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DISTRIBUTION OF MINIMAL VARIETIES IN SPHERES IN TERMS OF THE COORDINATE FUNCTIONS

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ABSTRACT. Let M be a compact k-dimensional Riemannian manifold minimally immersed in the unit n-dimensional sphere S^n. It is easy to show that for any $p \in S^n$ the boundary of the geodesic ball in S^n with radius $\pi/2$ and center at p (in this case this boundary is an equator) must intercept the manifold M. When the codimension is 1, i.e., $k = n-1$, it is known that the Ricci curvature is not greater than 1. We will prove that if the Ricci curvature is not greater than $1-\alpha^2/(n-2)$, then the boundary of every geodesic ball with radius $\cot^{-1}(\alpha)$ must intercept the manifold M. We give examples of manifolds for which the radius $\cot^{-1}(\alpha)$ is optimal. Next, for any codimension, i.e., for any $M^k \subset S^n$, we find a number r_1 that depends only on n such that for any collection of $n+1$ points $\{p_i\}_{i=1}^{n+1}$ in S^n that constitutes an orthonormal basis of $\mathbf{R}^{n+1}$, the union of the boundaries of the geodesic balls with radius r_1 and center p_i, $i = 1, 2, \dots, n+1$, must intercept the manifold M.

1. Introduction and preliminaries. Let M be a compact, oriented minimal hypersurface immersed in the n-dimensional unit sphere S^n. Let ν be a unit normal vector field along M. For any tangent vector $v \in T_mM$, $m \in M$, the shape operator A is given by $A(v) = -\overline{D}_v\nu$ where $\overline{D}$ denotes the Levi Civita connection in $\mathbf{R}^{n+1}$. With the same notation, for any tangent vector field W, the Levi Civita connection on M is given by $D_vW = (\overline{D}_vW)^T$ where $(\quad)^T$ denotes the orthogonal projection from $\mathbf{R}^{n+1}$ to T_mM. For a function $f : M \to \mathbf{R}$, ∇f will denote the gradient of f. For any pair of vectors $v, w \in T_mM$ the Hessian of f is given by $H(f)(v,w) = \langle D_v\nabla f, w\rangle$, where $\langle \quad , \quad \rangle$ denotes the inner product in $\mathbf{R}^{n+1}$. The Laplacian of f is given by $\Delta(f) = \sum_{i=1}^{n-1} H(f)(v_i,v_i)$ where $\{v_i\}_{i=1}^{n-1}$ is an orthonormal basis of T_mM.

For a given $w \in \mathbf{R}^{n+1}$, let us define the functions $l_w : M \to \mathbf{R}$ and $f_w : M \to \mathbf{R}$ by $l_w(m) = \langle m, w\rangle$ and $f_w(m) = \langle \nu(m), w\rangle$. Clearly

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the function l_w is the restriction to M of the linear function in $\mathbf{R}^{n+1}$ whose gradient is the constant vector w; therefore, the gradient of the function l_w at $m \in M$ is the projection of the vector w to T_mM, i.e.,

$$\nabla l_w = w^T = w - l_w(m)m - f_w(m)\nu(m).$$

Let v_i, v_j be two vectors in T_mM. The Hessian of l_w at $m \in M$ is given by

$$\begin{aligned} H(l_w)(v_i,v_j) &= \langle D_{v_i}\nabla l_w, v_j\rangle = \langle \overline{D}_{v_i}\nabla l_w, v_j\rangle \\ &= -l_w\langle v_i, v_j\rangle - f_w\langle \overline{D}_{v_i}\nu, v_j\rangle \\ &= -l_w\langle v_i, v_j\rangle + f_w\langle A(v_i), v_j\rangle. \end{aligned} \tag{1.1}$$

From the equation above and the fact that A is traceless (minimality of M) we get that

$$\Delta l_w = -(n-1)l_w. \tag{1.2}$$

Remark 1.3. As a corollary of equation (1.2) we get that every coordinate function l_w must change sign; therefore, the boundary of every geodesic ball with radius $\pi/2$ must intersect M.

Given any nonequatorial compact minimal hypersurface in S^n we know that there exists a radius r, $r < \pi/2$, such that M must intersect the boundary of *every* geodesic ball in S^n with radius r. Let γ_M be the minimum r with the property above. In Section 2 we will use the expression for the Hessian of the coordinate function l_w to find an upper bound for γ_M, namely we will show:

Theorem 1.4. *Let M^{n-1} be a minimal hypersurface immersed in S^n, and let $\{\lambda_i(m)\}_{i=1}^{n-1}$ be the eigenvalues of the shape operator at $m \in M$. Define $\bar{\alpha}(m) = \min\{|\lambda_i(m)|, i = 1, \dots, n-1\}$ and let α be the minimum over M of the function $\bar{\alpha}$. If r_0 satisfies that $\cot(r_0) = \alpha$ and $0 < r_0 \leq \pi/2$, i.e., $r_0 = \cot^{-1}(\alpha)$, then the boundary of every geodesic ball in S^n with radius r_0 must intersect M.*

Notice that if $\alpha = 0$, then Theorem 1.4 reduces to Remark 1.3. A direct computation shows that, if M is the minimal Clifford hypersurface, $M = \{(x,y) \in \mathbf{R}^{s+1} \times \mathbf{R}^{s+1} : \|x\|^2 = \|y\|^2 = 1/2\}$, then for any

$r < \pi/4$, the boundary of the geodesic ball with center at $(1, 0, \dots, 0)$ does not intersect M, therefore $\gamma_M \geq \pi/4$. The principal eigenvalues of the shape operator A of M are either 1 or -1 everywhere, then $\alpha = 1$ in this case and we get that $\pi/4 = \cot^{-1}(1) \geq \gamma M \geq \pi/4$. This example shows that the estimate in Theorem 1.4 is sharp.

Let us rewrite Theorem 1.4 in terms of the curvature of M. Denote by R and Ricci the curvature tensor and the Ricci curvature of M, respectively. The Gauss equation states that

$$\begin{aligned}\langle R(v,w)v,w\rangle = \langle v,v\rangle\langle w,w\rangle - \langle v,w\rangle\langle v,w\rangle + \langle A(v),v\rangle\langle A(w),w\rangle \\ - \langle A(w),v\rangle\langle A(v),w\rangle.\end{aligned}$$

Therefore, if $\{v_i\}_{i=1}^{i=n-1}$ is an orthonormal basis of T_mM, we have

$$\begin{aligned}\mathrm{Ricc}\,(v) &= \frac{1}{n-2}\left(\sum_{i=1}^{n-1}\langle R(v,v_i)v,v_i\right) \\ &= \frac{\sum_{i=1}^{n-1}(\langle v,v\rangle\langle e_i,e_i\rangle - \langle e_i,v\rangle\langle e_i,v\rangle}{n-2} \\ &\quad + \frac{\langle A(v),v\rangle\langle A(e_i),e_i\rangle - \langle A(e_i),v\rangle\langle A(e_i),v\rangle)}{n-2} \\ &= \frac{(n-1)|v|^2 - |v|^2 - |A(v)|^2}{n-2} \\ &= |v|^2 - \frac{|A(v)|^2}{n-2}.\end{aligned}$$

By the equation above, we get that another way to define α is given by

$$\max_{v\in T^1M} \mathrm{Ricc}\,(v) = 1 - \frac{\alpha^2}{n-2}$$

where $T^1M = \{v \in T_mM : m \in M \text{ and } |v| = 1\}$. By the observations made above, we get

Corollary 1.5. *Let $M \subset S^n$ be a minimal hypersurface. If $\mathrm{Ricc}\,(v) \leq 1 - \alpha^2/(n-2)$ for every $v \in T^1M$, then the boundary of every geodesic ball in the sphere with radius $\cot^{-1}(\alpha)$ must intersect M.*

The result in the previous corollary needs the minimality condition. To see this, it is enough to look at the following family of flat surfaces in S^3 given by

$$M_{r_1 r_2} = \{(x,y) \in \mathbf{R}^2 \times \mathbf{R}^2 : |x|^2 = r_1^2,\ |y|^2 = r_2^2,\ r_1^2 + r_2^2 = 1 \\ \text{and } r_1 \le r_2\}.$$

A direct computation shows that $\gamma_{M_{r_1 r_2}} = \sin^{-1}(r_2)$. We also have that α is 1 because all these surfaces are flat; therefore, if $r_1 < r_2$, then $\gamma_{M_{r_1 r_2}} > \pi/4$; hence, $\pi/4 = \cot^{-1}(\alpha)$ is not an upper bound for $\gamma_{M_{r_1 r_2}}$. The examples above show us that among all Euclidean products of circles in S^3, the minimal Clifford tori are the ones that best make the work of "trying to be as close to every point in S^3 as possible". In other words the minimal Clifford torus minimizes $\gamma_{M_{r_1 r_2}}$ in the family $M_{r_1 r_2}$.

Our second result states that if M is a minimal variety in S^n, then at least one of its coordinate functions must take the value $-(1 - \sqrt{(n-2)/(n+1)})$ at some point. Namely, we will show

Theorem 1.6. *Let M^k be a minimal k-dimensional manifold immersed in the n-dimensional unit sphere S^n. Then for every orthonormal basis of $\mathbf{R}^{n+1}$, $\{p_i\}_{i=1}^{i=n-1}$ for some i, M must intersect the boundary of geodesic ball with center at p_i and radius $\cos^{-1}(1 - \sqrt{(n-2)/(n+1)})$.*

Before I proceed, I would like to thank Professor Bruce Solomon for his lessons on mathematics and his comments on this paper. I would also like to thank Professor Peter Li for meeting with me to discuss mathematics; one of his comments motivated the idea for Theorem 1.4.

2. Proof of the theorems. We start this section stating and proving Theorem 1.4. This result is a consequence of equation (1.1) for the Hessian of the coordinate functions. Notice that in both of the Theorems, 1.4 and 1.5, we may assume that our manifold M is orientable, since otherwise the results follow by applying the theorem to the double covering of M.

Theorem 1.4. *Let M^{n-1} be a minimal hypersurface immersed in S^n, and let $\{\lambda_i(m)\}_{i=1}^{n-1}$ be the eigenvalues of the shape operator at $m \in M$. Define $\bar{\alpha}(m) = \min\{|\lambda_i(m)|,\ i = 1, \dots, n-1\}$ and let α be the minimum over M of the function $\bar{\alpha}$. If r_0 satisfies that $\cot(r_0) = \alpha$ and $0 < r_0 \leq \pi/2$, i.e., $r_0 = \cot^{-1}(\alpha)$, then the boundary of every geodesic ball in S^n with radius r_0 must intersect M.*

Proof. Since the result is trivial when M is totally geodesic we will assume that this is not the case. Notice that it is enough to prove that for every $v \in S^n$ the minimum of the coordinate function $l_v : M \to \mathbf{R}$ over M is less than or equal to $-\alpha/\sqrt{1+\alpha^2}$. Let m_0 be a point in M where the function l_v reaches its minimum. Since M is not an equator we have that $l_v(m_0) < 0$. We need to show that $l_v(m_0) \leq -\alpha/\sqrt{1+\alpha^2}$ or equivalently $|l_v(m_0)| \geq \alpha/\sqrt{1+\alpha^2}$. Since m_0 is a critical point of the function l_v we have $\nabla l_v = 0$, therefore,

$$1 = \|v\|^2 = \langle v, m_0\rangle^2 + \langle \nu(v), m_0\rangle^2 + \|\nabla l_v\|^2 = l_v(m_0)^2 + f_v(m_0)^2. \tag{2.1}$$

Let $\{v_i\}_{i=1}^{n-1}$ be an orthonormal basis of $T_{m_0}M$ that diagonalizes the shape operator A at m_0. Since m_0 is a minimum of l_v we get for $i = 1, \dots, n-1$ that

$$\begin{aligned} 0 &\leq H(l_v)(v_i, v_i) = -l_v(m_0)\langle v_i, v_i\rangle + f_v(m_0)\langle A(v_i), v_i\rangle \\ &= -l_v(m_0) + f_v(m_0)\lambda_i. \end{aligned} \tag{2.3}$$

Since $\sum_{i=1}^{n-1} \lambda_i = 0$ we can pick k such that $-f_v(m_0)\lambda_k$ is not negative. Using the definition of α, equation (2.1) and the inequalities (2.3) we get

$$|l_v(m_0)| = -l_v(m_0) \geq -f_v(m_0)\lambda_k = |-f_v(m_0)||\lambda_k| \geq \alpha\sqrt{1 - l_v(m_0)^2}.$$

Finally from the inequality above we can easily deduce the inequality we were looking for: $|l_v(m_0)| \geq \alpha/\sqrt{1+\alpha^2} = \beta$. $\square$

A direct computation shows that the formula for the Laplacian of the coordinate functions, (1.2), holds true for any codimension, i.e., we have that if M^k is a k-dimensional manifold minimally immersed in S^n, then $-\Delta l_w = k l_w$.

Now we will prove our second result. For the reader's convenience we will restate this theorem.

Theorem 1.6. *Let M^k be a minimal k-dimensional manifold immersed in the n-dimensional unit sphere S^n. Then for every orthonormal basis of $\mathbf{R}^{n+1}$, $\{p_i\}_{i=1}^{i=n-1}$ for some i, M must intersect the boundary of geodesic ball with center at p_i and radius $\cos^{-1}(1-\sqrt{(n-2)/(n+1)})$.*

Proof. Notice that it is enough to prove that at least one of the coordinate functions l_{p_i} takes a value less than or equal to $-(1-\sqrt{(n-2)/(n+1)})$. We will proceed by contradiction. Let us assume that all the functions $l_i = l_{p_i}$ are greater than or equal to $-(1-\sqrt{(n-2)/(n+1)})$. Therefore the vector fields $X_i = (1+l_i)^{-1}\nabla l_i$ are well defined. Let us compute the divergence of X_i:

$$\begin{aligned}\operatorname{div}(X_i) &= -(1+l_i)^{-2}\|\nabla l_i\|^2 + (1+l_i)^{-1}(-kl_i)\\ &= \frac{k}{2}\Big(-1+(1+l_i)^{-2}\Big(1-\frac{2}{k}\|\nabla l_i\|^2 - l_i^2\Big)\Big).\end{aligned}$$

Since $(1+l_i)^{-2} \le \big(1-(1-\sqrt{(n-2)/(n+1)})\big)^{-2} = (n+1)/(n-2)$ by assumption and $1-2/k\|\nabla l_i\|^2 - l_i^2 > 0$, we get from the expression above after using the divergence theorem that,

$$\begin{aligned}0 &= \int_M \Big(-1+(1+l_i)^{-2}\Big(1-\frac{2}{k}\|\nabla l_i\|^2 - l_i^2\Big)\Big)\\ &< \int_M \Big(-1+\frac{n+1}{n-2}\Big(1-\frac{2}{k}\|\nabla l_i\|^2 - l_i^2\Big)\Big).\end{aligned}$$

Notice that $\sum_{i=1}^{n+1} l_i^2 = 1$ and by Stoke's theorem $\int_M \|\nabla l_i\|^2 = k\int_M l_i^2$. Then if we sum the inequalities above from $i=1$ to $i=n+1$, we get

$$\begin{aligned}0 &< \sum_{i=1}^{n+1}\int_M \Big(-1+\frac{n+1}{n-2}\Big(1-\frac{2}{k}\|\nabla l_i\|^2 - l_i^2\Big)\Big)\\ &= \int_M \Big(-(n+1)+\frac{n+1}{n-2}(n+1-2-1)\Big) = 0.\end{aligned}$$

This contradiction proves the theorem. $\square$

Remark. The case $n = 3$ in the theorem above is also a consequence of the fact that any two minimal surfaces in S^3 must intersect [**1**]. For any minimal surface M in S^3 the surface $-M = \{p \in S^3 : -p \in M\}$ is also minimal. If $m_0 \in -M \cap M$ then one can check that either m_0 or $-m_0$ must intersect the union of the four geodesic balls with center at p_i and radius $1/2$.

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