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KARLIN'S CONJECTURE AND A QUESTION OF PÓLYA

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ABSTRACT. The paper answers an old question of Pólya involving Descartes' Rule of Signs and a related conjecture of Karlin involving the signs of Wronskians of entire functions and their derivatives. Counterexamples are given along with classes of functions for which the conjecture is valid.

0. Introduction. The purpose of this paper is to answer an old unsolved question of Pólya (c. 1934) and to resolve a related conjecture of Karlin (c. 1967). In Section 1 we state Pólya's question, Karlin's conjecture, provide some background information and recall the definitions and terminology that will be used in the sequel. For a general class of polynomials closed under differentiation, we prove in Section 2 that Descartes' Rule of Signs is equivalent to the sign regularity of certain Hankel determinants, Theorem 2.3. The counterexamples we give to Karlin's conjecture, Section 3, also provide a negative answer to Pólya's question. (While this manuscript was in preparation, Dr. Dimitar Dimitrov has kindly informed the authors that he has also obtained a counterexample to Karlin's conjecture.) In Section 4 we investigate some classes of entire functions for which Pólya's question has an affirmative answer, Theorem 4.6 and Corollary 4.8, and for which Karlin's conjecture is valid, Theorem 4.5.

1. Background information and definitions. In 1934, in connection with his investigation of the distribution of zeros of polynomials and the total positivity of certain matrices, Schoenberg [**24**, p. 562] cited the following question of Pólya.

Pólya's question. *Let $f(x)$ be a polynomial of degree n with only real and simple zeros, $x_1 < x_2 < \cdots < x_n$. Let $Z_{(x_n,\infty)}(f)$ denote the number of real zeros of f in the interval (x_n, ∞), and let*

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$V(\{a_0, a_1, \ldots, a_n\})$ denote the number of sign changes in the sequence of real numbers $\{a_0, a_1, \ldots, a_n\}$ (where, as usual, the zero terms of the sequence are deleted when counting the sign changes). Is it true that, for any sequence of real numbers $\{a_0, a_1, \ldots, a_n\}$ such that $\sum_{j=0}^{n} a_j f^{(j)} \neq 0$,

$$Z_{(x_n,\infty)}\left(\sum_{j=0}^{n} a_j f^{(j)}(x)\right) \leq V(\{a_j\}_{j=0}^{n})? \tag{1.1}$$

In other words, Pólya's question is whether or not Descartes' Rule of Signs holds for the sequence $f(x), f'(x), \ldots, f^{(n)}(x)$ on the interval (x_n, ∞). We recall, see, for example, [**23**, Part V, pp. 87–90] that a sequence of functions $f_0(x), f_1(x), \ldots, f_n(x)$ satisfies Descartes' Rule of Signs on an interval (a, b), if for any sequence of real numbers $\{a_0, a_1, \ldots, a_n\}$ such that $\sum_{j=0}^{n} a_j f_j \neq 0$,

$$Z_{(a,b)}\left(\sum_{j=0}^{n} a_j f_j(x)\right) \leq V(\{a_j\}_{j=0}^{n}), \tag{1.2}$$

where $Z_{(a,b)}(f)$ denotes the number of real zeros of f in the interval (a, b). The question of Pólya is natural in view of the plethora of examples of sequences of entire and meromorphic functions which satisfy (1.1), cf. [**23**, Part V, #36, #41, #77, #84 and #85]. While from the vast literature dealing with Descartes' Rule of Signs and its generalizations we cite here only [**9**], [**12**, Chapter 6], [**14**], [**16**, pp. 191–193], [**19**, pp. 53–87], [**21**, pp. 40–50] and [**22**, pp. 22–32], we specifically mention that it is known that (1.1) holds for sequences of orthogonal polynomials [**17, 20, 24**]. Until the 1950's various versions of Descartes' Rule of Signs (as for example Sylvester's theorem [**22**, Satz IV] were deemed interesting but isolated results. In the subsequent decades, these "isolated" theorems played a pivotal role in the theory of variation diminishing transformations, see, for example, [**12**, Chapters 5–6 and 9], in the theory of Chebyshev systems [**14**], [**4**, pp. 91–100] and the theory of total positivity and combinatorics [**2, 5, 12**].

Pólya's question can also be formulated with the aid of certain determinants. Indeed, Pólya and Szegö, see [**23**, Part V, #87 and #90] have provided a criterion, cf. Lemma 2.2 below, expressed in terms of Wronskian determinants, which is both necessary and sufficient for the validity of Descartes' Rule of Signs. We recall that the *Wronskian* (*determinant*) of the sufficiently smooth functions $f_0(x), f_1(x), \dots, f_n(x)$ is defined as

$$\begin{aligned} W(f_0(x), f_1(x), \dots, f_n(x)) &:= \det(f_i^{(j)}(x))_{i,j=0}^n \\ &= \begin{vmatrix} f_0(x) & f_0'(x) & \cdots & f_0^{(n)}(x) \\ f_1(x) & f_1'(x) & \cdots & f_1^{(n)}(x) \\ \vdots & \vdots & & \vdots \\ f_n(x) & f_n'(x) & \cdots & f_n^{(n)}(x) \end{vmatrix}. \end{aligned} \tag{1.3}$$

In order to motivate and state Karlin's conjecture referred to in the title of the paper, we will need to introduce the following definitions.

Definition 1.1. A *TP-sequence* (*totally positive sequence*) is a sequence $\{\alpha_k\}_{k=0}^\infty$, $\alpha_k \in \mathbb{R}$, for which $\phi(x) = \sum_{k=0}^\infty \alpha_k x^k$ is an *entire* function and all the minors, i.e., the determinants of square submatrices, of the matrix

$$M := \begin{pmatrix} \alpha_0 & 0 & 0 & 0 & 0 & \dots \\ \alpha_1 & \alpha_0 & 0 & 0 & 0 & \dots \\ \alpha_2 & \alpha_1 & \alpha_0 & 0 & 0 & \dots \\ \alpha_3 & \alpha_2 & \alpha_1 & \alpha_0 & 0 & \dots \\ & & \dots & & & \end{pmatrix} \tag{1.4}$$

are nonnegative.

Remark 1.2. (A remark on sign regularity.) We recall that a matrix $A_n = (a_{i,j})_{i=1,j=1}^n$ is a *Hankel matrix*, if the general term of A_n is of the form $a_{i,j} = \alpha_{i+j-2}$. By rearranging the columns of the matrix M defined by (1.4), we can form the Hankel (sub)matrices $A_n = (\alpha_{i+j-2})_{i=1,j=1}^n$, which, since $\{\alpha_k\}_{k=0}^\infty$ is a TP-sequence, enjoy the following sign regularity:

$$(-1)^{n(n-1)/2}\det A_n = (-1)^{n(n-1)/2} \begin{vmatrix} \alpha_0 & \alpha_1 & \dots & \alpha_{n-1} \\ \alpha_1 & \alpha_2 & \dots & \alpha_n \\ & & \dots & \\ \alpha_{n-1} & \alpha_n & \dots & \alpha_{2n-2} \end{vmatrix} \geq 0. \tag{1.5}$$

Definition 1.3. A real entire function $\phi(x) := \sum_{k=0}^{\infty}(\gamma_k/k!)x^k$ is said to be of *type* I in the *Laguerre-Pólya class*, denoted $\phi(x) \in \mathcal{L}\text{-}\mathcal{P}I$, if $\phi(x)$ or $\phi(-x)$ can be expressed in the form

$$\phi(x) = cx^n e^{bx} \prod_{k=1}^{\infty} \left(1 + \frac{x}{x_k}\right), \tag{1.6}$$

where $c, b, x_k \in \mathbb{R}$, $x_k > 0$, $b \geq 0$, n is a nonnegative integer and $\sum_{k=1}^{\infty} 1/x_k < \infty$. We will write $\phi \in \mathcal{L}\text{-}\mathcal{P}^+$, if $\phi \in \mathcal{L}\text{-}\mathcal{P}I$ and if $\gamma_k \geq 0$ for all $k = 0, 1, 2 \dots$. For reasons of convenience we will allow $0 \in \mathcal{L}\text{-}\mathcal{P}^+$.

With the foregoing terminology, Karlin's conjecture [**12**, p. 390], see also [**8**, p. 258] regarding a misprint in [**12**, p. 390], is as follows.

Karlin's Conjecture. *Let $\phi \in \mathcal{L}\text{-}\mathcal{P}^+$. Then for all $n \geq 1$*

$$(-1)^{n(n+1)/2} W(f(x), f'(x), \dots, f^{(n)}(x)) \geq 0, \quad \textit{for all } x \geq 0, \tag{1.7}$$

where the Wronskian determinant is defined by (1.3).

What is the *raison d'être* for this conjecture? In the first place, the results and numerous examples of Karlin and Szegö's 157 page paper [**15**] tend to suggest that (1.7) is true. Also, by a celebrated result of Schoenberg [**25**], see also [**1**] or [**12**, p. 412], if

$$\phi(x) := \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k := \sum_{k=0}^{\infty} \alpha_k x^k \in \mathcal{L}\text{-}\mathcal{P}^+,$$

then $\{\alpha_k\}_{k=0}^{\infty}$ is a totally positive sequence and consequently this sequence possesses the sign regularity condition given by (1.5). One can use this observation to construct nontrivial functions $\phi \in \mathcal{L}\text{-}\mathcal{P}^+$ for which Karlin's conjecture is valid. While there are many other known special cases when (1.7) is true, here we will confine ourselves to the remark that in 1989 the authors have investigated certain polynomial invariants and used them to prove that (1.7) is true for 3×3 Wronskian determinants [**8**, Theorem 2.13].

2. The equivalence of Descartes' Rule of Signs and the sign regularity of Hankel determinants.

Notation. Let $\mathcal{C}_p$ be a set of polynomials with nonnegative coefficients which is closed under differentiation.

Note that the set of all polynomials in $\mathcal{L}$-$\mathcal{P}^+$ forms such a set $\mathcal{C}_p$.

With the notation above, we consider the following two statements:

($\mathrm{D_p}$) (**Descartes' Rule of Signs**). For every polynomial $f \in \mathcal{C}_p$, every sequence $\{a_j\}_{j=0}^m$ of real numbers, such that $\sum_{j=0}^m a_j f^{(j)} \neq 0$, and for every $m \leq \operatorname{degree} f$,

$$Z_+\left(\sum_{j=0}^{m} a_j f^{(j)}(x)\right) \leq V(\{a_j\}_{j=0}^m), \tag{2.1}$$

where $Z_+(f)$ denotes the number of positive real zeros of f, counting multiplicities, and where $V(\{a_j\}_{j=0}^m)$ denotes the number of sign changes in the sequence $\{a_0, a_1, \ldots, a_m\}$.

($\mathrm{H_p}$) (**Sign Regularity of Hankel Determinants**). For every polynomial $f \in \mathcal{C}_p$, $f \neq 0$, and for each positive integer $m \leq \operatorname{degree} f$,

$$(-1)^{m(m+1)/2} W(f(x), f'(x), \ldots, f^{(m)}(x)) > 0, \quad \text{for all } x > 0. \tag{2.2}$$

We shall need the following two lemmas in order to prove that the statements (D_p) and (H_p) are equivalent.

Lemma 2.1 [**12**, Theorem 3.2, p. 59] or [**10**]. *Let $A = (a_{ij})$ be an $m \times n$ matrix with $2 \leq m \leq n$. Assume that*

(1) *all $m \times m$ minors with consecutive columns have the same sign, and*

(2) *all $(m-1) \times (m-1)$ minors of the submatrix B consisting of A with the last row deleted have the same (nonzero) sign.*

Then all $m \times m$ minors of A have the same sign. □

Lemma 2.2 [**23**, Part V, # 87 and #90]. *For any sufficiently smooth function f, Descartes' Rule of Signs for $f(x), f'(x), \ldots, f^{(m)}(x)$*

is equivalent to the statement that for any integers $0 \le i_1 < i_2 < \cdots < i_k \le m$*, all Wronskian determinants* $W(f^{(i_1)}, f^{(i_2)}, \ldots, f^{(i_k)})$ *of order* k *are nonzero on* $(0, \infty)$ *and have the same sign.* □

Theorem 2.3. *The statements* (D_p) *and* (H_p) *are equivalent.*

Proof. Lemma 2.2 gives us the following equivalent version of $\mathrm{D_p}$:

(D1) For every $f \in \mathcal{C}_p$, each $k = 1, \ldots, m = \operatorname{degree} f + 1$ and any integers $0 \le i_1 < i_2 < \cdots < i_k \le n = \operatorname{degree} f$, all Wronskian determinants $W(f^{(i_1)}, f^{(i_2)}, \ldots, f^{(i_k)})$ of order k are nonzero on $(0, \infty)$ and have the same sign.

(D1) $\implies$ ($\mathrm{H_p}$). Assume that the polynomial $f \in \mathcal{C}_p$ has degree n. Then the Wronskian $W(f^{(n-k+1)}(x), \ldots, f^{(n)}(x)) = (-1)^{k(k-1)/2} \times (f^{(n)}(x))^k$ has sign $(-1)^{k(k-1)/2}$ because $f^{(n)}(x)$ is a positive constant. Since all kth order Wronskians have the same sign by (D1), we have $(-1)^{k(k-1)/2} W(f(x), f'(x), \ldots, f^{(k-1)}(x)) > 0 \quad$ for all $x > 0$.

($\mathrm{H_p}$) $\implies$ (D1). Assume that the polynomial $f \in \mathcal{C}_p$ has degree n. We proceed by induction on the order k of the Wronskian. If $k = 1$, then we use the fact that $f \in \mathcal{C}_p$ implies that $f^{(i_1)} \in \mathcal{C}_p$. Hence $W(f^{(i_1)}(x)) = f^{(i_1)}(x) > 0$ on $(0, \infty)$ implies that $Z_+(f) = 0 = V(a_0)$. Now assume that (D1) holds for all integers less than k. Then ($\mathrm{H_p}$) together with the induction hypothesis implies that all Wronskians of order $k-1$ have the appropriate sign $(-1)^{(k-1)(k-2)/2}$ on the interval $(0, \infty)$. By ($\mathrm{H_p}$), we also have $(-1)^{k(k-1)/2} W(f^{(l)}, \ldots, f^{(k+l-1)}) > 0$ on $(0, \infty)$ for each $l = 0, 1, \ldots, n-k+1$. Thus we can now apply Lemma 2.1 to conclude the proof that (D1) holds for Wronskians of order k. □

We state and prove another equivalent condition in terms of determinants which plays an important role in the study of Descartes systems, Haar systems and generalized convexity [**12**, p. 25], [**14**] as well as in the theory of special functions dealing with determinants of Turán type [**15**, Chapter 3]. Theorem 2.4 can be used in conjunction with our examples in the next two sections to provide further results and examples in these contexts.

Theorem 2.4. *The statements* (D_p) *and* (H_p) *are equivalent to the following: For any* $f \in \mathcal{C}_p$, *any integers* $0 \le i_0 < i_1 < \cdots < i_k \le \deg f$ *and any real numbers* $0 < t_0 < t_1 < \cdots < t_k$,

$$(-1)^{k(k+1)/2} \begin{vmatrix} f^{(i_0)}(t_0) & f^{(i_1)}(t_0) & \cdots & f^{(i_k)}(t_0) \\ f^{(i_0)}(t_1) & f^{(i_1)}(t_1) & \cdots & f^{(i_k)}(t_1) \\ \vdots & \vdots & & \vdots \\ f^{(i_0)}(t_k) & f^{(i_1)}(t_k) & \cdots & f^{(i_k)}(t_k) \end{vmatrix} > 0.$$

Proof. The equivalence is nearly proved in [**14**, Theorem 4.4]. What is left for us to prove is only that the signs of the determinants, known to be dependent only on the size k, are in fact $(-1)^{k(k+1)/2}$. Assume inductively that this is true for determinants of smaller size. Note that the claimed sign is the sign of the permutation which reverses the numbers 0 through k; that is, the sign associated with the secondary diagonal of the matrix when one expands the determinant. Since $f^{(i_0)}$ has the largest degree of the polynomials in the matrix, when we expand the determinant along the first column, the dominant term involving the arbitrarily large number t_k is the number $(-1)^k f^{(i_0)}(t_k)$ multiplied by a determinant of sign $(-1)^{k(k-1)/2}$, using the induction hypothesis. Therefore the sign of the larger determinant is as claimed. □

3. Counterexamples. In this section we exhibit some examples, involving polynomials as well as transcendental entire functions having an infinite number of real negative zeros, which show that Karlin's conjecture, see (1.7), in the absence of additional assumptions, fails in general. The polynomials in Example 3.1 below show that (H_p), cf. (2.2), does not always hold, and whence, by virtue of Theorem 2.3, we have a negative answer to Pólya's question (1.1) as well.

Example 3.1. Let $f(x) := x^3(x+1)^5$. Then the 5×5 Wronskian of f,

$$\begin{aligned} &W(f(x), f'(x), f''(x), f'''(x), f^{(4)}(x)) \\ &= 414720(x+1)^5(175616x^{15} + 1317120x^{14} + 4751040x^{13} + 10885840x^{12} \\ &\quad + 17702475x^{11} + 21649908x^{10} + 20599540x^9 + 15556680x^8 \\ &\quad + 9410130x^7 + 4548200x^6 + 1727268x^5 + 496920x^4 \\ &\quad + 101015x^3 + 12540x^2 + 600x - 24), \end{aligned}$$

is negative for small values of x. Such examples are rather delicate; for example, $x^4(x+1)^5$ is not a counterexample in the 5×5 Wronskian case.

One can avoid the root at 0; $(50x+1)^3(x+1)^5$ and $(x+1)^3(x+50)^5$ also yield counterexamples. In fact, it can be shown that the 5×5 Wronskians of the polynomials of the form $(x+a)^3(x+b)^5$, where $0<a<b$ and $b/a\geq 42$, produce counterexamples. Moreover, a small perturbation of the zeros will give examples with simple zeros.

Example 3.2. A specific counterexample to (D_p), cf. (2.1), can be obtained as in the solution of [**23**, V #87]. Let $r>0$ be the positive zero of the Wronskian in Example 3.1. Then, evaluated at this point r, the five rows are linearly dependent, say with

$$c_0f^{(j)}(r)+c_1f^{(j+1)}(r)+c_2f^{(j+2)}(r)+c_3f^{(j+3)}(r)+c_4f^{(j+4)}(r)=0,$$

for $j=0,1,2,3,4$. Thus the function

$$c_0f(x)+c_1f'(x)+c_2f''(x)+c_3f'''(x)+c_4f^{(4)}(x)$$

has a positive root r of multiplicity 5, while there are at most 4 sign changes in the sequence $\{c_0,c_1,c_2,c_3,c_4\}$, contradicting Descartes' Rule of Signs.

Example 3.3. Counterexamples, to (2.1) and (2.2), of size 4×4 also exist, though they are not quite as nice as the 5×5 case. The lowest degree example we have found is $f(x)=x^2(x+1)^{11}$, for which the 4×4 Wronskian at $x=0$ is -1584.

We next consider examples involving transcendental entire functions in $\mathcal{L}$-$\mathcal{P}^+$. To this end, set $g(x)=e^{bx}h(x)$, where $h(x)$ is a sufficiently smooth function and $b\in\mathbf{R}$. Then

$$\begin{aligned}(3.1)\quad (-1)^{m(m-1)/2}&W(g,g',\ldots,g^{(m-1)})\\ &=(-1)^{m(m-1)/2}e^{mbx}W(h,h',\ldots,h^{(m-1)}),\end{aligned}$$

as one easily sees by factoring out the exponential factors and then doing row and column operations until the Wronskian in derivatives of

h is obtained: begin with the second row and subtract b times the first row; then subtract b times the first column from the second, obtaining the proper 2×2 matrix in the upper left corner. Proceed with the next row and column. Hence, it follows from Example 3.1 and equation (3.1) that $g(x) = e^{bx}x^3(x+1)^5$, where $b > 0$, is a transcendental entire function in $\mathcal{L}$-$\mathcal{P}^+$ for which (2.2) fails.

In the next example we construct counterexamples involving functions in $\mathcal{L}$-$\mathcal{P}^+$ which possess an infinite number of zeros. The intuitive idea behind the construction is that if we multiply the polynomial $f(x)$ in Example 3.3 by a function in $g(x) \in \mathcal{L}$-$\mathcal{P}^+$ all of whose zeros are located "far" away from the origin, then the qualitative behavior of the 4×4 Wronskian of the product $\phi(x) = f(x)g(x)$ near the origin differs "little" from the 4×4 Wronskian of $f(x)$. A precise formulation is given in the next example.

Example 3.4. (Functions in $\mathcal{L}$-$\mathcal{P}^+$ with an infinite number of zeros.) Set $f(x) = x^2(x+1)^{11}$ as in Example 3.3 and let $g(x) = \prod_{k=1}^{\infty}(1+x/\alpha_k)$. Assume that the zeros $\alpha_k > 0$ satisfy $\sum_{k=1}^{\infty} 1/\alpha_k < \varepsilon \le 0.2$ and consider the function $\phi(x) = f(x)g(x)$. We shall see that the 4×4 Wronskian is again negative at $x = 0$ and is positive at $x = 1$. To estimate the derivatives, we note that

$$\log g(x) = \sum_{k=1}^{\infty} \log(1 + x/\alpha_k) \le \sum_{k=1}^{\infty} x/\alpha_k$$

for $0 \le x < 2 < \alpha_1$, so that $1 < g(1) \le e^{\varepsilon} < 2$ and

$$g'(x) = \sum_{k=1}^{\infty} \frac{1}{\alpha_k} \prod_{l\neq k} \left(1 + \frac{x}{\alpha_l}\right) < 2\varepsilon$$

at $x = 1$ and is less than ε at $x = 0$. Similarly, one sees that $g^{(k)}(0) < \varepsilon^k$ and $g^{(k)}(1) < 2\varepsilon^k$ for $k \ge 2$. When the 4×4 Wronskian is evaluated at zero, we obtain

$$\begin{aligned}
&-48(5g^{(4)}(0) - 20g'(0)g'''(0) - 12g''(0)^2 \\
&\qquad + 54g'(0)^2g''(0) - 66g''(0) - 27g'(0)^4 + 66g'(0)^2 + 33) \\
&\qquad \le -48(33 - 20g'(0)g'''(0) - 12g''(0)^2 - 66g''(0) - 27g'(0)^4) \\
&\qquad \le -48(33 - 59\varepsilon^4 - 66\varepsilon^2) < 0,
\end{aligned}$$

since $\varepsilon \leq 0.61$. The corresponding computation at $x = 1$ has 91 terms. Dropping all but one positive term, the one with $g(1)^4 > 1$, and estimating the negative terms using $g^{(k)}(1) < 2\varepsilon^k$, yields

$$\begin{aligned} 68719476736(19001961 &- 49152\varepsilon^{12} - 700416\varepsilon^{10} - 1824768\varepsilon^9 \\ &- 3843584\varepsilon^8 - 29601792\varepsilon^7 - 46972928\varepsilon^6 - 68073984\varepsilon^5 \\ &- 299323392\varepsilon^4 - 276908544\varepsilon^3 - 406657152\varepsilon^2) \\ &> 0, \end{aligned}$$

since $\varepsilon \leq 0.2$.

Of course, this same technique can be used with any polynomial counterexample, though the precise determination of ε can be difficult.

Remark 3.5. (Polynomials of arbitrarily high degree.) We recall that the *Jensen polynomials*, $g_n(x)$, associated with $\phi(x) = \sum_{k=0}^{\infty} \gamma_k x^k/k! \in \mathcal{L}\text{-}\mathcal{P}^+$, are defined by

$$g_n(x) = \sum_{k=0}^{n} \binom{n}{k} \gamma_k x^k, \quad n = 0, 1, 2 \dots. \tag{3.2}$$

It is known, see for example [**19**, p. 40], that $g_n(x) \in \mathcal{L}\text{-}\mathcal{P}^+$, $n = 0, 1, 2 \dots$, and that the sequence $\{g_n(x/n)\}$ converges uniformly to $\phi(x)$ on compact subsets of $\mathbb{C}$. Now consider the Jensen polynomials associated with the function $\phi(x)$ defined in Example 3.4. Then elementary continuity considerations show that, for all n sufficiently large, the 4×4 Wronskian of $g_n(x/n)$ changes sign on the interval $[0, 1]$ and thus provide additional counterexamples to (2.1) and (2.2).

Example 3.6. Among the very nicest functions in the $\mathcal{L}\text{-}\mathcal{P}^+$ class are those which arise from an old theorem of Laguerre, cf. [**7**]. These are the functions $\sum_{k=0}^{\infty} \gamma_k x^k/k!$ for which the coefficients γ_k can be interpolated by a polynomial with only real nonpositive zeros. These also fail the Wronskian condition in general. As an example, consider $f(x) = \sum_{k=0}^{\infty} k^3(k+a)^5 x^k/k!$. The 5×5 Wronskian at $x = 0$ is a polynomial in a which is negative for $a \geq 163$.

4. Positive results. It was noted in Section 1 that there are many known classes of functions which satisfy Descartes' Rule of Signs

(1.2) and whose Wronskian determinants enjoy the sign regularity condition of (2.2). Nevertheless, today the problem of characterization of functions in $\mathcal{L}$-$\mathcal{P}^+$ which satisfy (1.2) or (2.2) remains open. Here our primary goal is to construct a class of functions in $\mathcal{L}$-$\mathcal{P}^+$ which satisfy both (2.1) and (2.2), without the degree restriction when the function has infinitely many zeros.

To begin with, we remark that Karlin's conjecture (1.7) is valid for functions in $\mathcal{L}$-$\mathcal{P}^+$ which have only one distinct positive zero, as the following example shows.

Example 4.1. Let $f(x) = (x+a)^r$, where r is an arbitrary positive integer and $a > 0$. Factoring common factors out of the rows, we obtain

$$\begin{aligned} &W(f(x), f'(x), \dots, f^{(m-1)}(x)) \\ &= (x+a)^{mr-m(m-1)} \prod_{k=0}^{m-2} (r-k)^{m-1-k} \\ &\quad \cdot \begin{vmatrix} 1 & r & r(r-1) & \cdots & r(r-1)\cdots(r-m+2) \\ 1 & r-1 & (r-1)(r-2) & \cdots & (r-1)(r-2)\cdots(r-m+1) \\ & & & \cdots & \\ 1 & r-m+1 & (r-m+1)(r-m) & \cdots & (r-m+1)\cdots(r-2m+3) \end{vmatrix}, \end{aligned}$$

in which the final determinant has the value $(-1)^{m(m-1)/2}\prod_{k=1}^{m-1} k!$ by [**18**, p. 106] (in which $d = -1$). Moreover by (3.1), the function $g(x) := e^{bx} f(x)$, where we assume that $b \geq 0$, also satisfies the sign regularity condition (2.2).

We introduce next the following class of entire functions in $\mathcal{L}$-$\mathcal{P}^+$,

(4.1)

$$\mathcal{C}_0 := \left\{ f(x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k \in \mathcal{L}\text{-}\mathcal{P}^+ \;\middle|\; \sum_{k=0}^{\infty} \gamma_{k+p} x^k \in \mathcal{L}\text{-}\mathcal{P}^+ \text{ for all } p \geq 0 \right\},$$

and proceed to derive several properties of this class of functions. In the first place, the class $\mathcal{C}_0$ is closed under differentiation. To see this, let $f(x) := \sum_{k=0}^{\infty} \gamma_k x^k/k! \in \mathcal{C}_0$. Then since $\mathcal{L}$-$\mathcal{P}^+$ is closed under differentiation [**21**, p. 119], $f'(x) = \sum_{k=0}^{\infty} \gamma_{k+1} x^k/k! \in \mathcal{L}$-$\mathcal{P}^+$. But then by the definition of the class $\mathcal{C}_0$, $\sum_{k=0}^{\infty} \gamma_{k+1+p} x^k \in \mathcal{L}$-$\mathcal{P}^+$ for all $p \geq 0$

and so, *a fortiori*, $f'(x) \in \mathcal{C}_0$. (Caveat. In general, the class $\mathcal{L}\text{-}\mathcal{P}^+$ is *not* closed under shift of indices in the sense that if $f(x) = \sum_{k=0}^{\infty} \alpha_k x^k \in \mathcal{L}\text{-}\mathcal{P}^+$, then the entire function $\sum_{k=0}^{\infty} \alpha_{k+1} x^k$ need not belong to $\mathcal{L}\text{-}\mathcal{P}^+$, as the following simple example shows. The function $e^x \in \mathcal{L}\text{-}\mathcal{P}^+$, but the entire function $\sum_{k=0}^{\infty} 1/(k+1)! x^k \notin \mathcal{L}\text{-}\mathcal{P}^+$). Moreover, $\mathcal{C}_0$ is closed under multiplication by x; that is, if $f(x) := \sum_{k=0}^{\infty} \gamma_k x^k/k! \in \mathcal{C}_0$, then $xf(x) \in \mathcal{C}_0$. In order to verify this, it suffices to check that for $p = 0, 1, \ldots,$

$$G_p(x) := \sum_{k=1}^{\infty} (k+p)\gamma_{k+p-1} x^k \in \mathcal{L}\text{-}\mathcal{P}^+.$$

Since $f(x) \in \mathcal{C}_0$, we have $F_p(x) := \sum_{k=0}^{\infty} \gamma_{k+p} x^k \in \mathcal{L}\text{-}\mathcal{P}^+$. But $\mathcal{L}\text{-}\mathcal{P}^+$ is closed under differentiation and hence $G_p(x) = x^{1-p} d/dx(x^{p+1} F_p(x)) \in \mathcal{L}\text{-}\mathcal{P}^+$. We remark also that if $f(x) := xg(x)$ with $f(x) \in \mathcal{C}_0$, then g need not belong to $\mathcal{C}_0$ as $f(x) = x(x^2/6 + x + 1)$ shows.

To facilitate the exposition of the properties of functions in class $\mathcal{C}_0$, it will be convenient to introduce the following definition.

Definition 4.2. Let $\{\gamma_k\}_{k=0}^{\infty}$ be a sequence of real numbers. The inequalities

$$\gamma_k^2 \geq \alpha \gamma_{k-1}\gamma_{k+1}, \text{ where } \alpha \geq 1, \quad k = 1, 2, 3, \ldots, \tag{4.2}$$

will be called *Turán inequalities* and the constant α will be referred to as the *Turán constant* associated with the sequence $\{\gamma_k\}_{k=0}^{\infty}$.

Proposition 4.3 (The order of a function in $\mathcal{C}_0$.) *Let $f(x) = \sum_{j=0}^{\infty} \gamma_j x^j/j! \in \mathcal{C}_0$ and for $p \geq 0$, let $F_p(x) = \sum_{j=0}^{\infty} \gamma_{j+p} x^j$. Then the sequence $\{\gamma_k\}_{k=0}^{\infty}$ satisfies the Turán inequalities* (4.2) *with Turán constant $\alpha \geq 2$. Moreover, both $f(x)$ and $F_p(x)$ are entire functions of order zero.*

Proof. We first suppose that $\gamma_0 > 0$. Since $f \in \mathcal{C}_0$ and $F_p \in \mathcal{L}\text{-}\mathcal{P}^+$, it follows from Remark 3.5 that the Jensen polynomials associated with $F_p(x)$, $p = 0, 1, 2, \ldots,$

$$g_{2,p}(x) = \sum_{k=0}^{2} \binom{2}{k} k! \gamma_{k+p} x^k = \gamma_p + 2\gamma_{p+1} x + 2\gamma_{p+2} x^2 \tag{4.3}$$

have only real negative zeros. Hence, $\gamma_{p+1}^2 \geq 2\gamma_p\gamma_{p+2}$, $p = 0, 1, 2, \ldots$. Then a calculation shows that $0 \leq \gamma_k \leq \gamma_0/2^{k(k-1)/2}(\gamma_1/\gamma_0)^k$, see [**1**, Lemma 4.2], where α^2 should be replaced by α. But then these estimates, together with the well-known formula which expresses the order of an entire function in terms of its Taylor coefficients, [**3**, p. 9, formula (2.2.3)], imply that the order of $f(x)$, as well as that of $F_p(x)$, is zero. If $\gamma_0 = \gamma_1 = \cdots = \gamma_{m-1} = 0$, but $\gamma_m \neq 0$, then we consider the Jensen polynomials associated with $f^{(p+m)}(x)$. Since the rest of the argument is, *mutatis mutandis*, the same as before, the proof of the proposition is complete. $\square$

The statement of the next theorem requires some additional notation. Let $f(x) = \sum_{k=0}^{\infty} \alpha_k x^k \in \mathcal{L}\text{-}\mathcal{P}^+$, $\alpha_0 \neq 0$, and let $A = (\alpha_{i-j})_{i=1,j=1}^{\infty}$, where $\alpha_{i-j} = 0$ if $i - j < 0$, denote the lower triangular matrix formed from the α_k's. Since $f(x) \in \mathcal{L}\text{-}\mathcal{P}^+$, it is known [**1**] or [**12**, p. 412] that the sequence $\{\alpha_k\}_{k=0}^{\infty}$ is a totally positive sequence and hence, cf. Definition 1.1, all the minors of the matrix A are nonnegative. For a positive integer n, let $1 \leq i_1 < i_2 < \cdots < i_n$ and $1 \leq j_1 < j_2 < \cdots < j_n$ be positive integers and let

$$A\big(i_1, i_2, \ldots, i_n \mid j_1, j_2, \ldots, j_n\big) \tag{4.4}$$

denote the $n \times n$ minor obtained from A by deleting all the rows and columns except those labeled $i_1, i_2, \ldots, i_n$ and $j_1, j_2, \ldots, j_n$, respectively. Using an induction argument, in conjunction with the Cauchy-Binet formula [**12**, p. 1], Karlin proved a general result, involving certain meromorphic functions, which gives a necessary and sufficient condition for the minors (4.4) to be *strictly* positive. In the sequel, we will make use of the following special case of the result of Karlin.

Theorem 4.4 [**12**, Theorem 10.1, p. 428]. *Let*

$$f(x) = \sum_{k=0}^{\omega} \alpha_k x^k \in \mathcal{L}\text{-}\mathcal{P}^+,$$

where $1 \leq \omega \leq \infty$ *and* $\alpha_k > 0$ *for all* k. *Then, with the notation above, the minor* (4.4) *is positive if and only if*

$$i_k - \omega \leq j_k \leq i_k, \quad k = 1, 2, \ldots, n. \tag{4.5}$$

$\square$

Preliminaries aside, we next show that functions in $\mathcal{C}_0$ satisfy the sign regularity condition (2.2), subject to certain restrictions to be made precise below.

Theorem 4.5. *Let $f(x) = \sum_{k=0}^{\omega}(\gamma_k/k!)x^k \in \mathcal{C}_0$, where $1 \le \omega \le \infty$. Let p be a nonnegative integer and let $t_0 \in \mathbf{R}$, $t_0 > 0$. For each nonnegative integer m, either the Wronskian $W(f^{(p)}, f^{(p+1)}, \ldots, f^{(p+m)})$ is identically zero or*

$$(-1)^{m(m+1)/2} W(f^{(p)}(t_0), f^{(p+1)}(t_0), \ldots, f^{(p+m)}(t_0)) > 0. \tag{4.6}$$

Proof. First suppose that $f(0) \ne 0$. Since $f(x) \in \mathcal{C}_0$, $\sum_{k=0}^{\infty} \gamma_{k+p} x^k \in \mathcal{L}\text{-}\mathcal{P}^+$ for all $p \ge 0$ and hence $\{\gamma_{k+p}\}_{k=0}^{\infty}$ is a totally positive sequence. Set $\alpha_k = \gamma_{k+p}$, $k = 0, 1, 2, \ldots$, and let $A = (\alpha_{i-j})_{i=1,j=1}^{\infty}$, where $\alpha_{i-j} = 0$ if $i - j < 0$, denote the lower triangular matrix formed from the α_k's. We will consider two cases. First suppose that $\omega = \infty$. For $r = 1, 2, \ldots, m$, set

$$A_r = (\alpha_{i+j-2})_{i=1,j=1}^{r} = (f^{(p+i+j-2)}(0))_{i=1,j=1}^{r}.$$

By Remark 1.2 we already know that $(-1)^{r(r-1)/2} \det A_r \ge 0$. We now claim that by Theorem 4.4, we have $(-1)^{r(r-1)/2} \det A_r > 0$. To see this, consider the lower triangular matrix A and note that for $r = 1, 2, \ldots, m$, the minor

$$A\big(r, r+1, \ldots, 2r-1 \mid 1, 2, \ldots, r\big) > 0, \tag{4.7}$$

since the inequalities (4.5) are satisfied. Thus, after rearranging the columns of the submatrix, whose minor is positive by (4.7), we obtain that for all $p = 0, 1, 2, \ldots$ and $r = 1, 2, \ldots, m$,

$$(-1)^{r(r-1)/2} \det(f^{(p+i+j-2)}(0))_{i=1,j=1}^{r} > 0. \tag{4.8}$$

But then it follows from a theorem of Karlin and Loewner [**13**, Theorem 3], that for all $p = 0, 1, 2, \ldots$, $r = 1, 2, \ldots, m$ and $x > 0$,

$$(-1)^{r(r-1)/2} \det(f^{(p+i+j-2)}(x))_{i=1,j=1}^{r} > 0$$

and so in particular (4.6) holds. In the second case, when ω is finite, so that f is a polynomial of degree ω, the argument is analogous to the first case except that now the validity of (4.6) is subject to the additional constraint that $m + p \leq \omega$ to avoid zero determinants caused by the higher order derivatives being identically zero.

To handle the case in which $f(0) = 0$, we proceed by induction on the multiplicity of the zero at the origin. Set $f(x) = \sum_{k=0}^{\omega}(\gamma_k/k!)x^k \in \mathcal{C}_0$ with $\gamma_0 = 0$ and assume that the theorem holds for functions with a zero of lower multiplicity at the origin. $f \in \mathcal{C}_0$ implies $x\sum_{k=1}\gamma_k x^{k-1} = \sum_{k=1}\gamma_k x^k \in \mathcal{L}\text{-}\mathcal{P}^+$, hence $\sum_{k=0}(\varepsilon\gamma_{k+1} + \gamma_k)x^k = (x+\varepsilon)\sum_{k=1}\gamma_k x^{k-1} \in \mathcal{L}\text{-}\mathcal{P}^+$, and therefore $g(x) = \sum_{k=0}(\varepsilon\gamma_{k+1} + \gamma_k)x^k/k! \in \mathcal{C}_0$. Since $g(x) = f(x) + \varepsilon f'(x)$ has a zero of lower multiplicity at the origin, the Wronskians of g have appropriate signs by the induction hypothesis. Expanding the Wronskian of g gives $W(g, g', \ldots, g^{(m)}) = \sum_{k=0}^{m} D_k\varepsilon^k$, where $D_0 = W(f, f', \ldots, f^{(m)})$, $D_1 = d/dx W(f(x), f'(x), \ldots, f^{(m)}(x))$, and in general, D_k is the $(m+1)\times(m+1)$ determinant in which the column headed by $f^{(m+1-k)}$ is deleted among columns headed by $f, \ldots, f^{(m+1)}$. If the Wronskian of g is identically zero for all sufficiently small ε, then so is the Wronskian of f. Otherwise, for sufficiently small ε, $(-1)^{m(m+1)/2}W(g(x), g'(x), \ldots, g^{(m)}(x)) > 0$ for every $x > 0$, whence $(-1)^{m(m+1)/2}W(f(x), f'(x), \ldots, f^{(m)}(x)) \geq 0$. Assume the Wronskian of f equals zero at some positive x_0. Then x_0 must be a multiple root of the Wronskian, so $D_0 = D_1 = 0$. We now have the first $m+1$ of $m+2$ column vectors being linearly dependent, and also the linear dependence of all columns but the one headed by $f^{(m)}$, from $D_1 = 0$. It easily follows that all sets of $m+1$ column vectors are linearly dependent, that is, that all D_k are zero at x_0, which contradicts the sum being nonzero. Therefore the Wronskian of f, when not identically zero, is never zero for any positive value of x. In the case of derivatives $f^{(p)}$, $p \geq 1$, the conclusion is immediate from the induction hypothesis because the zero of $f^{(p)}$ at the origin has lower multiplicity. □

We next show that functions in $\mathcal{C}_0$ satisfy Descartes' Rule of Signs.

Theorem 4.6. *Let $f \in \mathcal{C}_0$. Then $Z_+\left(\sum_{k=0}^{m} a_k f^{(k)}(x)\right) \leq V(\{a_k\})$ for any sequence $\{a_k\}_{k=0}^{m}$ of real numbers such that $\sum_{j=0}^{m} a_j f^{(j)} \neq 0$.*

Proof. Let $f \in \mathcal{C}_0$ and first suppose that f is a polynomial of degree n. Then by Theorem 4.5 the sign regularity condition (4.6) holds when $m \leq n$. Since $\mathcal{C}_0$ is closed under differentiation, Theorem 2.3 implies that f satisfies Descartes' Rule of Signs. If $m > n$, then the assertion clearly remains valid. Next suppose that f is a transcendental entire function in $\mathcal{C}_0$. By the Pólya-Szegö criterion, see Lemma 2.2, a sufficient condition for the validity of Descartes' Rule of Signs for $f(x), f'(x), \ldots, f^{(m)}(x)$ is that for any integers $0 \leq i_1 < i_2 < \cdots < i_k \leq m$, all Wronskian determinants $W(f^{(i_1)}, f^{(i_2)}, \ldots, f^{(i_k)})$ of order k be nonzero on $(0, \infty)$ and have the same sign. Since $f^{(j)}(x) > 0$ for $x > 0$, an easy induction argument in conjunction with Lemma 2.1 shows that the sufficient condition above remains valid if it is only assumed that the indices $0 \leq i_1 < i_2 < \cdots < i_k \leq m$ are consecutive integers; $i_2 = i_1 + 1, i_3 = i_2 + 1, \ldots, i_k = i_{k-1} + 1$, cf. [**24**, p. 555]. Since by Theorem 4.5 the Wronskian determinants are nonzero on $(0, \infty)$, it follows that Descartes' Rule of Signs holds in this case as well. □

If $f(x) \in \mathcal{L}\text{-}\mathcal{P}^+$, then it is clear that for $a > 0,\ b \geq 0,\ f(ax + b)$ is also in $\mathcal{L}\text{-}\mathcal{P}^+$. While the invariance of $\mathcal{C}_0$ under the dilation $x \mapsto ax$, $a > 0$, is evident, Example 4.7 below shows that the class $\mathcal{C}_0$ is *not* closed under the translation $x \mapsto x + b$ for $b \geq 0$.

Example 4.7. Let $f_0(x) = \sum_{k=0}^{\infty} \gamma_k x^k / k!$, where $\gamma_k > 0$ and suppose that the sequence $\{\gamma_k\}_{k=0}^{\infty}$ satisfies the *Turán inequalities* $\gamma_k^2 \geq \alpha \gamma_{k-1} \gamma_{k+1}$, $k = 1, 2, 3, \ldots$, with Turán constant $\alpha \geq 4$. For $p = 0, 1, 2, \ldots$, set $F_p(x) = \sum_{k=0}^{\infty} \gamma_{k+p} x^k$. Then the method used to prove Proposition 4.3 shows that $f_0(x)$ and $F_p(x)$ are entire functions of order zero. Moreover, a result of Hutchinson [**11**, p. 327], see also [**7**, Section 4], implies that $F_p \in \mathcal{L}\text{-}\mathcal{P}^+$. Thus $f_0 \in \mathcal{C}_0$. We can also use Hutchinson's result to show that, for any positive integer n, the polynomial $p_n(x) = \sum_{k=0}^{n} x^k / (2^{k^2} k!)$ is in $\mathcal{C}_0$. But it is easy to see that, for $t_0 > 0$ sufficiently large, $\sum_{k=0}^{2} (x + t_0)^k / 2^{k^2} \notin \mathcal{L}\text{-}\mathcal{P}^+$ and consequently $p_2(x + t_0) \notin \mathcal{C}_0$.

We note that both Descartes' Rule of Signs and the sign regularity of Hankel determinants, when they hold for a given function $f(x)$, will also hold for $f(x + a)$ for any $a \geq 0$. This observation, in conjunction with Theorem 4.6 and (3.1), yields the following immediate corollary.

Corollary 4.8. *Let*

$$\mathcal{C}_1 = \left\{ \frac{d^n}{dx^n}\, e^{bx} f(x+a) \;\middle|\; f \in \mathcal{C}_0,\ a, b \geq 0,\ n = 0, 1, 2, \ldots \right\}.$$

If $f \in \mathcal{C}_1$, then $Z_+\left(\sum_{k=0}^m a_k f^{(k)}(x)\right) \leq V(\{a_k\}_{k=0}^m)$ for any sequence $\{a_k\}_{k=0}^m$ of real numbers such that $\sum_{j=0}^m a_j f^{(j)} \neq 0$. □

Corollary 4.9. *Let $f(x) = \sum_{k=0}^\infty \gamma_k x^k/k! \in \mathcal{C}_0$ be a transcendental entire function and let $g_n^*(t)$ be the nth Appell polynomial associated with $f(x)$; that is, the set $\{g_n^*(t)\}_{n=0}^\infty$ is generated by $e^{xt} f(x) = \sum_{n=0}^\infty g_n^*(t) x^n/n!$. Then, for $t \geq 0$, $n = 1, 2, \ldots,$*

$$(-1)^{n(n-1)/2} \begin{vmatrix} g_0^*(t) & g_1^*(t) & \cdots & g_{n-1}^*(t) \\ g_1^*(t) & g_2^*(t) & \cdots & g_n^*(t) \\ \vdots & \vdots & & \vdots \\ g_{n-1}^*(t) & g_n^*(t) & \cdots & g_{2n-2}^*(t) \end{vmatrix} > 0$$

or the determinant is identically zero.

Proof. Define the matrix

$$B = \left(\binom{i-1}{j-1} t^{i-j} \right)_{i,j=1}^n = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ t & 1 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ t^{n-1} & \binom{n-1}{1} t^{n-2} & \ldots & 1 \end{pmatrix}.$$

Then we obtain

$$\begin{vmatrix} g_0^*(t) & g_1^*(t) & \cdots & g_{n-1}^*(t) \\ g_1^*(t) & g_2^*(t) & \cdots & g_n^*(t) \\ \vdots & \vdots & & \vdots \\ g_{n-1}^*(t) & g_n^*(t) & \cdots & g_{2n-2}^*(t) \end{vmatrix}$$

$$= \det \left(B \begin{pmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_{n-1} \\ \gamma_1 & \gamma_2 & \cdots & \gamma_n \\ \vdots & \vdots & & \vdots \\ \gamma_{n-1} & \gamma_n & \cdots & \gamma_{2n-2} \end{pmatrix} B^T \right) = \begin{vmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_{n-1} \\ \gamma_1 & \gamma_2 & \cdots & \gamma_n \\ \vdots & \vdots & & \vdots \\ \gamma_{n-1} & \gamma_n & \cdots & \gamma_{2n-2} \end{vmatrix},$$

where we note, in particular, that the result is independent of t. Since $f(x)$ is a transcendental entire function in $\mathcal{C}_0$, it follows from the proof of Theorem 4.5, see inequality (4.8), that

$$(-1)^{n(n-1)/2}\det(f^{(i+j-2)}(0))_{i=1,j=1}^{n} = (-1)^{n(n-1)/2}\det(\gamma_{i+j-2})_{i=1,j=1}^{n} \geq 0,$$

proving the corollary. □

How special is the Turán constant α in Example 4.7? Hutchinson [**11**] shows that α must be at least 2 and also indicates that 4 is not necessary. For example, Hutchinson's comments [**11**, p. 331] lead one to note that $1 + 13.5x + 54x^2/2! + 54x^3/3! \in \mathcal{C}_0$, though the Turán inequalities involve ratios 4 and 3.375. We examine this question more carefully, in the next example, in the case of transcendental entire functions.

Example 4.10. Let $f(x) = \sum_{k=0}^{\infty} x^k/a^{k^2}$. We show $f(x) \in \mathcal{L}\text{-}\mathcal{P}^+$ for $a \geq 1.85$. This will be accomplished by showing that the Jensen polynomials

$$\begin{aligned} g_n(x) &= \sum_{k=0}^{n} \binom{n}{k} \frac{k!}{a^{k^2}} x^k \\ &= 1 + \frac{nx}{a} + \frac{nx}{a}\frac{(n-1)x}{a^3} + \cdots + \frac{nx}{a}\frac{(n-1)x}{a^3}\cdots\frac{x}{a^{2n-1}} \end{aligned} \tag{4.9}$$

have only real zeros for each $n = 1, 2, \ldots$. We use a refinement of a technique in [**23**, V, #176]. Since, $a/n < a^3/(n-1) < \cdots$, we can apply the argument of [**23**, I, #117] to conclude that for any given x, the absolute values of the terms of $g_n(x)$ monotonically increase from the first term 1 to the maximum term and then monotonically decrease thereafter (allowing, of course, for the case that the initial or final term is the maximal one). Fix $n \geq 2$ and $x > 0$. Assume that the kth term is the maximal one for some k, $1 \leq k \leq n-1$, and consider

$$\begin{aligned} &(-1)^k g_n(-x) \\ &\quad\geq \frac{n!\,x^k}{(n-k)!\,a^{k^2}} - \frac{n!\,x^{k-1}}{(n-k+1)!\,a^{(k-1)^2}} - \frac{n!\,x^{k+1}}{(n-k-1)!\,a^{(k+1)^2}} \\ &\qquad + \frac{n(n-1)\cdots(n-k-1)x^{k+2}}{a^{(k+2)^2}} - \frac{n(n-1)\cdots(n-k-2)x^{k+3}}{a^{(k+3)^2}}. \end{aligned} \tag{4.10}$$

From (4.9), we can see that the kth term is dominant precisely when $a^{2k-1}/(n-k+1) < x < a^{2k+1}/(n-k)$, see [**23**, I, #117]. Accordingly, we evaluate the right-hand side of (4.10) at $x = a^{2k}/(n-k)$, obtaining

$$(-1)^k g_n\left(-\frac{a^{2k}}{n-k}\right) \geq \frac{a^{k^2-9} n(n-1)\cdots(n-k+2)}{(n-k)^{k-1}} G(n,k),$$

where

(4.11)

$$\begin{aligned} G(n,k) := a^9 \frac{n-k+1}{n-k} - a^8\left(1+\frac{n-k+1}{n-k}\right) \\ + a^5 \frac{(n-k+1)(n-k-1)}{(n-k)^2} - \frac{(n-k+1)(n-k-1)(n-k-2)}{(n-k)^3}. \end{aligned}$$

We claim that $G(n,k)$ is positive for $a \geq 1.85$. If we can establish that, then we shall have

$$\begin{gathered} g_n(0) > 0, \quad g_n\left(-\frac{a^2}{n-1}\right) < 0, \qquad g_n\left(-\frac{a^4}{n-2}\right) > 0, \ldots, \\ (-1)^{n-1} g_n(-a^{2(n-1)}) > 0, \quad (-1)^n g_n(x) > 0 \quad \text{as } x \to -\infty, \end{gathered}$$

demonstrating that $g_n(x)$ has n negative real zeros.

To show that $G(n,k)$ is positive, we first note that it increases with k. Indeed, we have

$$\begin{aligned} \frac{\partial}{\partial k} G(n,k) = \frac{a^8}{(n-k)^2}(a-1.5) + \frac{a^5}{2(n-k)^2}\left(a^3 - \frac{4}{n-k}\right) \\ + \frac{2((n-k)^2+n-k-3)}{(n-k)^4}. \end{aligned}$$

For $a > 1.6$ this is always positive, with the only difficulty being when $k = n-1$ makes the last term negative; in this case one has $a^9 - a^8 - 2a^5 - 2$ whose largest real root is less than 1.6. Therefore we need only deal with $k=1$ in (4.11). Since $n \geq 2$, we have

(4.12)
$$\begin{aligned} G(n,1) &= \frac{n}{n-1}\,a^9 - \frac{2n-1}{n-1}\,a^8 + \frac{n(n-2)}{(n-1)^2}\,a^5 - \frac{n(n-2)(n-3)}{(n-1)^3} \\ &= a^9 - 2a^8 + a^5 - 1 + \frac{1}{n-1}\Big(-\frac{2}{(n-1)^2} + \frac{1-a^5}{n-1} + 2 - a^8 - a^9\Big) \\ &\geq a^9 - 2a^8 + a^5 - 1 + \frac{1}{n-1}(-2 + (1-a^5) + 2 - a^8 + a^9) \\ &> a^9 - 2a^8 + a^5 - 1 \qquad \text{if } a \geq 1.33, \end{aligned}$$

since the coefficient of $1/(n-1)$ is positive for $a \geq 1.33$. The last polynomial in (4.12), $a^9 - 2a^8 + a^5 - 1$, has only one real root, that being just under 1.85.

Will smaller values of a suffice? A computation shows that if $a = 1.7$, then $g_9(x)$ has only 7 real zeros. The exact threshold value of a is unknown.

Example 4.11. From Example 4.10, we know that if $a \geq 1.85$, then the function
$$\sum_{k=0}^{\infty} \frac{x^k}{a^{(k+p)^2}} = \frac{1}{a^{p^2}} \sum_{k=0}^{\infty} \frac{1}{a^{k^2}} \left(\frac{x}{a^{2p}}\right)^k$$
is in $\mathcal{L}\text{-}\mathcal{P}^+$ for all $p \geq 0$. Therefore
$$\sum_{k=0}^{\infty} \frac{x^k}{a^{k^2} k!} \in \mathcal{C}_0 \qquad \text{for } a \geq 1.85$$
and, in particular, a^2 is somewhat less than the Turán constant $\alpha = 4$ of Example 4.7. This provides new examples of functions for which (D_p) and (H_p) both hold.

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