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EXISTENCE AND BEHAVIOR OF SOLUTIONS OF THE RATIONAL EQUATION $x_{n+1} = (ax_{n-1} + bx_n)/(cx_{n-1} + dx_n)x_n$, $n = 0, 1, 2, \ldots$

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ABSTRACT. We investigate the existence and behavior of the solutions of the equation in the title, where a, b, c, and d are real numbers, and the initial conditions are real numbers.

1. Introduction and preliminaries. Consider the equation

$$x_{n+1} = \frac{ax_{n-1} + bx_n}{cx_{n-1} + dx_n}\, x_n, \quad n = 0, 1, \ldots \tag{1}$$

where the parameters

$$a, b, c, d$$

are given real numbers and the initial conditions x_{-1}, x_0 are arbitrary real numbers.

This work is motivated by Problem 1572 in *Mathematics Magazine*, April 1999, [**5**].

Our first goal is to give a detailed description of the set

$$\mathcal{G} = \{(x_{-1}, x_0) \in \mathbf{R}^2 : \text{Eq. (1) is well defined for all } n \geq 0\}.$$

The set $\mathcal{G} \subset \mathbf{R}^2$ is the set of *good* initial conditions. The complement of $\mathcal{G} \subset \mathbf{R}^2$ is called the *forbidden* set of equation (1) and is denoted by $\mathcal{F}$. That is,

$$\mathcal{F} = \{(x_{-1}, x_0) \in \mathbf{R}^2 : \text{Eq. (1) is not well defined for some } n \geq 0\}.$$

Our second goal is to understand the short and long term behavior of the solutions of equation (1) when $(x_{-1}, x_0) \in \mathcal{G}$.

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It follows from equation (1) that, formally,

$$\frac{x_{n+1}}{x_n} = \frac{ax_{n-1} + bx_n}{cx_{n-1} + dx_n} = \frac{a + b(x_n/x_{n-1})}{c + d(x_n/x_{n-1})}, \quad n = 0, 1, \dots .$$

By the change of variables

$$z_n = \frac{x_n}{x_{n-1}},$$

the above equation reduces to the Riccati equation

$$z_{n+1} = \frac{a + bz_n}{c + dz_n}, \quad n = 0, 1, \dots$$

with

$$z_0 = \frac{x_0}{x_{-1}}.$$

When studying the asymptotic behavior of solutions of equation (1), infinite products are utilized. We shall use the following results from [**3**].

Theorem A. *Let $\{a_k\}_{k=0}^{\infty}$ be a sequence of positive numbers less than one. Consider the following*

$$\prod_{k=0}^{\infty} (1 + a_k)\,, \quad \prod_{k=0}^{\infty} (1 - a_k)\,.$$

Then the following statements are true.

(1) *If $\sum_{k=0}^{\infty} a_k$ converges, then both products converge to finite, nonzero limits.*

(2) *If $\sum_{k=0}^{\infty} a_k$ diverges, then the first product diverges to $+\infty$ and the second product diverges to 0.*

Theorem B. *Consider the following*

$$\prod_{k=0}^{\infty} (1 + a_k). \tag{2}$$

Then the following statement is true.

If $\sum_{k=0}^{\infty} |a_k|$ *converges, then* (2) *converges.*

2. The Riccati equation. In this section we recall known results of the Riccati difference equation:

$$z_{n+1} = \frac{a + bz_n}{c + dz_n}, \quad n = 0, 1, \dots \tag{3}$$

where the parameters a, b, c, d are real numbers. These results will be useful in the sequel.

To avoid degenerate cases, we assume throughout this section without further mention that

$$|a| + |b| \neq 0 \quad \text{and} \quad |c| + |d| \neq 0.$$

We will also assume throughout this section, unless otherwise mentioned, that

$$d \neq 0 \quad \text{and} \quad bc - ad \neq 0.$$

Indeed when $d = 0$, equation (3) is a linear equation, while if

$$d \neq 0 \quad \text{and} \quad bc - ad = 0,$$

equation (3) reduces to the trivial equation

$$z_{n+1} = \frac{bz_n + (bc/d)}{dz_n + c} = \frac{b(dz_n + c)}{d(dz_n + c)} = \frac{b}{d}, \quad n = 0, 1, \dots .$$

Finally, we note that when

$$b + c = 0 \quad \text{and} \quad z_0 \neq -\frac{c}{d}, \tag{4}$$

the solution $\{z_n\}_{n=0}^{\infty}$ of equation (3) is periodic with period two.

Unless stated otherwise, throughout the remainder of this section, we shall assume that

$$d \neq 0, \quad bc - ad \neq 0, \quad \text{and} \quad b + c \neq 0. \tag{5}$$

The change of variables

$$z_n = \frac{b+c}{d} w_n - \frac{c}{d} \quad \text{for} \quad n = 0, 1, \dots$$

transforms equation (3) into the difference equation with one parameter

$$w_{n+1} = 1 - \frac{\mathcal{R}}{w_n}, \quad n = 0, 1, \dots \tag{6}$$

where the parameter $\mathcal{R}$, which we call the *Riccati number* of equation (3), is the nonzero real number given by

$$\mathcal{R} = \frac{bc - ad}{(b+c)^2},$$

and where the initial condition w_0 of equation (6) is

$$w_0 = \frac{dz_0 + c}{b+c}.$$

We make the further change of variables

$$\begin{cases} w_n = \dfrac{u_{n+1}}{u_n} & \text{for } n = 0, 1, \dots \\ u_0 = 1 \end{cases}$$

which reduces equation (6) to the second order linear difference equation

$$u_{n+2} - u_{n+1} + \mathcal{R} u_n = 0, \quad n = 0, 1, \dots \tag{7}$$

with initial conditions

$$u_0 = 1 \quad \text{and} \quad u_1 = w_0.$$

Finally, we denote by λ_1 and λ_2 the roots of the characteristic equation of equation (7),

$$\lambda_1 = \frac{1 - \sqrt{1 - 4\mathcal{R}}}{2} \quad \text{and} \quad \lambda_2 = \frac{1 + \sqrt{1 - 4\mathcal{R}}}{2}.$$

2.1 The forbidden set and explicit solution of the Riccati equation (2). Let G be the set of all initial conditions $z_0 \in \mathbf{R}$ such that the solution $\{z_n\}_{n=0}^{\infty}$ of equation (3) exists for all $n \geq 0$. Set

$$F = \mathbf{R} - G.$$

Then F is the set of initial conditions $z_0 \in \mathbf{R}$ such that the solution of equation (3) with initial condition z_0 fails to exists after a finite number of terms. That is, F is the *forbidden set* of the Riccati difference equation (3).

When $b + c = 0$, the forbidden set of equation (3) is the singleton

$$F = \left\{ -\frac{c}{d} \right\},$$

while in the degenerate cases where $d(bc - ad) = 0$, the forbidden set of equation (3) is empty.

The next three theorems give an explicit description of the forbidden set F of equation (3) when (5) holds. The first theorem gives an explicit description of F and also provides a closed form expression for the solutions of equation (3), when

$$\mathcal{R} < \frac{1}{4}.$$

Theorem 2.1. *Assume that* (5) *holds and that* $\mathcal{R} < 1/4$. *Then the forbidden set* F *of equation* (3) *is given by*

$$F = \left\{ \frac{b+c}{d} \left(\frac{\lambda_1 \lambda_2^n - \lambda_2 \lambda_1^n}{\lambda_2^n - \lambda_1^n} \right) - \frac{c}{d} : n \geq 1 \right\}. \tag{8}$$

For any initial condition $z_0 \notin F$, *the solution of equation* (3) *is given by*

$$z_n = \frac{b+c}{d} \left(\frac{c_1 \lambda_1^{n+1} + c_2 \lambda_2^{n+1}}{c_1 \lambda_1^n + c_2 \lambda_2^n} \right) - \frac{c}{d} \quad \textit{for} \quad n = 0, 1, \ldots \tag{9}$$

where

$$c_1 = \frac{\lambda_2(b+c) - (dz_0 + c)}{(b+c)(\lambda_2 - \lambda_1)} \quad \textit{and} \quad c_2 = \frac{(dz_0 + c) - \lambda_1(b+c)}{(b+c)(\lambda_2 - \lambda_1)}.$$

Corollary 2.1. *Assume that $\mathcal{R} < 1/4$, and let $\{z_n\}_{n=0}^{\infty}$ be a solution of equation* (3) *that exists forever. Then the following statements are true.*

(i) *Suppose*

$$z_0 = \frac{b+c}{d}\,\lambda_1 - \frac{c}{d}.$$

Then

$$z_n = \frac{b+c}{d}\,\lambda_1 - \frac{c}{d} \quad \text{for all} \quad n \geq 0.$$

(ii) *Suppose*

$$z_0 = \frac{b+c}{d}\,\lambda_2 - \frac{c}{d}.$$

Then

$$z_n = \frac{b+c}{d}\,\lambda_2 - \frac{c}{d} \quad \textit{for all } n \geq 0.$$

(iii) *Suppose*

$$z_0 \neq \frac{b+c}{d}\,\lambda_1 - \frac{c}{d} \quad \textit{and} \quad z_0 \neq \frac{b+c}{d}\,\lambda_2 - \frac{c}{d}.$$

Then

$$\lim_{n\to\infty} z_n = \frac{b+c}{d}\,\lambda_2 - \frac{c}{d}.$$

Corollary (2.1) states that if we choose an initial condition on an equilibrium point, the solution remains there. Otherwise, the solution converges to the larger (in absolute value) of the two equilibrium points.

The next theorem gives an explicit description of F and also provides a closed form expression for the solutions of equation (3), when

$$\mathcal{R} = \frac{1}{4}.$$

Theorem 2.2. *Assume* (5) *holds, and that $\mathcal{R} = 1/4$. Then the forbidden set F of equation* (3) *is given by*

$$F = \left\{ \frac{n(b-c) - b - c}{2dn} : n \geq 1 \right\}. \tag{10}$$

For any initial condition $z_0 \notin F$*, the solution of equation* (3) *is given by*

$$z_n = \frac{b+c}{d}\left(\frac{(b+c)+(n+1)(2dz_0+c-b)}{2(b+c)+2n(2dz_0+c-b)}\right) - \frac{c}{d} \quad \text{for} \quad n = 0, 1, \ldots . \tag{11}$$

Corollary 2.2. *Assume that* $\mathcal{R} = 1/4$*, and let* $\{z_n\}_{n=0}^{\infty}$ *be a solution of equation* (3) *that exists forever. Then the following statements are true.*

(i) *Suppose* $z_0 = (b-c)/2d$*; then* $z_n = (b-c)/2d$ *for all* $n \geq 0$*.*

(ii) *Suppose* $z_0 \neq (b-c)/2d$*; then* $\lim_{n\to\infty} z_n = (b-c)/2d$*.*

Corollary (2.2) states that, if we choose an initial condition on the unique equilibrium point it remains there. Otherwise, the solution converges to the unique equilibrium point. Finally, we give an explicit description of F and also provide a closed form expression for the solutions of equation (3), when

$$\mathcal{R} > \frac{1}{4}.$$

Theorem 2.3. *Assume that* (5) *holds, and that* $\mathcal{R} > 1/4$*. Let* $\theta \in (0, \pi/2)$ *be such that*

$$\cos\theta = \frac{1}{2\sqrt{\mathcal{R}}} \quad \text{and} \quad \sin\theta = \frac{\sqrt{4\mathcal{R}-1}}{2\sqrt{\mathcal{R}}}.$$

Then the forbidden set F *of equation* (3) *is given by*

$$F = \left\{\frac{b-c}{2d} - \frac{(b+c)\sqrt{4\mathcal{R}-1}}{2d}\ \cot(n\theta) : n \geq 1 \text{ and } \sin(n\theta) \neq 0\right\}. \tag{12}$$

For any initial condition $z_0 \notin F$*, the solution of equation* (3) *is given by*

(13)
$$z_n = \frac{b+c}{d}\left(\frac{\sqrt{\mathcal{R}}\left(\cos((n+1)\theta) + (2w_0-1)/(\sqrt{4\mathcal{R}-1})\sin((n+1)\theta)\right)}{\cos(n\theta) + (2w_0-1)/\sqrt{4\mathcal{R}-1}\sin(n\theta)}\right) - \frac{c}{d}$$

where $w_0 = (dz_0 + c)/b + c$.

Corollary 2.3. *Assume that* $\mathcal{R} > 1/4$*, and let* $\{z_n\}_{n=0}^{\infty}$ *be a solution of equation* (3) *that exists forever. Then the following statements are true.*

(i) *Assume that* $\theta = (p/q)\pi$*, where* p *and* q *are relatively prime. Let* k *be the first positive integer such that* $(kq)/p$ *is an integer. Then* $\{z_n\}_{n=0}^{\infty}$ *is periodic with prime period* $(kq)/p$.

(ii) *Assume that* θ *is an irrational multiple of* π*. Then* $\{z_n\}_{n=0}^{\infty}$ *is dense in the reals.*

2.2 The zero set of the Riccati equation (3)**.** In this section we give a complete description of the zero set Z of equation (3)

$$Z = \{z_0 \in \mathbf{R} : \text{Eq. (3) equals zero for some } n \geq 0\}.$$

As we shall see, Z plays an important role in obtaining the forbidden set $\mathcal{F}$ of equation (1).

First assume that

$$d = 0.$$

As $|c| + |d| \neq 0$, equation (3) reduces to the linear equation

$$z_{n+1} = \frac{b}{c}\, z_n + \frac{a}{c}, \quad n = 0, 1, \dots . \tag{14}$$

For $n \geq 1$, the solution of equation (14) is

$$z_n = \begin{cases} \dfrac{a}{c} & \text{if } b = 0 \\ z_0 + \dfrac{na}{c} & \text{if } b \neq 0 \text{ and } b = c \\ \left(z_0 - \dfrac{a}{c-b}\right)\left(\dfrac{b}{c}\right)^n + \dfrac{a}{c-b} & \text{if } b \neq 0 \text{ and } b \neq c. \end{cases}$$

Therefore when $d = 0$, the zero set Z of equation (3) is given by

$$Z = \begin{cases} \{0\} & \text{if } b = 0 \\ \left\{-\dfrac{na}{c} : n \geq 0\right\} & \text{if } b \neq 0 \text{ and } b = c \\ \left\{\dfrac{a}{c-b}\left[1 - \left(\dfrac{c}{b}\right)^n\right] : n \geq 0\right\} & \text{if } b \neq 0 \text{ and } b \neq c. \end{cases} \tag{15}$$

Clearly, when $d \neq 0$ and $bc - ad = 0$, the zero set of equation (3) is

$$Z = \{0\}\,. \tag{16}$$

Next assume that $d \neq 0$, $bc - ad \neq 0$ and $b + c = 0$. Then every solution of equation (3) with $z_0 \neq b/d$, is periodic with period two; namely, the solution is the two cycle

$$\ldots, z_0, \frac{a + bz_0}{-b + dz_0}, \ldots .$$

Hence, when $d \neq 0$, $bc - ad \neq 0$ and $b + c = 0$, the zero set of equation (3) is

$$Z = \begin{cases} \{0\} & \text{if } b = 0 \\ \left\{0, \ -\dfrac{a}{b}\right\} & \text{if } b \neq 0. \end{cases} \tag{17}$$

The following theorem gives an explicit description of Z when (5) holds.

Theorem 2.4. *Assume that* (5) *holds. Then the following statements are true*:

(i) *Suppose* $\mathcal{R} < 1/4$. *Then the zero set* Z *of equation* (3) *is given by*

$$Z = \{0\} \cup \left\{ \frac{(b+c)}{d} \left(\frac{\mathcal{R}(b+c)(\lambda_2^n - \lambda_1^n) + c(\lambda_1^n \lambda_2 - \lambda_1 \lambda_2^n)}{(b+c)(\lambda_2^{n+1} - \lambda_1^{n+1}) + c(\lambda_1^n - \lambda_2^n)} \right) - \frac{c}{d} : \right.$$
$$\left. n \geq 1 \right\}.$$

(ii) *Suppose $\mathcal{R} = 1/4$. Then the zero set Z of equation* (3) *is given by*

$$Z = \{0\} \cup \left\{ \frac{b+c}{2d} \left(\frac{2c+(b-c)n}{b+c+(b-c)n} \right) - \frac{c}{d} : n \geq 1 \right\}.$$

(iii) *Suppose $\mathcal{R} > 1/4$. Let $\theta \in (0, \pi/2)$ be such that*

$$\cos\theta = \frac{1}{2\sqrt{\mathcal{R}}} \quad \text{and} \quad \sin\theta = \frac{\sqrt{4\mathcal{R}-1}}{2\sqrt{\mathcal{R}}}.$$

For $n \geq 0$, set

$$K_n = \left(\frac{b+c}{2d}\right)\left(\frac{\sqrt{4\mathcal{R}-1}(c\,\cos(n\theta) - \sqrt{\mathcal{R}}\,(b+c)\cos((n+1)\theta))}{\sqrt{\mathcal{R}}\,(b+c)\sin((n+1)\theta) - c\sin(n\theta)} + 1\right) - \frac{c}{d}.$$

Then the zero set Z of equation (3) *is given by*

$$Z = \{0, K_0, K_1, \dots, \}.$$

3. Special cases of equation (1). In this section we describe the forbidden set and behavior of solutions of the special cases of equation (1). Throughout this section we will assume that

$$|a| + |b| \neq 0 \quad \text{and} \quad |c| + |d| \neq 0.$$

3.1 The case $d = 0$. In the case $d = 0$ equation (1) reduces to

$$x_{n+1} = \frac{ax_{n-1} + bx_n}{cx_{n-1}}\, x_n, \quad n = 0, 1, \dots . \tag{18}$$

Theorem 3.1. *Assume that $d = 0$ and $b = 0$. Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of equation* (1). *Then*

$$x_{n+1} = \frac{a}{c}\, x_n, \quad n = 0, 1, \dots .$$

Theorem 3.2. *Assume $d = 0$, $b \neq 0$, $a \neq 0$ and $b = c$. Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of equation* (1). *Then the following statements are true.*

1. *The forbidden set $\mathcal{F}$ of equation* (1) *is given by*

$$\mathcal{F} = \left\{(x_{-1}, x_0) : x_{-1} = 0 \quad \textit{or} \quad \frac{x_0}{x_{-1}} \in \left\{\frac{-na}{c},\ \textit{for } n \geq 0\right\}\right\}.$$

2. $\lim_{n\to\infty} |x_n| = +\infty$.

Proof. Let

$$x_n = x_{n-1} z_n \quad \text{for} \quad n = 0, 1, \dots .$$

Then equation (18) reduces to

$$z_{n+1} = z_n + \frac{a}{c} \quad \text{for} \quad n = 0, 1, \dots ,$$

whose solution is given by

$$z_n = z_0 + \frac{na}{c}, \quad \text{for} \quad n \geq 1.$$

Thus,

$$x_n = x_{-1} \prod_{k=0}^{n} z_k = x_{-1} \prod_{k=0}^{n} \left(z_0 + \frac{ka}{c}\right) \quad \text{for} \quad n = 0, 1, \dots ,$$

from which the results follow. $\square$

Theorem 3.3. *Assume that $d = 0$, $b \neq 0$, $a \neq 0$ and $b \neq c$. Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of equation* (1). *Then the following statements are true.*

1. *The forbidden set $\mathcal{F}$ of equation* (1) *is given by*

$$\mathcal{F} = \left\{(x_{-1}, x_0) : x_{-1} = 0 \quad \textit{or} \quad \frac{x_0}{x_{-1}} \in \left\{\frac{a}{c-b}\left[1 - \left(\frac{c}{b}\right)^n\right], \right.\right.$$
$$\left.\left. \textit{for } n \geq 0\right\}\right\}.$$

2. *Assume* $x_0/x_{-1} = a/(c-b)$. *Then the following statements are true.*

(i) *If* $|a/(c-b)| < 1$, *then* $\lim_{n\to\infty} x_n = 0$.

(ii) *If* $|a/(c-b)| > 1$, *then* $\lim_{n\to\infty} |x_n| = +\infty$.

(iii) *If* $a/(c-b) = 1$, *then* $x_n = x_{-1}$ *for* $n \geq 0$.

(iv) *If* $a/(c-b) = -1$, *then* $\{x_n\}_{n=-1}^{\infty}$ *is periodic with period two.*

3. *Assume* $x_0/x_{-1} \neq a/(c-b)$. *Then the following statements are true.*

(i) *If* $|b/c| < 1$ *and* $|a/(c-b)| < 1$, *then* $\lim_{n\to\infty} x_n = 0$.

(ii) *If* $|b/c| < 1$ *and* $|a/(c-b)| > 1$, *then* $\lim_{n\to\infty} |x_n| = +\infty$.

(iii) *If* $|b/c| < 1$ *and* $a/(c-b) = 1$, *then* $\{x_n\}_{n=-1}^{\infty}$ *converges.*

(iv) *If* $|b/c| < 1$ *and* $a/(c-b) = -1$, *then* $\{x_n\}_{n=-1}^{\infty}$ *converges to a prime period two solution.*

(v) *If* $|b/c| > 1$, *then* $\lim_{n\to\infty} |x_n| = +\infty$.

Proof. Let

$$x_n = x_{n-1} z_n \quad \text{for} \quad n = 0, 1, \dots .$$

Then equation (18) reduces to

$$z_{n+1} = \frac{b}{c}\, z_n + \frac{a}{c} \quad \text{for} \quad n = 0, 1, \dots ,$$

whose solution is given by

$$z_n = \left(z_0 - \frac{a}{c-b}\right)\left(\frac{b}{c}\right)^n + \frac{a}{c-b}, \quad \text{for} \quad n \geq 0.$$

Proof of statement (2). Observe that

$$x_n = x_{-1} \prod_{k=0}^{n} z_k = x_{-1} \prod_{k=0}^{n} \left(\frac{a}{c-b}\right) \quad \text{for} \quad n = 0, 1, \dots$$

from which the results follow.

Proof of statement (3). Observe that

$$x_n = x_{-1}\prod_{k=0}^{n} z_k = x_{-1}\prod_{k=0}^{n}\left(\left(z_0 - \frac{a}{c-b}\right)\left(\frac{b}{c}\right)^n + \frac{a}{c-b}\right)$$

$$\text{for} \quad n = 0, 1, \dots .$$

The proofs of statements (i), (ii) and (v) are clear.

The proofs of statements (iii) and (iv) will follow.

Proof of statement (iii). Observe that

$$\begin{aligned} z_n &= \frac{b^n\left(z_0(c-b)-a\right)+ac^n}{(c-b)c^n} \\ &= \frac{z_0 b^n(c-b)+a(c^n-b^n)}{(c-b)c^n} \\ &= 1+\frac{(c-b)\left[z_0 b^n - c^n\right]+a(c^n-b^n)}{(c-b)c^n} \\ &= 1+\frac{(c-b)\left[z_0(b/c)^n-1\right]+a\left(1-(b/c)^n\right)}{(c-b)}. \end{aligned}$$

Let

$$a_k = \frac{(c-b)\left[z_0(b/c)^k-1\right]+a\left(1-(b/c)^k\right)}{(c-b)}.$$

We will show that the infinite series

$$\sum_{k=0}^{\infty} |a_k|$$

is convergent.

We will use the limit comparison test with the convergent geometric series

$$b_k = \sum_{k=0}^{\infty}\left|\frac{(b/c)^k}{c-b}\right|.$$

$$\begin{aligned} \lim_{k\to\infty}\frac{|a_k|}{b_k} &= \left|(c-b)\left[z_0-\left(\frac{c}{b}\right)^k\right]+a\left(\left(\frac{c}{b}\right)^k-1\right)\right| \\ &= \left|a\left[z_0-\left(\frac{c}{b}\right)^k\right]+a\left(\left(\frac{c}{b}\right)^k-1\right)\right| \\ &= |a(z_0-1)|. \end{aligned}$$

Observe that we assumed $a \neq 0$, $z_0 = x_0/x_{-1} \neq a/(c-b) = 1$.

Thus, we have a positive limit, and the proof follows from Theorem (B) and the fact that

$$x_n = x_{-1} \prod_{k=0}^{n} (1 + a_k),$$

where a_k is defined above.

Proof of statement (iv). Observe that

$$z_n = \frac{b^n(z_0(c-b) - a) + ac^n}{(c-b)c^n}.$$

Thus,

$$\begin{aligned} -z_n &= \frac{ab^n - z_0 b^n(c-b) - ac^n}{(c-b)c^n} \\ &= \frac{a(b^n - c^n) - z_0 b^n (c-b)}{(c-b)c^n} \\ &= 1 + \frac{a(b^n - c^n) - (c-b)\left[z_0 b^n + c^n\right]}{(c-b)c^n} \\ &= 1 + \frac{(a/c^n)(b^n - c^n) - (c-b)/c^n[z_0 b^n + c^n]}{(c-b)} \\ &= 1 + \frac{a((b/c)^n - 1) - (c-b)[z_0(b/c)^n + 1]}{(c-b)}. \end{aligned}$$

Let

$$a_k = \frac{a((b/c)^k - 1) - (c-b)[z_0(b/c)^k + 1]}{(c-b)}.$$

We will show that the infinite series

$$\sum_{k=0}^{\infty} |a_k|$$

is convergent.

We will use the limit comparison test with the convergent geometric series

$$b_k = \sum_{k=0}^{\infty} \left| \frac{(b/c)^k}{c-b} \right|.$$

$$\begin{aligned}
\lim_{k\to\infty} \frac{|a_k|}{b_k} &= \left| a\left(1 - \left(\frac{c}{b}\right)^k\right) - (c-b)\left[z_0 + \left(\frac{c}{b}\right)^k\right] \right| \\
&= \left| a\left(1 - \left(\frac{c}{b}\right)^k\right) + a\left[z_0 + \left(\frac{c}{b}\right)^k\right] \right| \\
&= |a(1+z_0)|.
\end{aligned}$$

Observe that we assumed $a \neq 0$, $z_0 = x_0/x_{-1} \neq a/(c-b) = -1$.

Thus, we have a positive limit and the proof follows from Theorem (B) and the fact that

$$x_n = x_{-1}(-1)^{n-1} \prod_{k=0}^{n} (1 + a_k)$$

where a_k is defined above. $\square$

Theorem 3.4. *Assume that $a = 0$ and $d = 0$. Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of equation (1). Then the following statements are true.*

1. *The forbidden set $\mathcal{F}$ of equation (1) is given by*

$$\mathcal{F} = \{(x_{-1}, x_0) : x_{-1} = 0 \quad \text{or} \quad x_0 = 0\}.$$

2. *If $|b/c| < 1$, then $\lim_{n\to\infty} x_n = 0$.*
3. *If $|b/c| > 1$, then $\lim_{n\to\infty} |x_n| = +\infty$.*
4. *If $|b/c| = 1$ and $|x_0/x_{-1}| < 1$, then $\lim_{n\to\infty} x_n = 0$.*
5. *If $|b/c| = 1$ and $|x_0/x_{-1}| > 1$, then $\lim_{n\to\infty} |x_n| = +\infty$.*
6. *If $b/c = 1$ and $x_0/x_{-1} = 1$, then $\{x_n\}_{n=-1}^{\infty} = x_{-1}$ for $n \geq 0$.*
7. *If $b/c = 1$ and $x_0/x_{-1} = -1$, then $\{x_n\}_{n=-1}^{\infty}$ is periodic with period two.*
8. *If $b/c = -1$ and $|x_0/x_{-1}| = 1$, then $\{x_n\}_{n=-1}^{\infty}$ is periodic with period four.*

Proof. Observe that

$$x_n = x_{-1} \prod_{k=0}^{n} z_k = x_{-1} \prod_{k=0}^{n} \left(\frac{x_0}{x_{-1}} \left(\frac{b}{c} \right)^n \right) \quad \text{for} \quad n = 0, 1, \dots ,$$

from which the results follow. □

3.2 The case $bc - ad = 0$.

Theorem 3.5. *Assume $bc - ad = 0$. Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of equation* (1)*. Then the following statements are true.*

1. *The forbidden set $\mathcal{F}$ of equation* (1) *is given by*

$$\mathcal{F} = \{(x_{-1}, x_0) : x_{-1} = 0 \quad or \quad x_0 = 0\}.$$

2. *If $|b/d| < 1$, then $\lim_{n\to\infty} x_n = 0$.*
3. *If $|b/d| > 1$, then $\lim_{n\to\infty} |x_n| = +\infty$.*
4. *If $b/d = 1$, then $\{x_n\}_{n=-1}^{\infty} = x_{-1}$ for $n \geq 0$.*
5. *If $b/c = -1$, then $\{x_n\}_{n=-1}^{\infty}$ is periodic with period two.*

Proof. Observe that

$$x_n = x_{-1} \prod_{k=0}^{n} z_k = x_{-1} \prod_{k=0}^{n} \left(\frac{b}{d} \right) \quad \text{for} \quad n = 0, 1, \dots ,$$

from which the results follow. □

3.3 The case $b + c = 0$.

Theorem 3.6. *Assume $b + c = 0$. Let $\{z_n\}_{n=0}^{\infty}$ be the solution of the associated Riccati equation. Then the following statements are true.*

1. *The forbidden set $\mathcal{F}$ of equation* (1) *is given by*

$$\mathcal{F} = \left\{ (x_{-1}, x_0) : x_{-1} = 0 \quad or \quad x_0 = 0 \quad or \quad \frac{x_0}{x_{-1}} = \frac{b}{d} \right. \\ \left. or \quad \frac{x_0}{x_{-1}} = \frac{-a}{d} \right\}.$$

2. *If* $z_0z_1 < -1$, *then* $\lim_{n\to\infty} |x_n| = +\infty$.
3. *If* $z_0z_1 = -1$, *then* $\{x_n\}_{n=-1}^{\infty}$ *is periodic with period four.*
4. *If* $z_0z_1 \in (-1, 1)$, *then* $\lim_{n\to\infty} x_n = 0$.
5. *If* $z_0z_1 = 1$, *then* $\{x_n\}_{n=-1}^{\infty}$ *is periodic with period two.*
6. *If* $z_0z_1 > 1$, *then* $\lim_{n\to\infty} |x_n| = +\infty$.

Throughout the remainder of the paper we assume that

$$abcd \neq 0, \quad bc - ad \neq 0 \quad \text{and} \quad b + c \neq 0. \tag{19}$$

4. The forbidden set of equation (1). In this section we describe the forbidden set, $\mathcal{F}$, of equation (1). Recall that the forbidden set is the set of initial conditions $(x_{-1}, x_0) \in \mathbf{R}^2$ such that the solution of equation (1) with initial conditions (x_{-1}, x_0) fails to exist after a finite number of terms. Recall that F is the forbidden set and Z is the zero set of the Riccati equation.

Theorem 4.1. *Assume that* $\mathcal{R} < 1/4$. *The forbidden set* $\mathcal{F}$ *of equation* (1) *is given by*

$$\mathcal{F} = \{(0,0)\} \cup \left\{ (x_{-1}, x_0) : x_{-1} \neq 0 \quad \textit{and} \quad \frac{x_0}{x_{-1}} \in F \right\} \cup \left\{ (x_{-1}, x_0) : x_{-1} \neq 0 \quad \textit{and} \quad \frac{x_0}{x_{-1}} \in Z \right\},$$

where

$$F = \left\{ \frac{b+c}{d} \left(\frac{\lambda_1 \lambda_2^n - \lambda_2 \lambda_1^n}{\lambda_2^n - \lambda_1^n} \right) - \frac{c}{d} : n \geq 1 \right\}$$

and

$$Z = \{0\} \cup \left\{ \frac{(b+c)}{d} \left(\frac{\mathcal{R}(b+c)(\lambda_2^n - \lambda_1^n) + c(\lambda_1^n \lambda_2 - \lambda_1 \lambda_2^n)}{(b+c)(\lambda_2^{n+1} - \lambda_1^{n+1}) + c(\lambda_1^n - \lambda_2^n)} \right) - \frac{c}{d} : n \geq 1 \right\}.$$

Proof. Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of equation (1), and set $z_n = x_n/x_{n-1}$ for $n \geq 0$. Note that $\{z_n\}_{n=0}^{\infty}$ is a solution of equation (3).

Clearly $(x_{-1}, x_0) \in \mathcal{F}$ precisely when $x_0/x_{-1} \in F \cup Z$ from which the proof follows. $\square$

The proof of the next two theorems is similar and will be omitted.

Theorem 4.2. *Assume that $\mathcal{R} = 1/4$. The forbidden set $\mathcal{F}$ of equation* (1) *is given by*

$$\mathcal{F} = \{(0,0)\} \cup \left\{(x_{-1}, x_0) : x_{-1} \neq 0 \quad \textit{and} \quad \frac{x_0}{x_{-1}} \in F\right\} \cup \left\{(x_{-1}, x_0) : x_{-1} \neq 0 \quad \textit{and} \quad \frac{x_0}{x_{-1}} \in Z\right\}$$

where

$$F = \left\{\frac{n(b-c) - b - c}{2dn} : n \geq 1\right\}$$

and

$$Z = \{0\} \cup \left\{\frac{b+c}{2d}\left(\frac{2c + (b-c)n}{b+c+(b-c)n}\right) - \frac{c}{d} : n \geq 1\right\}.$$

Theorem 4.3. *Assume that $\mathcal{R} > 1/4$. The forbidden set $\mathcal{F}$ of equation* (1) *is given by*

$$\mathcal{F} = \{(0,0)\} \cup \left\{(x_{-1}, x_0) : x_{-1} \neq 0 \quad \textit{and} \quad \frac{x_0}{x_{-1}} \in F\right\} \cup \left\{(x_{-1}, x_0) : x_{-1} \neq 0 \quad \textit{and} \quad \frac{x_0}{x_{-1}} \in Z\right\},$$

where F is given by equation (12) *and Z is given in Theorem* (2.4) (iii).

In the subsequent section we describe the behavior of solutions of equation (1) for all values of the Riccati number $\mathcal{R}$. Note that we may assume without loss of generality that $d > 0$.

5. Dynamics of equation (1). In this section we describe the behavior of solutions of equation (1) for all the values of the Riccati number $\mathcal{R}$.

5.1 The case $\mathcal{R} < 1/4$.

Theorem 5.1. *Assume that* (19) *holds and that* $\mathcal{R} < 1/4$. *Let* $\{x_n\}_{n=-1}^{\infty}$ *be a solution of equation* (1) *such that* $x_0/x_{-1} = (b+c)/d\lambda_1 -(c/d)$. *Then the following statements are true.*

(i) *Suppose that* $(b+c)/d\lambda_1 - c/d < -1$. *Then* $\lim_{n\to\infty} |x_n| = +\infty$.

(ii) *Suppose that* $(b+c)/d\lambda_1 - c/d = -1$. *Then* $\{x_n\}_{n=-1}^{\infty}$ *is periodic with period two.*

(iii) *Suppose that* $(b+c)/d\lambda_1 - c/d \in (-1,1)$. *Then* $\lim_{n\to\infty} x_n = 0$.

(iv) *Suppose that* $(b+c)/d\lambda_1 - c/d = 1$. *Then* $x_n = x_{-1}$ *for all* $n \geq 0$.

(v) *Suppose that* $(b+c)/d\lambda_1 - c/d > 1$. *Then* $\lim_{n\to\infty} |x_n| = +\infty$.

Proof. Recall that

$$x_n = x_{n-1} z_n \quad \text{for} \quad n = 0, 1, \dots .$$

Thus

$$x_n = x_{-1} \prod_{k=0}^{n} z_k = x_{-1} \prod_{k=0}^{n} \frac{b+c}{d}\, \lambda_1 - \frac{c}{d} \quad \text{for} \quad n = 0, 1, \dots .$$

This observation, which follows from Corollary (2.1), completes the proof. □

The proof of the next theorem is similar and will be omitted.

Theorem 5.2. *Assume that* (19) *holds and* $\mathcal{R} < 1/4$. *Let* $\{x_n\}_{n=-1}^{\infty}$ *be a solution of equation* (1) *such that* $x_0/x_{-1} = (b+c)/c\lambda_2 - c/d$. *Then the following statements are true.*

(i) *Suppose that* $(b+c)/c\lambda_2 - c/d < -1$. *Then* $\lim_{n\to\infty} |x_n| = +\infty$.

(ii) *Suppose that* $(b+c)/c\lambda_2 - c/d = -1$. *Then* $\{x_n\}_{n=-1}^{\infty}$ *is periodic with period two.*

(iii) *Suppose that* $(b+c)/c\lambda_2 - c/d \in (-1,1)$. *Then* $\lim_{n\to\infty} x_n = 0$.

(iv) *Suppose that* $(b+c)/c\lambda_2 - c/d = 1$. *Then* $x_n = x_{-1}$ *for all* $n \geq 0$.

(v) *Suppose that* $(b+c)/c\lambda_2 - c/d > 1$. *Then* $\lim_{n\to\infty} |x_n| = +\infty$.

Theorem 5.3. *Assume* (19) *holds, and* $\mathcal{R} < 1/4$. *Let* $\{x_n\}_{n=-1}^{\infty}$ *be a solution of equation* (1) *such that* $x_0/x_{-1} \neq (b+c)/d\lambda_1 - c/d$ *and* $x_0/x_{-1} \neq (b+c)/d\lambda_2 - c/d$.

Then the following statements are true.

(i) *Suppose that* $(b+c)/d\lambda_2 - c/d < -1$. *Then* $\lim_{n\to\infty} |x_n| = +\infty$.

(ii) *Suppose that* $(b+c)/d\lambda_2 - c/d = -1$. *Then* $\{x_n\}_{n=-1}^{\infty}$ *converges to a prime period two solution.*

(iii) *Suppose that* $(b+c)/d\lambda_2 - (c/d) \in (-1, 1)$. *Then* $\lim_{n\to\infty} x_n = 0$.

(iv) *Suppose* $(b+c)/d\lambda_2 - c/d = 1$. *Then* $\{x_n\}_{n=-1}^{\infty}$ *converges.*

(v) *Suppose that* $(b+c)/d\lambda_2 - c/d > 1$. *Then* $\lim_{n\to\infty} |x_n| = +\infty$.

Proof. Statements (i), (iii) and (v). Recall that

$$x_n = x_{n-1} z_n \quad \text{for} \quad n = 0, 1, \dots .$$

Thus

$$x_n = x_{-1} \prod_{k=0}^{n} z_k \quad \text{for} \quad n = 0, 1, \dots .$$

It follows from Corollary 2.1 (iii) that $\lim_{k\to\infty} z_k = (b+c)/d\lambda_2 - c/d$, from which the result follows.

Statement (ii). The proof of statement (iv) is similar.

Note the following:

$$z_n = \frac{b+c}{d}\left(\frac{c_1\lambda_1^{n+1} + c_2\lambda_2^{n+1}}{c_1\lambda_1^n + c_2\lambda_2^n}\right) - \frac{c}{d} \quad \text{for} \quad n = 0, 1, \dots .$$

We assumed $(\lambda_2(b+c) - c)/d = -1$, which implies that $\lambda_2 = (c-d)/(b+c)$.

By the definition of λ_1 and λ_2 we have $\lambda_1 + \lambda_2 = 1$.

Solving for $\lambda_1 = 1 - \lambda_2 = (b + c - (c-d))/(b+c) = (b+d)/(b+c)$.

Using the definition of c_1 and c_2, we have:

$$c_1 = \frac{d(1 - (x_0/x_{-1})) + b - c}{c - b - 2d}$$
$$c_2 = \frac{d(x_0/x_{-1}) + c - b - d}{c - b - 2d}.$$

Consider the following simplification.

$$
\begin{aligned}
-z_n &= \frac{c}{d} - \frac{b+c}{d}\left(\frac{c_1\lambda_1^{n+1} + c_2\lambda_2^{n+1}}{c_1\lambda_1^n + c_2\lambda_2^n}\right) \\
&= \frac{c(c_1\lambda_1^n + c_2\lambda_2^n) - (b+c)(c_1\lambda_1^{n+1} + c_2\lambda_2^{n+1})}{d(c_1\lambda_1^n + c_2\lambda_2^n)} \\
&= \frac{d(c_1\lambda_1^n + c_2\lambda_2^n)}{d(c_1\lambda_1^n + c_2\lambda_2^n)} \\
&\quad + \frac{-d(c_1\lambda_1^n + c_2\lambda_2^n) + c(c_1\lambda_1^n + c_2\lambda_2^n) - (b+c)(c_1\lambda_1^{n+1} + c_2\lambda_2^{n+1})}{d(c_1\lambda_1^n + c_2\lambda_2^n)} \\
&= 1 + \left(\frac{(c-d)(c_1\lambda_1^n + c_2\lambda_2^n) - (b+c)(c_1\lambda_1^{n+1} + c_2\lambda_2^{n+1})}{d(c_1\lambda_1^n + c_2\lambda_2^n)}\right).
\end{aligned}
$$

Let

$$
a_n = \frac{(c-d)(c_1\lambda_1^n + c_2\lambda_2^n) - (b+c)(c_1\lambda_1^{n+1} + c_2\lambda_2^{n+1})}{d(c_1\lambda_1^n + c_2\lambda_2^n)}.
$$

We will now simplify a_n.

Divide numerator and denominator of a_n by the expression $1/\lambda_2^{n+1}$. Thus

$$
\begin{aligned}
a_n &= \frac{(c-d)(c_1/\lambda_2(\lambda_1/\lambda_2)^n + c_2/\lambda_2) - (b+c)(c_1(\lambda_1/\lambda_2)(\lambda_1/\lambda_2)^n + c_2)}{d((c_1/\lambda_2)(\lambda_1/\lambda_2)^n + c_2/\lambda_2)} \\
&= \frac{(c-d)/\lambda_2(c_1(\lambda_1/\lambda_2)^n + c_2) - (b+c)(c_1(\lambda_1/\lambda_2)(\lambda_1/\lambda_2)^n + c_2)}{d(c_1/\lambda_2(\lambda_1/\lambda_2)^n + c_2/\lambda_2)} \\
&= \frac{(b+c)(c_1(\lambda_1/\lambda_2)^n + c_2) - (b+c)(c_1\lambda_1\lambda_2(\lambda_1/\lambda_2)^n + c_2)}{d(c_1/\lambda_2(\lambda_1/\lambda_2)^n + c_2/\lambda_2)} \\
&= \frac{(b+c)[c_1(\lambda_1/\lambda_2)^n[(\lambda_2-\lambda_1)/\lambda_2]\,]}{d(c_1/\lambda_2(\lambda_1/\lambda_2)^n + c_2/\lambda_2)} \\
&= \frac{(b+c)[c_1(\lambda_1/\lambda_2)^n[c-b-2d/c+d]\,]}{d(c_1/\lambda_2(\lambda_1/\lambda_2)^n + c_2/\lambda_2)} \\
&= \frac{(b+c)[c_1(\lambda_1/\lambda_2)^n[c-b-2d/c+d]\,]}{d/\lambda_2[c_1(\lambda_1/\lambda_2)^n + c_2]} \\
&= \frac{(c-b-2d)[c_1(\lambda_1/\lambda_2)^n]}{d[c_1(\lambda_1/\lambda_2)^n + c_2]}.
\end{aligned}
$$

We will show that the infinite series

$$\sum_{n=0}^{\infty} |a_n|$$

is convergent. We will use the limit comparison test with the convergent geometric series

$$b_n = \sum_{n=0}^{\infty} \left| \left(\frac{\lambda_1}{\lambda_2}\right)^n \right|$$

$$\lim_{n\to\infty} \frac{b_n}{|a_n|} = \frac{|dc_2|}{|(c-b-2d)c_1|}.$$

Observe that we assumed

$$\frac{x_0}{x_{-1}} \neq \frac{\lambda_1(b+c)-c}{d}$$

and

$$\frac{x_0}{x_{-1}} \neq \frac{\lambda_2(b+c)-c}{d}.$$

These assumptions are equivalent to $c_2 \neq 0$ and $c_1 \neq 0$, respectively.

Lastly, $c-b-2d \neq 0$, otherwise $\lambda_1 = \lambda_2$.

Thus, we have a positive limit and the proof follows from Theorem (B) and the fact that

$$x_n = x_{-1}(-1)^{n-1} \prod_{k=0}^{n} (1+a_k),$$

where a_k is defined above. $\square$

5.2 The case $\mathcal{R} = 1/4$.

Theorem 5.4. *Assume that* (19) *holds and* $\mathcal{R} = 1/4$. *Let* $\{x_n\}_{n=-1}^{\infty}$ *be a solution of equation* (1) *such that* $(b-c)/2d < -1$. *Then* $\lim_{n\to\infty} |x_n| = +\infty$.

Proof. Recall that

$$x_n = x_{n-1} z_n \quad \text{for} \quad n = 0, 1, \dots .$$

Thus,

$$x_n = x_{-1} \prod_{k=0}^{n} z_k \quad \text{for} \quad n = 0, 1, \dots .$$

The result follows from Corollary 2.2. □

Theorem 5.5. *Assume that* (19) *holds and* $\mathcal{R} = 1/4$. *Let* $\{x_n\}_{n=-1}^{\infty}$ *be a solution of equation* (1) *such that* $(b-c)/2d = -1$. *Then the following statements are true.*

(i) *Suppose that* $x_0/x_{-1} = (b-c)/2d$. *Then* $\{x_n\}_{n=-1}^{\infty}$ *is periodic of period two.*

(ii) *Suppose that* $x_0/x_{-1} \neq (b-c)/2d$ *and* $b + d < 0$. *Then* $\lim_{n\to\infty} |x_n| = +\infty$.

(iii) *Suppose that* $x_0/x_{-1} \neq (b-c)/2d$ *and* $b + d > 0$. *Then* $\lim_{n\to\infty} x_n = 0$ *for all* $n \geq -1$.

(iv) *Suppose that* $x_0/x_{-1} \neq (b-c)/2d$ *and* $b+d = 0$. *Then* $\{x_n\}_{n=-1}^{\infty}$ *is periodic of period two.*

Proof.

Statement (i). Recall that

$$x_n = x_{n-1} z_n \quad \text{for} \quad n = 0, 1, \dots ,$$

and so

$$x_n = x_{-1} \prod_{k=0}^{n} (-1) \quad \text{for} \quad n = 0, 1, \dots ,$$

from which statement (i) follows.

Statement (ii). Recall that

$$x_n = x_{n-1} z_n \quad \text{for} \quad n = 0, 1, \dots ,$$

and so

$$x_n = x_{-1} \prod_{k=0}^{n} z_k = x_{-1}(-1)^{n-1} \prod_{k=0}^{n} (-z_k) = x_{-1}(-1)^{n-1} \prod_{k=0}^{n} (1 + a_k)$$

where

$$a_k = \frac{-(b+d)((x_0/x_{-1}) + 1)}{(b+d) + dk((x_0/x_{-1}) + 1)} \quad \text{for} \quad k = 0, 1, \dots .$$

Claim. *There exists an* $N \geq 1$ *such that* $a_k \in (0,1)$ *for* $k \geq N + 1$. *Thus,*

$$x_n = \left(x_{-1}(-1)^{n-1} \prod_{k=0}^{N} (1 + a_k) \right) \prod_{k=N+1}^{n} (1 + a_k).$$

The proof of statement (ii) *now follows from Theorem* (A).

Statement (iii). The proof of statement (iii) is similar to statement (ii).

Statement (iv). The proof of statement (iv) is similar to statement (i). $\square$

The proofs of the following theorems are similar to Theorem 5.5 and will be omitted.

Theorem 5.6. *Assume that* (19) *holds and that* $\mathcal{R} = 1/4$. *Let* $\{x_n\}_{n=-1}^{\infty}$ *be a solution of equation* (1) *such that* $(b-c)/2d \in (-1, 1)$. *Then* $\lim_{n\to\infty} x_n = 0$.

Theorem 5.7. *Assume that* (19) *holds and that* $\mathcal{R} = 1/4$. *Let* $\{x_n\}_{n=-1}^{\infty}$ *be a solution of equation* (1) *such that* $(b-c)/2d = 1$. *Then the following statements are true.*

(i) *Suppose* $x_0/x_1 = (b-c)/2d$. *Then* $x_n = x_{-1}$ *for all* $n \geq -1$.

(ii) *Suppose* $x_0/x_1 \neq (b-c)/2d$ *and* $b-d < 0$. *Then* $\lim_{n\to\infty} x_n = 0$.

(iii) *Suppose* $(x_0/x_1) \neq (b-c)/2$ *and* $b-d > 0$. *Then* $\lim_{n\to\infty} |x_n| = +\infty$.

(iv) *Suppose* $x_0/x_1 \neq (b-c)/2d$ *and* $b-d = 0$. *Then* $x_n = x_{-1}$ *for all* $n \geq 1$.

Theorem 5.8. *Assume that* (19) *holds and* $\mathcal{R} = 1/4$. *Let* $\{x_n\}_{n=-1}^{\infty}$ *be a solution of equation* (1) *such that* $(b-c)/2d > 1$. *Then* $\lim_{n\to\infty} |x_n| = +\infty$.

5.3 The case $\mathcal{R} > 1/4$. In this section we describe the behavior of solutions of equation (1) when $(x_{-1}, x_0) \in \mathcal{G}$ and $\mathcal{R} > 1/4$.

Theorem 5.9. *Assume that* $\mathcal{R} > 1/4$. *Let* $\{x_n\}_{n=-1}^{\infty}$ *be a solution of equation* (1) *and let* $\{z_n\}_{n=0}^{\infty}$ *be the solution of the associated Riccati equation. Assume* $\{z_n\}_{n=0}^{\infty}$ *is periodic with period* P. *Then the following statements are true.*

(i) *If* $\prod_{k=0}^{P-1} z_k < -1$, *then* $\lim_{n\to\infty} |x_n| = +\infty$.

(ii) *If* $\prod_{k=0}^{P-1} z_k = -1$, *then* $\{x_n\}_{n=-1}^{\infty}$ *is periodic with period* $2P$.

(iii) *If* $\prod_{k=0}^{P-1} z_k \in (-1, 1)$, *then* $\lim_{n\to\infty} x_n = 0$.

(iv) *If* $\prod_{k=0}^{P-1} z_k = 1$, *then* $\{x_n\}_{n=-1}^{\infty}$ *is periodic with period* P.

(v) *If* $\prod_{k=0}^{P-1} z_k > 1$, *then* $\lim_{n\to\infty} |x_n| = +\infty$.

Proof.

$$x_n = x_{n-1} z_n \quad \text{for} \quad n = 0, 1, \dots ,$$

and so

$$x_n = x_{-1} \prod_{k=0}^{n} (z_k). \qquad \square$$

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