta(|z|)) w(2^m z + y) |\hat{\Psi}(z)|\, dz \\
&\quad \le C\beta(2^{-m+j})^{-1} w(y).
\end{aligned}
$$

To get the last inequality, we have used the growth condition (2) of β. From this we can easily get the conclusion of Lemma 3. □

Next we prove Lemma 4. We have

$$
\begin{aligned}
\left| \int \tau(x,\xi) \hat{f}(\xi) e^{2\pi i \langle x,\xi\rangle}\, d\xi \right| &= \left| \int B(x,k) f(x+k)\, dk \right| \\
&\le C \int \rho(k-x)^{-1} |f(k)|\, dk.
\end{aligned}
$$

Integrating with respect to $w(x)\,dx$, we get the L^1_w boundedness. The L^2_w boundedness can be proved as in the last paragraph of Section 2. □

We see that $\tilde{\sigma}(x, D)$ (see Lemma 3) is bounded on L^2_w by the L^2_w boundedness of $\tau(x, D)$ (see Lemma 4) and $\sigma(x, D)$. Therefore, by Lemma 4 and Lemma 3 along with Proposition 1, now we can conclude the proof of Theorem 6.

4. Proofs of Theorems 3 and 4. We first prove Theorem 3. We prove the validity of the conditions (1.11), (1.12) and (1.13) with $\rho(k) = (1 + |k|^2)^s$, $s = [n/2] + d$, where d satisfies $a > d$ and $[n/2] + d > n/2$. By integration by parts,

$$
\begin{aligned}
A_j(x,k) = (2\pi i k_m)^{-[n/2]} \int_{\mathbf{R}^n} \Big[\Big(\frac{\partial}{\partial \xi_m}\Big)^{[n/2]} (\sigma(x, 2^j\xi)\Psi(\xi)) \Big] \\
\cdot \exp(-2\pi i \langle k, \xi\rangle) d\xi.
\end{aligned}
$$

Let ψ be as in Section 2. Then by applying Plancherel's theorem, we have for $l \geq 0$,

$$\begin{aligned}
&\int_{|k|\approx|k_m|, 2^l\leq|k|\leq 2^{l+1}} |A_j(x,k)|^2(1+|k|^2)^s\,dk \\
(4.1)\qquad &\quad \leq C2^{2sl}\int_{|k|\approx|k_m|}|\psi(2^{-l}k)A_j(x,k)|^2\,dk \\
&\quad \leq C2^{2dl}\int_{\mathbf{R}^n}\Big|\hat{\psi}_{2^{-l}} * \Big[\Big(\frac{\partial}{\partial\xi_m}\Big)^{[n/2]}(\sigma(x,2^j\cdot)\Psi(\cdot))\Big](\xi)\Big|^2\,d\xi.
\end{aligned}$$

Put $F(x,\xi) = (\partial/\partial\xi_m)^{[n/2]}(\sigma(x,2^j\xi)\Psi(\xi))$. Then by (1.4) and (1.5) with $L = [n/2]$ we have $|F(x,\xi)| \leq C$ and

$$(4.2)\qquad |F(x,\xi+\eta) - F(x,\xi)| \leq C|\eta|^a.$$

When $|\xi| \geq 1$, by (4.2) we see that

$$\begin{aligned}
&\Big|\hat{\psi}_{2^{-l}} * \Big[\Big(\frac{\partial}{\partial\xi_m}\Big)^{[n/2]}(\sigma(x,2^j\cdot)\Psi(\cdot))\Big](\xi)\Big| \\
&\qquad = \Big|\int[F(x,\xi+\eta)-F(x,\xi)]\hat{\psi}_{2^{-l}}(\eta)\,d\eta\Big| \\
&\qquad \leq \Big|\int_{|\eta|<|\xi|/2}[F(x,\xi+\eta)-F(x,\xi)]\hat{\psi}_{2^{-l}}(\eta)\,d\eta\Big| \\
&\qquad\quad + \Big|\int_{|\eta|\geq|\xi|/2}[F(x,\xi+\eta)-F(x,\xi)]\hat{\psi}_{2^{-l}}(\eta)\,d\eta\Big| \\
&\qquad \leq C\chi_0(\xi)\int|\eta|^a|\hat{\psi}_{2^{-l}}(\eta)|\,d\eta + C(2^l|\xi|)^{-2n} \\
&\qquad \leq C2^{-al}(1+|\xi|)^{-2n},
\end{aligned}$$

where χ_0 is the characteristic function of the ball $\{|\xi| \leq 5\}$. We also have this estimate for $|\xi| < 1$. Using this in (4.1) we have

$$(4.3)\qquad \int_{\substack{|k|\approx|k_m|\\|k|\geq 1}}|A_j(x,k)|^2(1+|k|^2)^s\,dk \leq \sum_{l\geq 0}C2^{2dl}2^{-2al} \leq C.$$

It is easier to get the estimate

$$\int_{|k|\leq 1}|A_j(x,k)|^2(1+|k|^2)^s\,dk \leq C.$$

Using this and (4.3) for $m = 1, \dots, n$, we see that the condition (1.11) holds.

Next we show that the condition (1.12) holds. By integration by parts,

$$\begin{aligned}
&A_j(x+y,k) - A_j(x,k)\\
&= \int_{\mathbf{R}^n} (\sigma(x+y,2^j\xi) - \sigma(x,2^j\xi))\Psi(\xi)\exp(-2\pi i\langle k,\xi\rangle)\,d\xi\\
&= (2\pi i k_m)^{-[n/2]} \int_{\mathbf{R}^n} \Big[\Big(\frac{\partial}{\partial\xi_m}\Big)^{[n/2]}((\sigma(x+y,2^j\xi) - \sigma(x,2^j\xi))\Psi(\xi))\Big]\\
&\qquad\qquad \cdot \exp(-2\pi i\langle k,\xi\rangle) d\xi.
\end{aligned}$$

Put $G(x,y,\xi) = (\partial/\partial\xi_m)^{[n/2]}((\sigma(x+y,2^j\xi) - \sigma(x,2^j\xi))\Psi(\xi))$. Then by Plancherel's theorem we have, as above, for $l \geq 0$,

$$\begin{aligned}
\int_{\substack{|k|\approx|k_m|\\ 2^l\leq|k|\leq 2^{l+1}}} |A_j(x+y,k) - A_j(x,k)|^2(1+|k|^2)^s\,dk\\
\leq C2^{2dl}\int_{\mathbf{R}^n} |[\hat{\psi}_{2^{-l}} * G(x,y,\cdot)](\xi)|^2\,d\xi.
\end{aligned} \tag{4.4}$$

By (1.6) and (1.7) with $M = [n/2]$ and $a = b$ we have $|G(x,y,\xi)| \leq C\omega(2^j,|y|)$ and

$$|G(x,y,\xi+\eta) - G(x,y,\xi)| \leq C|\eta|^a\omega(2^j,|y|). \tag{4.5}$$

Using (4.5) and arguing as in the proof for (1.11) above, we can see that

$$|[\hat{\psi}_{2^{-l}} * G(x,y,\cdot)](\xi)| \leq C2^{-al}\omega(2^j,|y|)(1+|\xi|)^{-2n}.$$

Using this in (4.4) and summing up in $l \geq 0$, we have

$$\int_{\substack{|k|\approx|k_m|\\ |k|\geq 1}} |A_j(x+y,k) - A_j(x,k)|^2(1+|k|^2)^s\,dk \leq C\omega(2^j,|y|)^2. \tag{4.6}$$

We also have

$$\int_{|k|\leq 1} |A_j(x+y,k) - A_j(x,k)|^2(1+|k|^2)^s\,dk \leq C\omega(2^j,|y|)^2.$$

Using this and (4.6) for $m = 1, \dots, n$, we can get (1.12).

The condition (1.13) can be proved similarly. Since $\rho(x) = (1+|x|^2)^s$ satisfies (1.10) for all $w \in A_1$, now Theorem 3 follows from Theorem 5. $\square$

Next we prove Theorem 4. By integration by parts and estimates similar to (4.2), under the assumption of Theorem 4, we have

$$\begin{aligned} |A_j(x,k)| &\le C(1+|k|^2)^{-(n+a)/2}, \quad j \ge 1, \\ |B(x,k)| &\le C(1+|k|^2)^{-(n+a)/2}. \end{aligned}$$

Also by Theorem 3, $\sigma(x,D)$ is bounded on L^2_w for $w \in A_1$. Furthermore, we see that $\rho(x) = (1+|x|^2)^{(n+a)/2}$ satisfies all the requirements of Theorem 6 with any $w \in A_1$ and, for example, $\beta(t) = t^{a/2}$ for (1.14). Therefore we can apply Theorem 6 to get Theorem 4. $\square$

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