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FORCING LINEARITY NUMBERS FOR FINITELY GENERATED MODULES

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ABSTRACT. Forcing linearity numbers for finitely generated free modules over fields, integral domains and local rings have previously been determined. In this paper we find the forcing linearity numbers for finitely generated modules over commutative Noetherian rings.

1. Introduction and basic results. Throughout this paper R will be a commutative Noetherian ring with identity and V a finitely generated (fg) nonzero unital R-module. The set $M_R(V) := \{f\colon V \to V \mid f(rv) = rf(v),\ r \in R,\ v \in V\}$ is the collection of homogeneous functions determined by the R-module V. This set is a nearring under point-wise addition and composition of functions. We say a collection $\mathcal{S} = \{W_\alpha\}_{\alpha\in\mathcal{A}}$ of proper, nonzero submodules W_α of V *forces linearity* on V if whenever $f \in M_R(V)$ and $f \in \mathrm{Hom}_R(W_\alpha, V)$ for each $\alpha \in \mathcal{A}$ then $f \in \mathrm{End}_R(V)$. The following definition is taken from [**4**].

Definition A. To each nonzero R-module V we assign a number $\mathrm{fln}\,(V) \in \mathbf{N} \cup \{0\} \cup \{\infty\}$ called the forcing linearity number of V as follows:

(i) If $M_R(V) = \mathrm{End}_R(V)$, then $\mathrm{fln}\,(V) = 0$.

(ii) If $M_R(V) \neq \mathrm{End}_R(V)$ and there is some finite collection $\mathcal{S}$ of proper submodules of V which forces linearity with, say, $|\mathcal{S}| = s$ but no collection $\mathcal{T}$ of proper submodules of V with $|\mathcal{T}| < s$ forces linearity, then we say $\mathrm{fln}\,(V) = s$.

(iii) If neither of the above conditions hold we say $\mathrm{fln}\,(V) = \infty$.

The number $\mathrm{fln}\,(V)$ measures how far the nearring $M_R(V)$ is from being the endomorphism ring. The following results were obtained in [**4**]. We include them for the sake of reference and completeness.

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Theorem B. *Let R be an integral domain, not a field. Then* $\mathrm{fln}\,(R^m) = 0$ *if and only if* $m = 1$. *Otherwise* $\mathrm{fln}\,(V) = 1$.

Theorem C. *Let R be a field and V a vector space over R.*

(i) $\mathrm{fln}\,(V) = 0$ *if and only if* $\dim_R(V) = 1$.

(ii) *If* $|R| = \infty$ *and* $\dim_R(V) > 1$, *then* $\mathrm{fln}\,(V) = \infty$.

(iii) *If* $|R| < \infty$ *and* $\dim_R(V) = 2$, *then* $\mathrm{fln}\,(V) = \infty$.

(iv) *If* $|R| = q < \infty$ *and* $2 < \dim_R(V)$, *then* $\mathrm{fln}\,(V) = q + 2$.

Theorem D. *Let R be a local ring, not a field, not an integral domain, with unique maximal ideal J, and let $V = R^m$, $m \in \mathbf{N}$.*

(i) $\mathrm{fln}\,(V) = 0$ *if and only if* $m = 1$.

(ii) *If* $\mathrm{Ann}_R(J) = \{0\}$ *and* $m \geq 2$, *then* $\mathrm{fln}\,(V) = 1$.

We now take $\mathrm{Ann}_R(J) \neq \{0\}$ *and* $m \geq 2$.

(iii) *If* $|R/J| = \infty$, *then* $\mathrm{fln}\,(V) = \infty$.

(iv) *If* $|R/J| < \infty$ *and* $m = 2$, *then* $\mathrm{fln}\,(V) = \infty$.

(v) *If* $|R/J| = q < \infty$ *and* $m \geq 3$, *then* $\mathrm{fln}\,(V) = q + 2$.

In this paper we determine the fln for fg modules over arbitrary commutative Noetherian rings. As an application we complete the above determination of forcing linearity numbers for all fg free modules over commutative Noetherian rings.

2. Forcing linearity numbers for finitely generated modules. In this section we consider finitely generated (fg) modules over commutative Noetherian rings. The section is divided into two parts. We first consider rings that are not local and then restrict to local rings. Throughout this section all rings are commutative and Noetherian. We start with a general result.

Theorem 2.1. *Let V be an fg faithful R-module. If R has at least two maximal ideals, then* $\mathrm{fln}\,(V) \leq 2$.

Proof. Let $I_1 \neq \{0\} \neq I_2$ be proper ideals of R such that $I_1 + I_2 = R$. We know $I_1V \neq \{0\} \neq I_2V$. Since V is faithful we have $I_1V \neq V \neq I_2V$ from [**3**, Theorem 76]. Thus I_1V and I_2V are proper nonzero submodules of V. We show that $\{I_1V, I_2V\}$ forces linearity. To this end, let $u, v \in V$ and $f \in M_R(V)$ be such that f is linear on I_1V and I_2V. If $w := f(u+v) - f(u) - f(v)$, then $I_1w = \{0\}$ and $I_2w = \{0\}$. Hence, $Rw = (I_1 + I_2)w = \{0\}$ so $w = 0$ and f is linear on V. □

Following Kaplansky [**3**], we define $\mathcal{Z}(V) = \{r \in R \mid rw = 0$ for some nonzero $w \in V\}$ for any R-module V and as usual we let $\mathcal{U}(R)$ denote the group of units in R. Several of our results will depend on whether $\mathcal{Z}(V) = R - \mathcal{U}(R)$ or not.

Theorem 2.2. *If V is an fg R-module, then $\mathcal{Z}(V) \subsetneqq R - \mathcal{U}(R)$ if and only if there exists a nonzero ideal I of R with* $\mathrm{Ann}_V(I) := \{v \in V \mid Iv = \{0\}\} = \{0\}$.

Proof. If $\mathcal{Z}(V) \subsetneqq R - \mathcal{U}(R)$, then there exists $d \in R - \mathcal{U}(R)$ such that $dv \neq 0$ for each nonzero $v \in V$. Thus $\mathrm{Ann}_V(Rd) = \{0\}$. Conversely if $\mathcal{Z}(V) = R - \mathcal{U}(R)$ and I is any nonzero ideal of R, then since $I \subseteq \mathcal{Z}(V)$ we have from Kaplansky, [**3**, Theorem 82], that there exists a nonzero $u \in V$ such that $Iu = \{0\}$, i.e., $\mathrm{Ann}_V(I) \neq \{0\}$. □

We turn our attention to determining when $\mathrm{fln}(V) = 0$. Recall that, for any R-module V, $\mathrm{Ass}(V) = \{P | P$ is a prime ideal of R and $P = (0 : w)$ for some nonzero $w \in V\}$. The collection of maximal elements in $\mathrm{Ass}(V)$ is denoted by Max-Ass(V). The following result also appeared in [**7**], but here we provide a proof that is significantly less involved.

Theorem 2.3. *The following are equivalent for an fg module V:*

i) $\mathrm{fln}(V) = 0$;

ii) $M_R(V) = \mathrm{End}_R V$;

iii) *V_P is R_P-cyclic for each $P \in$ Max-Ass(V).*

Proof. Suppose $M_R(V) \neq \mathrm{End}_R(V)$ but that V_P is R_P-cyclic for each $P \in$ Max-Ass(V). Let $x, y \in V$ and $f \in M_R(V)$ such that $w := f(x+y) - f(x) - f(y) \neq 0$. From [**5**, Lemma 9.34], there exists $P \in$ Max-Ass(V) such that $(0 : w) \subseteq P$. Since V_P is R_P-cyclic there exists $a/1 \in V_P$ such that $x/1 = r_1/s_1 \cdot a/1$ and $y/1 = r_2/s_2 \cdot a/1$. Thus $t_1 x s_1 = t_1 r_1 a$ and $t_2 y s_2 = t_2 r_2 a$ for some $t_1, t_2 \notin P$. Because $t_1 s_1 t_2 s_2 \notin P$ and $(0 : w) \subseteq P$, $t_1 s_1 t_2 s_2 w \neq 0$. But $t_1 s_1 t_2 s_2 w = t_1 s_1 t_2 s_2 [f(x+y) - f(x) - f(y)] = f(t_2 s_2 t_1 r_1 a + t_1 s_1 t_2 r_2 a) - f(t_2 s_2 t_1 r_1 a) - f(t_1 s_1 t_2 r_2 a) = 0$, a contradiction, so V_P is not R_P-cyclic.

Conversely suppose that V_P is not R_P-cyclic for some $P \in$ Max-Ass(V). We show $M_R(V) \neq \mathrm{End}_R(V)$. For an R-module U, denote by $E(U)$ the injective hull of U. From [**7**, Theorem 2.1 and Corollary 2.6] we have that $E(V) = E(R/P_1) \oplus \cdots \oplus E(R/P_n)$, where Ass$(V) = \{P_1, \dots, P_n\}$, and that each element of $E(R/P_i)$ is annihilated by a power of P. Thus, since V is finitely generated, we have from [**5**, Lemma 3.55] that there exists an $a \in \bigcap_{P_i \not\subseteq P} P_i \setminus P$ such that the component of av in $E(R/P_i)$ is zero for all $v \in V$ if $P_i \not\subseteq P$. Let $W = aV$. Since we know that multiplication by $s \in R \setminus P_i$ acts as an isomorphism on $E(R/P_i)$, we have that $(0 : w) \subseteq P$ if w is a nonzero element in W. Moreover, $W_P = V_P$ so W_P is not R_P-cyclic. Note also that the map $w \mapsto w/1$ for $W \to W_P$ is injective so we consider W as an R-submodule of W_P. If J denotes the Jacobson radical of R_P, then an application of Nakayama's lemma gives $JW_P \neq W_P$ so let $x/1 \in W_P - JW_P$ where $x \in W$.

Let $X = R_P x \cap W$, say X is generated by $\{(r_1/s_1)x, \dots, (r_n/s_n)x\}$, and let $t = s_1 s_2 \cdots s_n$. Since $P \in$ Max-Ass(V), there exists $v_0 \in V$ with $P = (0 : v_0)$. Let $y = a v_0$. Then $P = (0 : y)$ and $y \in W$. Define $f\colon V \to V$ by

$$f(w) = \begin{cases} t(r/s)y & \text{if } w = (r/s)x \in X \\ 0 & \text{if } w \in W \backslash X. \end{cases}$$

Since $(r_1/s_1)x = (r_2/s_2)x$ implies $((r_1/s_1) - (r_2/s_2)) \in J$ we see that f is well defined. We now show $f \in M_R(W)$. First we note that if $w \in X$, $r \in R$ then $rw \in X$ so $f(rw) = rf(w)$, so we take $w \in W \backslash X$, $r \in R$. If $r \notin P$, then $rw \notin X$ for otherwise $(1/r)rw = w \in X$. Also, if $w \notin X$ and $r \in P$ with $rw = (c/d)x \in W$ then $c \in P$ since if $c \notin P$, $x/1 = rd/c \cdot w/1 \in JW_P$, a contradiction to the choice of x. Thus $f \in M_R(W)$.

We next note that, if $X = W$, then $W_P = R_P x$, a contradiction to the fact that W_P is not R_P-cyclic. So we let $y_0 \in W \backslash X$ and note that $f(y_0 + x) - f(y_0) - f(x) = -ty \neq 0$. Define g: $V \to V$ by $g(v) = f(av), v \in V$. Then $g \in M_R(V)$ but $g(y_0+x)-g(y_0)-g(x) = -aty \neq 0$, i.e., $M_R(V) \neq \mathrm{End}_R(V)$. □

If, in particular, $\mathcal{Z}(V) = R - \mathcal{U}(R)$, then Max-Ass$(V)$ is just the collection of maximal ideals of R and in this case $\mathrm{fln}(V) = 0$ if and only if V_I is R_I-cyclic for every maximal ideal I of the fg module V.

Suppose now that $\mathrm{fln}(V) \neq 0$ and $\mathcal{Z}(V) \subsetneqq R - \mathcal{U}(R)$. Thus there exists $0 \neq d \in R - \mathcal{U}(R)$ such that $dw \neq 0$ for each nonzero $w \in V$. Then dV is a nonzero submodule of V. If we assume that V is a faithful R-module, then it follows from Theorem 76 of Kaplansky [**3**] that $dV \neq V$. Now if $f \in M_R(V)$ is such that f is linear on dV, then, for $u, v \in V$, if $w := f(u + v) - f(u) - f(v)$ we see that $dw = 0$. But then $w = 0$ so f is linear on V. This establishes our next result.

Theorem 2.4. *If V is an fg faithful R-module such that $\mathcal{Z}(V) \subsetneqq R - \mathcal{U}(R)$ and $\mathrm{fln}(V) \neq 0$, then $\mathrm{fln}(V) = 1$.*

We continue with V being an fg module. If $\mathcal{Z}(V) = R - \mathcal{U}(R)$, then for any maximal submodule M of V and any $v \in V - M$, $(M : v) \subseteq \mathcal{Z}(V)$, where $(M : v) := \{r \in R | rv \in M\}$, so there exists $0 \neq w \in V$ with $(M : v)w = 0$, i.e., $(M : v) \subseteq \mathrm{Ann}_R(w)$. Since M is maximal, $(M : v) = (M : M + Rv) = (M : V)$. Thus $V/M \cong R/(M : V)$ as R-modules, and therefore $(M : V)$ is a maximal ideal of R.

Theorem 2.5. *Let V be an fg R-module with $\mathcal{Z}(V) = R - \mathcal{U}(R)$ and $\mathrm{fln}(V) \neq 0$. Then $\mathrm{fln}(V) = 1$ if and only if there is a maximal submodule M of V such that $V_{(M:V)}$ is a cyclic $R_{(M:V)}$-module.*

Proof. If $\mathrm{fln}(V) = 1$, then there exists a maximal submodule M of V which forces linearity. From [**2**, Theorem III.8], for every prime ideal P of R either $M_P = V_P$ or V_P is a cyclic R_P-module or $\mathrm{Soc}(V_P) = \{0\}$. As we have noted above, $(M : V)$ is a maximal ideal of R. On the other hand, $M_{(M:V)} \neq V_{(M:V)}$ since $(M : V) \cap (R - (M : V)) = \varnothing$. Since

$(M:V) \subseteq \mathcal{Z}(V)$ there is some $0 \neq w \in V$ with $(M:V)w = 0$, [**3**, Theorem 82]. A straightforward calculation gives $\mathrm{Ann}_{R_{(M:V)}}(w/1) = (M:V)_{(M:V)}$ so $R_{(M:V)}(w/1)$ is a minimal submodule of $V_{(M:V)}$, i.e., $\mathrm{Soc}\,(V_{(M:V)}) \neq \{0\}$. Thus we must have that $V_{(M:V)}$ is a cyclic $R_{(M:V)}$-module.

If Q is any prime ideal different from $(M:V)$, then since $(M:V) \cap (R-Q) \neq \varnothing$ we have $M_Q = V_Q$. The reverse direction now also follows from Theorem III.8 of [**2**]. □

Corollary 2.6. *Let V be an fg faithful R-module with* $\mathrm{fln}\,(V) \neq 0$ *where R is not a local ring.*

1) *If* $\mathcal{Z}(V) \subsetneqq R - \mathcal{U}(R)$, $\mathrm{fln}\,(V) = 1$.

2) *If* $\mathcal{Z}(V) = R - \mathcal{U}(R)$, *then* $\mathrm{fln}\,(V) = 1$ *if and only if* $V_{(M:V)}$ *is a cyclic* $R_{(M:V)}$*-module for some maximal submodule M of V. Otherwise* $\mathrm{fln}\,(V) = 2$.

Proof. The result follows from Theorems 2.1, 2.4, and 2.5. □

We now let R be a local ring, i.e., a commutative Noetherian ring with a unique maximal ideal J consisting of the nonunits of R. For local rings the analogue of Theorem 2.2 is given in the next result. This result is most probably part of the folklore in commutative algebra, but since we could not find a suitable reference, we will provide a proof.

Theorem 2.7. *Let R be a local ring with maximal ideal J, and let V be an fg R-module. The following are equivalent*:

(i) $\mathcal{Z}(V) = J$;

(ii) $\mathrm{Ann}_V(J) \neq \{0\}$;

(iii) $\mathrm{Soc}(V) \neq \{0\}$.

Proof. Since $J = R - \mathcal{U}(R)$, the equivalence of (i) and (ii) is as in Theorem 2.2. Suppose $\mathrm{Ann}_V(J) = W \neq \{0\}$. Then W may be regarded as a R/J-vector space. Let Y be a one-dimensional subspace of W. Since $JY = \{0\}$ we see that Y is also a simple R-module so $\mathrm{Soc}\,(V) \neq \{0\}$. Conversely, suppose W is a nonzero simple submodule of V. From Nakayama's lemma, $JW \subsetneq W$ and since W is simple, $JW = \{0\}$, i.e., $\mathrm{Ann}_V(J) \neq \{0\}$. $\square$

Suppose $\mathcal{Z}(V) = J$. From Theorem 2.3, $\mathrm{fln}\,(V) = 0$ if and only if V_J is a cyclic R-module. But, since $V \cong V_J$ as R-modules, we have the following result.

Theorem 2.8. *If V is an fg R-module where R is a local ring with $\mathcal{Z}(V) = J$, then $\mathrm{fln}\,(V) = 0$ if and only if V is a cyclic R-module.*

We note that the above theorem generalizes Theorem I.1 of [**1**] for if J is nilpotent then $\mathrm{Ann}_V(J) \neq \{0\}$. We show in the next example that if $\mathrm{Ann}_V(J) = \{0\}$ it may happen that $\mathrm{fln}\,(V) = 0$ while V is not cyclic.

Example 2.9. Let R be a local ring with $P_1, \ldots, P_n$ incomparable prime ideals in R. Denote by V the R-module $R/P_1 \oplus \cdots \oplus R/P_n$. Then $\mathrm{Ann}_V(J) = \{0\}$ and V is not cyclic, since the direct sum of nontrivial modules over a local ring is not cyclic. We have that $\text{Max-Ass}\,(V) = \{P_1, \ldots, P_n\}$, and thus since $V_{P_i} \cong (R/P_i)_{P_i}$, we have from Theorem 2.3 that $\mathrm{fln}\,(V) = 0$.

Theorem 2.10. *Let V be an fg faithful module over the local ring R with $\mathrm{fln}\,(V) \neq 0$. The following are equivalent*:

i) $\mathrm{fln}\,(V) = 1$;

ii) $\mathrm{Soc}\,(V) = \{0\}$;

iii) $\mathrm{Ann}_V(J) = \{0\}$;

iv) $\mathcal{Z}(V) \subsetneq J$.

Proof. The equivalence of ii), iii) and iv) is Theorem 2.7. It follows from Theorem 2.4 that (iv) implies (i). In order to show that (i) implies (iv), we assume that $\mathcal{Z}(V) = J$ and that $\mathrm{fln}\,(V) = 1$. It follows from Theorem 2.5 that there is a maximal submodule M of V such that $V_{(M:V)}$ is a cyclic $R_{(M:V)}$-module. Since $(M : V)$ is a maximal ideal of R, $(M : V) = J$. Thus, since R is local by assumption, V is cyclic. This contradicts the assumption that $\text{fln}\,(V) \neq 0$, and therefore (i) implies (iv). □

Henceforth in this section we take $\text{Ann}_V(J) \neq \{0\}$. When this is the case we have the following nice relationship.

Theorem 2.11. *Let V be an fg module over a local ring R. Suppose further that $\text{Ann}_V(J) \neq \{0\}$. Then $\text{fln}\,(V/JV) \leq \text{fln}\,(V)$ where we consider V/JV as a vector space over R/J.*

Proof. If $\dim_{R/J}(V/JV) = 1$, then $\text{fln}\,(V/JV) = 0$ and the result clearly holds. Thus, we take $\dim_{R/J}(V/JV) \geq 2$. Let s be a nonnegative integer with $s < \text{fln}\,(V/JV)$. It suffices to show that $s < \text{fln}\,(V)$. Moreover, when determining $\text{fln}\,(V)$ one only has to consider maximal submodules. If M is a maximal submodule of V, then from Nakayama's lemma, $JV \subseteq M$. Now let C be a collection of s maximal submodules of V where we take $C = \varnothing$ if $s = 0$. Let D be the corresponding collection in V/JV. Since $\text{fln}\,(V/JV) > s$, there is some $f \in M_{R/J}(V/JV)$ which is linear on the subspaces in D but is not linear on V/JV. Let $n = \dim_{R/J}(V/JV)$ so we may consider f as a map in $M_{R/J}((R/J)^n)$. Let $x, y \in (R/J)^n$, or V/JV, be such that $a := f(x+y) - f(x) - f(y) \neq 0$. Suppose that a is nonzero in the ith component. Using the projection map from $(R/J)^n$ to the i-component, we obtain a nonlinear homogeneous map g: $(R/J)^n \to R/J$ which is linear on D.

Now let $0 \neq b \in \text{Ann}_V(J)$. Suppose $g(v + JV) = r_v + J$ where r_v is determined modulo J. Define F: $V \to V$ by $F(v) = r_v b$. It is straightforward to verify that F is well defined, that $F \in M_R(V)$ and that F is linear on C. Since $g(x+y+JV) \neq g(x+JV) + g(y+JV)$, we have $r_{x+y} + J \neq r_x + r_y + J$ so $r_{x+y}b \neq (r_x + r_y)b$, i.e., $F(x+y) \neq F(x) + F(y)$. Thus, $\text{fln}\,(V) > s$ as desired. □

Corollary 2.12. *If $\dim_{R/J}(V/JV) = 2$, then $\text{fln}\,(V) = \infty$. If R/J is infinite and $\dim_{R/J}(V/JV) \geq 2$, then $\text{fln}\,(V) = \infty$.*

Proof. From Theorem C of the introduction, $\text{fln}\,(V/JV) = \infty$ in these situations. □

If $\dim_{R/J}(V/JV) = 1$, then, from Nakayama's lemma, V is a cyclic R-module, so $\text{fln}\,(V) = 0$. As in the case of fg free modules over local rings, [**4**], it remains to consider the case where $|R/J| < \infty$ and $2 < \dim_{R/J}(V/JV)$. We do this in the final result of this section.

Theorem 2.13. *Let V be an fg module over a local ring R such that $|R/J| = q < \infty$ and $2 < \dim_{R/J}(V/JV)$ and further $\text{Ann}_V(J) \neq \{0\}$. Then $\text{fln}\,(V) = q + 2$.*

Proof. From Nakayama's lemma, $V = \langle e_1, \dots, e_m \rangle$ if and only if $\{e_1 + JV, \dots, e_m + JV\}$ is a basis for V/JV over R/J. The proof of Theorem 4.7 of [**4**] works here so $\text{fln}\,(V) \leq q + 2$. On the other hand, from the introduction we know that $\text{fln}\,(V/JV) = q + 2$ in this case. Hence, from Theorem 2.11, $\text{fln}\,(V) = q + 2$. □

3. Applications. In this section we apply results of the previous section to complete the determination of the forcing linearity numbers for finitely generated free modules over an arbitrary commutative Noetherian ring. Further we specialize some of the above results to fg modules over non-local integral domains.

We first let $V = R^m$ where R is not a local ring. From Theorem 2.1, $\text{fln}\,(R^m) \leq 2$. Moreover, as in [**4**], $\text{fln}\,(R^m) = \{0\}$ if and only if $m = 1$. Thus we take $m \geq 2$. From Theorem 2.4, if $\mathcal{Z}(R^m) \subsetneq R - \mathcal{U}(R)$, then $\text{fln}\,(R^m) = 1$ so we take $\mathcal{Z}(R^m) = R - \mathcal{U}(R)$. Suppose $\text{fln}\,(R^m) = 1$. From Theorem 2.5, $R^m_{(M:R^m)}$ is a cyclic $R_{(M:R^m)}$-module for some maximal submodule M of R^m. Let $P := (M : R^m)$. Then $(R^m)_P$ is isomorphic as an R_P-module to $(R_P)^m$ and, since $m \geq 2$, $(R^m)_P$ is not a cyclic R_P-module. Hence M cannot force linearity which means $\text{fln}\,(R^m) \geq 2$. We summarize in the following:

Theorem 3.1. *Let R be a commutative Noetherian ring, not a local ring. Let $V = R^m$ be an fg free R-module.*

1) $\text{fln}\,(R^m) = 0$ *if and only if* $m = 1$.

2) *For $m \geq 2$, if $\mathcal{Z}(R^m) \subsetneq R - \mathcal{U}(R)$, then* $\text{fln}\,(R^m) = 1$ *while, if $\mathcal{Z}(R^m) = R - \mathcal{U}(R)$, then* $\text{fln}\,(R^m) = 2$.

We note that $\mathcal{Z}(R^m)$ is the set of divisors of zero in R so we see that when $m \neq 1$ then $\mathrm{fln}\,(R^m) = 2$ if and only if every nonunit is a divisor of zero in R.

We conclude this section and the paper with a specialization to (Noetherian) integral domains, not local rings.

Theorem 3.2. *Let R be a Noetherian integral domain, not a local ring, and let V be an fg faithful R-module. Then $\mathrm{fln}\,(V) = 0$ if and only if V is torsion free and uniform.*

Proof. Since V is an fg faithful R-module, V cannot be a torsion module. If $\mathrm{fln}\,(V) = 0$, then V must be R-connected [**6**, Proposition 2.3] and therefore V must be torsion free. The result now follows from Corollary 3.2 of [**6**]. $\square$

Now if $\mathrm{fln}\,(V) \neq 0$ and $\mathcal{Z}(V) \subsetneqq R - \mathcal{U}(R)$, we know from Theorem 2.4 that $\mathrm{fln}\,(V) = 1$. As in the case for free modules we show that $\mathcal{Z}(V) = R - \mathcal{U}(R)$ implies $\mathrm{fln}\,(V) = 2$. Note that $\mathcal{Z}(V) = R - \mathcal{U}(R)$ implies that V is not torsion free (and not torsion since V is faithful by assumption). Assume $\mathrm{fln}\,(V) = 1$ when $\mathcal{Z}(V) = R - \mathcal{U}(R)$, and let M be a maximal submodule which forces linearity. As in the proof of Theorem 2.5, if we let P denote the maximal ideal $(M : V)$, then we know $M_P \neq V_P$ and $\mathrm{Soc}\,(V_P) \neq \{0\}$ so we must have that V_P is a cyclic R_P-module, say $(Rw)_P = V_P$. Thus for each $v \in V$ there exists $r \in R, l, t \notin (M : V)$ such that $t(rw - lv) = 0$. If w is a torsion element we choose v to be torsion free and obtain a contradiction. If w is torsion free we choose v to be torsion and again obtain a contradiction. Hence no maximal submodule forces linearity.

Theorem 3.3. *Let R be a Noetherian integral domain, not a local ring, and let V be an fg faithful R-module with $\mathrm{fln}\,(V) \neq 0$. If $\mathcal{Z}(V) \subsetneqq R - \mathcal{U}(R)$, then $\mathrm{fln}\,(V) = 1$. Otherwise $\mathrm{fln}\,(V) = 2$.*

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