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Φ-INEQUALITIES OF NONCOMMUTATIVE MARTINGALES

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ABSTRACT. In the recent article [**10, 11**], Pisier and Xu showed that, among other things, the noncommutative analogue of the classical Burkholder-Gandy inequalities in martingale theory. We prove the noncommutative analogue of the classical Φ-inequalities for commutative martingale.

1. Preliminaries. Let E be a rearrangement invariant space on $[0,\infty)$, cf. [**5**] for the definition. We denote by $\mathcal{N}$ a semi-finite von Neumann algebra with a semi-finite normal faithful trace σ. The set of all $\sigma-$measurable operators will be denoted by $\widetilde{\mathcal{N}}$. For $x \in \widetilde{\mathcal{N}}$, let $\mu_.(x)$ be the generalized singular value function of x, cf. [**4**]. We define

$$L_E(\mathcal{N},\sigma) = \{x \in \widetilde{\mathcal{N}} :\ \mu_.(x) \in E\}$$
$$\|x\|_{L_E(\mathcal{N},\sigma)} = \|\mu_.(x)\|_E \quad \text{for} \quad x \in L_E(\mathcal{N},\sigma).$$

Then $(L_E(\mathcal{N},\sigma), \|.\|_{L_E(\mathcal{N},\sigma)})$ is a Banach space, [**2, 12**]. For $E = L^p(0,\infty)$, we recover the noncommutative L^p-space $L^p(\mathcal{N},\sigma)$ associated with $(\mathcal{N},\sigma)$. We will denote $L_E(\mathcal{N},\sigma)$ simply by $L_E(\mathcal{N})$. Let $a = (a_n)_{n\geq 0}$ be a finite sequence in $L_E(\mathcal{N})$, define

$$\|a\|_{L_E(\mathcal{N},l^2_C)} = \Big\| \Big(\sum_{n\geq 0} |a_n|^2 \Big)^{1/2} \Big\|_{L_E(\mathcal{N})},$$
$$\|a\|_{L_E(\mathcal{N},l^2_R)} = \Big\| \Big(\sum_{n\geq 0} |a_n^*|^2 \Big)^{1/2} \Big\|_{L_E(\mathcal{N})}.$$

This gives two noms on the family of all finite sequences in $L_E(\mathcal{N})$. To see this, denoting by $\mathcal{B}(l^2)$ the algebra of all bounded operators on l^2 with its usual trace tr, let us consider the von Neumann algebra tensor

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product $\mathcal{N} \otimes \mathcal{B}(l^2)$ with the product trace $\sigma \otimes \mathrm{tr}$. $\sigma \otimes \mathrm{tr}$ is a semifinite normal faithful trace, the associated noncommutative L_E space is denoted by $L_E(\mathcal{N} \otimes \mathcal{B}(l^2))$. Now, any finite sequence $a = (a_n)_{n\geq 0}$ in $L_E(\mathcal{N})$ can be regarded as an element in $L_E(\mathcal{N} \otimes \mathcal{B}(l^2))$ via the following map

$$a \longmapsto T(a) = \begin{pmatrix} a_0 & 0 & \dots \\ a_1 & 0 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix},$$

that is, the matrix of $T(a)$ has all vanishing entries except those in the first column which are the a_n's. Such a matrix is called a column matrix, and the closure in $L_E(\mathcal{N} \otimes \mathcal{B}(l^2))$ of all column matrices is called the column subspace of $L_E(\mathcal{N} \otimes \mathcal{B}(l^2))$. Then

$$\|a\|_{L_E(\mathcal{N},l^2_C)} = \||T(a)|\|_{L_E(\mathcal{N}\otimes\mathcal{B}(l^2))} = \|T(a)\|_{L_E(\mathcal{N}\otimes\mathcal{B}(l^2))}.$$

Therefore $\|.\|_{L_E(\mathcal{N},l^2_C)}$ defines a norm on the family of all finite sequences of $L_E(\mathcal{N})$. The corresponding completion is a Banach space, denoted by $L_E(\mathcal{N}, l^2_C)$. It is clear that a sequence $a = (a_n)_{n\geq 0}$ in $L_E(\mathcal{N})$ belongs to $L_E(\mathcal{N}, l^2_C)$ if and only if

$$\sup_{n\geq 0} \Big\| \Big(\sum_{k=0}^{n} |a_k|^2 \Big)^{1/2} \Big\|_E < \infty.$$

If this is the case, $(\sum_{k=0}^{\infty} |a_k|^2)^{1/2}$ can be appropriately defined as an element of $L_E(\mathcal{N})$. Similarly, we may show that $\|.\|_{L_E(\mathcal{N},l^2_R)}$ is a norm on the family of all finite sequences in $L_E(\mathcal{N})$. As above, it defines a Banach space $L_E(\mathcal{N}, l^2_R)$, which now is isometric to the row subspace of $L_E(\mathcal{N}\otimes\mathcal{B}(l^2))$ consisting of matrices whose nonzero entries lie only in the first row. Observe that the column and row subspaces of $L_E(\mathcal{N} \otimes \mathcal{B}(l^2))$ are 1-complemented subspaces (by the definition of E and Theorem 3.4 in [**3**]). If E is $q-$concave, $q < \infty$, cf. [**5**], then $L_{E^*}(\mathcal{N} \otimes \mathcal{B}(l^2)) = L^*_E(\mathcal{N} \otimes \mathcal{B}(l^2))$, [**8**, p. 362]. Then we deduce that, if E is q-concave,

$$(1) \quad (L_E(\mathcal{N}, l^2_C))^* = L_{E^*}(\mathcal{N}, l^2_C) \quad \text{and} \quad (L_E(\mathcal{N}, l^2_R))^* = L_{E^*}(\mathcal{N}, l^2_R).$$

We now turn to the description of noncommutative martingales and their square functions. Let $\mathcal{M}$ be a finite von Neumann algebra with a normalized normal faithful trace τ. Let $(\mathcal{M}_n)_{n\geq 0}$ be an increasing sequence of von Neumann subalgebras of $\mathcal{M}$ such that $\cup_{n\geq 0}\mathcal{M}_n$ generates $\mathcal{M}$, in the w^*-topology. $(\mathcal{M}_n)_{n\geq 0}$ is called a filtration of $\mathcal{M}$. The restriction of τ to $\mathcal{M}_n$ is still denoted by τ. Let $\mathcal{E}_n = \mathcal{E}(.|\mathcal{M}_n)$ be the conditional expectation of $\mathcal{M}$ with respect to $\mathcal{M}_n$, $\mathcal{E}_n$ a norm 1 projection of $L_E(\mathcal{M})$ onto $L_E(\mathcal{M}_n)$, by the definition of E and Theorem 3.4 in [**3**], and $\mathcal{E}_n(x) \geq 0$ whenever $x \geq 0$.

A non-commutative L_E-martingale with respect to $(\mathcal{M}_n)_{n\geq 0}$ is a sequence $x = (x_n)_{n\geq 0}$ such that $x_n \in L_E(\mathcal{M}_n)$ and

$$\mathcal{E}_m(x_n) = x_m, \quad \forall\, m = 0, 1, 2, \ldots, n.$$

Let $\|x\|_{L_E(\mathcal{M})} = \sup_{n\geq 0} \|x_n\|_{L_E(\mathcal{M}_n)}$. If $\|x\|_{L_E(\mathcal{M})} < \infty$, x is said to be bounded.

Remark. (i) Let $x_\infty \in L_E(\mathcal{M})$. Set $x_n = \mathcal{E}_n(x_\infty)$ for all $n \geq 0$. Then $x = (x_n)$ is a bounded L_E-martingale and $\|x\|_{L_E(\mathcal{M})} = \|x_\infty\|_{L_E(\mathcal{M})}$.

(ii) Suppose E is p-convex and q-concave for some $1 < p,\ q < \infty$ with the relevant constants equal to 1. But then $L_E(\mathcal{M})$ is uniformly convex and so reflexive. Then, by standard argument, any bounded noncommutative martingale $x = (x_n)$ in $L_E(\mathcal{M})$ converges to some x_∞ in $L_E(\mathcal{M})$ and $x_n = \mathcal{E}_n(x_\infty)$ for all $n \geq 0$.

Let x be a martingale; its difference sequence, denoted by $dx = (dx_n)_{n\geq 0}$, is defined as

$$dx_0 = x_0, \quad dx_n = x_n - x_{n-1}, \quad n \geq 1.$$

Set

$$S_{C,n}(x) = \left(\sum_{k=0}^{n} |dx_k|^2\right)^{1/2} \quad \text{and} \quad S_{R,n}(x) = \left(\sum_{k=0}^{n} |dx_k^*|^2\right)^{1/2}.$$

By the preceding discussion dx belongs to $L_E(\mathcal{M}, l_C^2)$, respectively $L_E(\mathcal{M}, l_R^2)$), if and only if $(S_{C,n}(x))_{n\geq 0}$, respectively $(S_{R,n}(x))_{n\geq 0}$, is a bounded sequence in $L_E(\mathcal{M})$; in this case,

$$S_C(x) = \left(\sum_{k=0}^{\infty} |dx_k|^2\right)^{1/2} \quad \text{and} \quad S_R(x) = \left(\sum_{k=0}^{\infty} |dx_k^*|^2\right)^{1/2}$$

are elements in $L_E(\mathcal{M})$. These are noncommutative analogues of the usual square functions in the commutative martingale theory. It should be pointed out that the two sequences $S_{C,n}(x)$ and $S_{R,n}(x)$ may not be bounded in $L_E(\mathcal{M})$ at the same time. Define $H_C^E(\mathcal{M})$, respectively $H_R^E(\mathcal{M})$, to be the space of all L_E-martingales with respect to $(\mathcal{M}_n)_{n\geq 0}$ such that $dx \in L_E(\mathcal{M}, l_C^2)$, respectively $dx \in L_E(\mathcal{M}, l_R^2)$, and set

$$\|x\|_{H_C^E(\mathcal{M})} = \|dx\|_{L_E(\mathcal{M},l_C^2)}, \quad \text{resp. } \|x\|_{H_R^E(\mathcal{M})} = \|dx\|_{L_E(\mathcal{M},l_R^2)}.$$

Equipped respectively with the previous norms $H_C^E(\mathcal{M})$ and $H_R^E(\mathcal{M})$ are Banach spaces. Note that, if $x \in H_C^E(\mathcal{M})$,

$$\|x\|_{H_C^E(\mathcal{M})} = \sup_{n\geq 0} \|S_{C,n}(x)\|_{L_E(\mathcal{M})} = \|S_C(x)\|_{L_E(\mathcal{M})}$$

and similar equalities hold for $H_R^E(\mathcal{M})$. Then we define the Hardy space of noncommutative martingales as follows: If E is 2-cotype,

$$H_E(\mathcal{M}) = H_C^E(\mathcal{M}) + H_R^E(\mathcal{M}),$$

equipped with the norm

$$\|x\| = \inf\big\{\|y\|_{H_C^E(\mathcal{M})} + \|z\|_{H_R^E(\mathcal{M})} : x = y + z,\ y \in H_C^E(\mathcal{M}),\ z \in H_R^E(\mathcal{M})\big\}.$$

If E is 2-type,

$$H_E(\mathcal{M}) = H_C^E(\mathcal{M}) \cap H_R^E(\mathcal{M}),$$

equipped with the norm

$$\|x\| = \max\big\{\|x\|_{H_C^E(\mathcal{M})},\ \|x\|_{H_R^E(\mathcal{M})}\big\}.$$

The reason that we have defined $H_E(\mathcal{M})$ differently according to whether E has 2-cotype or 2-type will become clear in the next section. This was used in [**10, 11**] and also in [**9**].

For every $0 < s < \infty$, we define a linear operator D_s : for a measurable function f on $[0,\infty)$

$$(D_s f)(t) = f\left(\frac{t}{s}\right), \quad 0 < s < \infty, \quad \forall\, t \in [0,\infty).$$

The Boyd indices p_E, q_E of E are defined by

$$p_E = \lim_{s\to\infty} \frac{\log s}{\log \|D_s\|} = \sup_{s>1} \frac{\log s}{\log \|D_s\|},$$
$$q_E = \lim_{s\to 0^+} \frac{\log s}{\log \|D_s\|} = \inf_{0<s<1} \frac{\log s}{\log \|D_s\|}.$$

Then $1 \le p_E \le q_E \le \infty$. The 2-convexification $E^{(2)}$ of E is defined as $\|a\|_{E^{(2)}} = \||a|^2\|_E^{1/2}$, [**5**, p. 54].

Let Φ be a convex nondecreasing function defined on $[0,\infty)$ with $\Phi(0) = 0$, $\lim_{t\to\infty} \Phi(t) = \infty$ and such that $\Phi'(t) = \phi(t)$ is left-continuous and $\phi(0) = \phi(0^+)$. Φ is said to be moderate. If there is a constant $C > 0$ such that $\Phi(2t) \le C\Phi(t)$, for all $t > 0$, Φ is called a Young function if $\lim_{t\to\infty} t^{-1}\Phi(t) = \infty$. A Young function is called strictly convex if $\inf_{t>0} t\phi(t)/\Phi(t) > 1$. Consider the left-inverse ψ of ϕ which is defined by $\psi(s) = \inf\{t, \phi(t) \ge s\}$. It is easily verified that if Φ is a Young function, then $\phi(t) \uparrow \infty$, $t \to \infty$. In this case ψ is well defined on $[0.\infty)$. Put $\Phi^*(t) = \int_0^t \psi(s)\, ds$. Then Φ^* is also a convex nondecreasing function. The function Φ^*, defined in this way, is called the Young complementary function of Φ. It is clear that Φ is the Young complementary function of Φ^*, i.e., $\Phi^{**} = \Phi$. We let

$$p_\Phi = \sup_{t>0} t\phi(t)/\Phi(t), \quad q_\Phi = \inf_{t>0} t\phi(t)/\Phi(t);$$

then $p_{\Phi^*} = q'_\Phi$ where $1/q'_\Phi + 1/q_\Phi = 1$, see [**1, 6**]. Given a Young function Φ, we consider the function space on $[0,\infty)$ which is defined by

$$L_\Phi = \big\{f, \|f\|_\Phi < \infty\big\},$$

where

$$\|f\|_\Phi = \inf\big\{\lambda > 0,\ E\Phi(|f|/\lambda) < 1\big\}.$$

If Φ is a moderate function, then L_Φ is a rearrangement invariant space. Note that $L_\Phi(\mathcal{M}) = L_{L_\Phi}(\mathcal{M})$.

2. The main results. In this section $(\mathcal{M},\tau)$ always denotes a finite von Neumann algebra equipped with a normalized normal faithful trace, and $(\mathcal{M}_n)_{n\ge 0}$ an increasing filtration of subalgebras of $\mathcal{M}$ which generate $\mathcal{M}$. We keep all notations introduced in the previous section.

Theorem 2.1. *Let E be a rearrangement invariant space with $1 < p_E \leq q_E < \infty$. Then there is a positive constant β_E such that, for all finite martingales x in $L_E(\mathcal{M})$, we have*

$$\Big\| \sum \varepsilon_n dx_n \Big\|_{L_E(\mathcal{M})} \leq \beta_E \Big\| \sum dx_n \Big\|_{L_E(\mathcal{M})}, \quad \forall\, \varepsilon_n = \pm 1. \tag{2}$$

Proof. Theorem 2.b.11 in [**5**] gives that E is an interpolation space for the couple (L_p, L_q) where $1 < p < p_E \leq q_E < q < \infty$. Then, by Theorem 3.4 in [**3**], we have that $L_E(\mathcal{M})$ is an interpolation space for the couple $(L_p(\mathcal{M}), L_q(\mathcal{M}))$. We define

$$T : L_p(\mathcal{M}) + L_q(\mathcal{M}) \longrightarrow L_p(\mathcal{M}) + L_q(\mathcal{M})$$

by

$$Tx = \sum \varepsilon_n\, dx_n \quad \text{for} \quad x \in L_p(\mathcal{M}) + L_q(\mathcal{M}) \quad \text{and} \quad x_n = \mathcal{E}_n(x).$$

Then Theorem 2.1 in [**11**] gives

$$\|T\|_p \leq \beta_p, \quad \|T\|_q \leq \beta_q,$$

where β_p, β_q are positive constants. Using the fact that $L_E(\mathcal{M})$ is an interpolation space for the couple $(L_p(\mathcal{M}), L_q(\mathcal{M}))$, we obtain that there is a constant β_E such that

$$\|Tx\| \leq \beta_E \|x\|.$$

Hence (2) holds. $\square$

Corollary. *Let Φ be a strictly convex and moderate Young function, i.e., $1 < q_\Phi \leq p_\Phi < \infty$. Then there is a positive constant β_Φ such that for all finite martingales x in $L_\Phi(\mathcal{M})$, we have*

$$\Big\| \sum \varepsilon_n\, dx_n \|_{L_\Phi(\mathcal{M})} \leq \beta_\Phi \Big\| \sum dx_n \|_{L_\Phi(\mathcal{M})}, \quad \forall\, \varepsilon_n = \pm 1.$$

Lemma 2.1. *Let E be a q-concave rearrangement invariant space with $q < \infty$ and $(\mathcal{N}, \sigma)$ a semi-finite von Neumann algebra with a normal semi-finite faithful trace.*

(i) *If* $1 \le q < 2$, *then for any finite sequence* $a = (a_n)_{n\ge 0}$ *in* $L_E(\mathcal{N})$, *we have*

$$\int_G \Big\| \sum \varepsilon_n a_n \Big\|_{L_E(\mathcal{N})} d\varepsilon \approx \|a\|_{L_E(\mathcal{N},l^2_C)+L_E(\mathcal{N},l^2_R)}. \tag{3}$$

(ii) *If* E *is a* p*-convex with* $p > 2$, *then, for any finite sequence* $a = (a_n)_{n\ge 0}$ *in* $L_E(\mathcal{N})$, *we have*

$$\int_G \| \sum \varepsilon_n a_n \|_{L_E(\mathcal{N})} d\varepsilon \approx \|a\|_{L_E(\mathcal{N},l^2_C)\cap L_E(\mathcal{N},l^2_R)}. \tag{4}$$

Proof. (i) Let $E^* = F$. Then F is q'-convex with q' the conjugate index of q, so F is 2-convex and there is a rearrangement invariant space F_1 such that $F_1^{(2)} = F$. It is clear that F_1 is $q'/2$-convex. Hence we use Theorem IV.4 in [**9**] and Theorem V.5 in [**8**] to obtain the desired result, see [**9**, p. 254].

(ii) E^* satisfies the condition of (i). Then, for any finite sequence $a = (a_n)_{n\ge 0}$ in $L_{E^*}(\mathcal{N})$, we have

$$\int_G \Big\| \sum \varepsilon_n a_n \Big\|_{L_{E^*}(\mathcal{N})} d\varepsilon \approx \|a\|_{L_{E^*}(\mathcal{N},l^2_C)+L_{E^*}(\mathcal{N},l^2_R)}.$$

By Kahane's inequality, [**5**, Theorem 1.e.13], it follows that

$$\left(\int_G \| \sum \varepsilon_n a_n \|^2_{L_{E^*}(\mathcal{N})} d\varepsilon \right)^{1/2} \approx \|a\|_{L_{E^*}(\mathcal{N},l^2_C)+L_{E^*}(\mathcal{N},l^2_R)}. \tag{5}$$

Since E is q-concave, by (1)

$$(L_E(\mathcal{N}, l^2_C))^* = L_{E^*}(\mathcal{N}, l^2_C) \quad \text{and} \quad (L_E(\mathcal{N}, l^2_R))^* = L_{E^*}(\mathcal{N}, l^2_R).$$

On the other hand, we have

$$(L^2(L_E(\mathcal{N})))^* = L^2(L_{E^*}(\mathcal{N})),$$

see [**8**, p. 362]. The condition of (ii) implies that $L_{E^*}(\mathcal{N})$ is K-convex. Then there exists a constant C such that, for all $f \in L^2(L_{E^*}(\mathcal{N}))$,

$$\Big\| \sum \varepsilon_n b_n \Big\|_{L^2(L_{E^*}(\mathcal{N}))} \le C \|f\|_{L^2(L_{E^*}(\mathcal{N}))},$$

where $b_n = \int_G \varepsilon_n f d\varepsilon, n \geq 0$. Hence,

$$\Big(\int_G \Big\|\sum_{n\geq 0}\varepsilon_n a_n\Big\|^2_{L_E(\mathcal{N})} d\varepsilon\Big)^{1/2}$$
$$= \sup\Big\{\Big|\int_G <\sum_{n\geq 0}\varepsilon_n a_n,\ f> d\varepsilon\Big| : f\in L^2(L_{E^*}(\mathcal{N})),\ \|f\|_{L^2(L_{E^*}(\mathcal{N}))}\leq 1\Big\}$$
$$= \sup\Big\{\Big|\int_G <\sum_{n\geq 0}\varepsilon_n a_n,\ \sum_{n\geq 0}\varepsilon_n b_n> d\varepsilon\Big| : b_n = \int_G \varepsilon_n f d\varepsilon,\ \|f\|_{L^2(L_{E^*}(\mathcal{N}))}\leq 1\Big\}$$
$$\leq \sup\Big\{\Big|\int_G <\sum_{n\geq 0}\varepsilon_n a_n,\ \sum_{n\geq 0}\varepsilon_n b_n> d\varepsilon\Big| : \Big\|\sum_{n\geq 0}\varepsilon_n b_n\Big\|_{L^2(L_{E^*}(\mathcal{N}))}\leq C\Big\}$$
$$\leq \sup\Big\{\Big|\sum_{n\geq 0}\langle a_n, b_n\rangle\Big| : \|(b_n)_{n\geq 0}\|_{L_{E^*}(\mathcal{N},l^2_C)+L_{E^*}(\mathcal{N},l^2_R)}\leq C_1\Big\}$$
$$\leq \beta_E\|(a_n)_{n\geq 0}\|_{L_E(\mathcal{N},l^2_C)\cap L_E(\mathcal{N},l^2_R)}.$$

Since E is 2-convex, we use (I.7) in [**9**, p. 247] to obtain that

$$\|(a_n)_{n\geq 0}\|_{L_E(\mathcal{N},l^2_C)\cap L_E(\mathcal{N},l^2_R)} \leq \Big(\int_G \Big\|\sum \varepsilon_n a_n\Big\|^2_{L_E(\mathcal{N})} d\varepsilon\Big)^{1/2}.$$

So we get (4). □

Corollary. *Let Φ be a convex function and $(\mathcal{N},\sigma)$ a semi-finite von Neumann algebra with a normal semi-finite faithful trace.*

(i) *If $1 < q_\Phi \leq p_\Phi < 2$, then for any finite sequence $a = (a_n)_{n\geq 0}$ in $L_\Phi(\mathcal{N})$, we have*

$$\int_G \Big\|\sum \varepsilon_n a_n\Big\|_{L_\Phi(\mathcal{N})} d\varepsilon \approx \|a\|_{L_\Phi(\mathcal{N},l^2_C)+L_\Phi(\mathcal{N},l^2)}. \tag{6}$$

(ii) *If $2 < q_\Phi \leq p_\Phi < \infty$, then for any finite sequence $a = (a_n)_{n\geq 0}$ in $L_\Phi(\mathcal{N})$, we have*

$$\int_G \Big\|\sum \varepsilon_n a_n\Big\|_{L_\Phi(\mathcal{N})} d\varepsilon \approx \|a\|_{L_\Phi(\mathcal{N},l^2_C)\cap L_\Phi(\mathcal{N},l^2)}. \tag{7}$$

Proof. We prove only (i). Let $E = L_\Phi$, $F = L_{\Phi^*}$. Then $E = F^*$ and Φ^* is a convex function with $2 < q_{\Phi^*} \le p_{\Phi^*} < \infty$. Then F is q_{Φ^*}-convex. By (3), we obtain (6). $\square$

Lemma 2.2. *Let E be a rearrangement invariant space with $1 < p_E \le q_E < \infty$. Define the map Q on the family of all finite sequences $a = (a_n)_{n\ge 0}$ in $L_E(\mathcal{M})$ by*

$$Q(a) = (\mathcal{E}_n(a_n))_{n\ge 0}.$$

Then there exists r_E such that

$$\|Q(a)\|_{L_E(\mathcal{M},l^2_C)} \le r_E \|a\|_{L_E(\mathcal{M},l^2_C)},$$
$$\|Q(a)\|_{L_E(\mathcal{M},l^2_R)} \le r_E \|a\|_{L_E(\mathcal{M},l^2_R)}.$$

Thus Q extends to a bounded projection on $L_E(\mathcal{M}, l^2_C)$ and $L_E(\mathcal{M}, l^2_R)$; consequently, $H_E(\mathcal{M})$ is complemented in $L_E(\mathcal{M}, l^2_C) + L_E(\mathcal{M}, l^2_R)$ or $L_E(\mathcal{M}, l^2_C) \cap L_E(\mathcal{M}, l^2_R)$ according to whether E is 2-cotype or E is 2-type.

Proof. Let us consider the von Neumann algebra tensor product $\mathcal{M} \otimes \mathcal{B}(l^2)$ with the product trace $\tau \otimes \mathrm{tr}$; then $\tau \otimes \mathrm{tr}$ is a semi-finite normal faithful trace. Let $L_E(\mathcal{M} \otimes \mathcal{B}(l^2))$ be the associated noncommutative L_E space. Then $L_E(\mathcal{M} \otimes \mathcal{B}(l^2))$ is an interpolation space for the couple $(L_p(\mathcal{M} \otimes \mathcal{B}(l^2)), L_q(\mathcal{M} \otimes \mathcal{B}(l^2)))$ where $1 < p < p_E \le q_E < q < \infty$. We define

$$T : L_p(\mathcal{M}\otimes\mathcal{B}(l^2)) + L_q(\mathcal{M}\otimes\mathcal{B}(l^2)) \longrightarrow L_p(\mathcal{M}\otimes\mathcal{B}(l^2)) + L_q(\mathcal{M}\otimes\mathcal{B}(l^2)),$$

by

$$T\begin{pmatrix} a_{11} & \dots & a_{1n} & \dots \\ a_{21} & \dots & a_{2n} & \dots \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \dots & a_{nn} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} \mathcal{E}_1(a_{11}) & 0 & 0 & \dots \\ \mathcal{E}_2(a_{21}) & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots \\ \mathcal{E}_n(a_{n1}) & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Theorem 2.3 in [**11**] gives that T is a bounded operator on $L_p(\mathcal{M} \otimes \mathcal{B}(l^2))$ into $L_p(\mathcal{M} \otimes \mathcal{B}(l^2))$ and on $L_q(\mathcal{M} \otimes \mathcal{B}(l^2))$ into $L_q(\mathcal{M} \otimes \mathcal{B}(l^2))$.

Then T is a bounded operator on $L_E(\mathcal{M}\otimes\mathcal{B}(l^2))$ into $L_E(\mathcal{M}\otimes\mathcal{B}(l^2))$. This gives Q is a bounded operator on $L_E(\mathcal{M},l^2_C)$ into $L_E(\mathcal{M},l^2_C)$. Similarly, we may show that Q is a bounded operator on $L_E(\mathcal{M},l^2_R)$ into $L_E(\mathcal{M},l^2_R)$ too. □

Theorem 2.2. *Let E be a q-concave rearrangement invariant space with $q<\infty$ and $x=(x_n)_{n\geq 0}$ an L_E-martingale with respect to $(\mathcal{M}_n)_{n\geq 0}$ as above.*

(i) *If $1<q<2$ and $1<p_E$, then x is bounded in $L_E(\mathcal{M})$ if and only if x belongs to $H_E(\mathcal{M})$; moreover, if this is the case, we have*

$$\alpha_E\|x\|_{H_E(\mathcal{M})}\leq\|x\|_{L_E(\mathcal{M})}\leq\beta_E\|x\|_{H_E(\mathcal{M})} \tag{8}$$

where α_E,β_E are positive constants.

(ii) *If E is a p-convex with $p>2$, then x is bounded in $L_E(\mathcal{M})$ if and only if x belongs to $H_E(\mathcal{M})$; moreover, if this is the case, we have*

$$\alpha_E\|x\|_{H_E(\mathcal{M})}\leq\|x\|_{L_E(\mathcal{M})}\leq\beta_E\|x\|_{H_E(\mathcal{M})},$$

where α_E,β_E are positive constants.

Proof. (i) The Boyd indices of E satisfy $1<p_E\leq q_E<\infty$. So Theorem 1 holds for E. Let x be any finite martingale in $L_E(\mathcal{M})$; then we have (2). Applying (2) to the martingale difference sequence $(\varepsilon_n dx_n)$ instead of (dx_n), we obtain the converse inequality

$$\|x\|_{L_E(\mathcal{M})}\leq\beta_E\Big\|\sum\varepsilon_n\,dx_n\Big\|_{L_E(\mathcal{M})},\quad\forall\,\varepsilon_n=\pm1.$$

Therefore, integrating in ε over G, we have

$$\|x\|_{L_E(\mathcal{M})}\approx\int_G\Big\|\sum\varepsilon_n\,dx_n\Big\|_{L_E(\mathcal{M})}d\varepsilon.$$

It follows from (i) of Lemma 2.1 that

$$\|x\|_{L_E(\mathcal{M})}\approx\|dx\|_{L_E(\mathcal{M},l^2_C)+L_E(\mathcal{M},l^2)}.$$

Then using Lemma 2.2, we get (8).

(ii) The proof is similar to (i). Using (ii) of Lemma 2.1, Lemma 2.2 and Theorem 2.1, we obtain the desired result. □

Corollary 1. *Let E satisfy the condition of Theorem* 2.2. *Then*

$$H_E(\mathcal{M}) = L_E(\mathcal{M})$$

with equivalent norms.

Corollary 2. *Let Φ be a convex function such that $1 < q_\Phi \leq p_\Phi < 2$ or $2 < q_\Phi \leq p_\Phi < \infty$. Then*

$$H_\Phi(\mathcal{M}) = L_\Phi(\mathcal{M})$$

with equivalent norms.

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