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A STABILITY THEOREM FOR A CLASS OF DISTRIBUTED PARAMETER CONTROL SYSTEMS

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ABSTRACT. This paper presents an optimal control problem governed by a hyperbolic equation. The control may appear in the cost functional and in the right side of this equation. The difference approximations problem for the considered problem is obtained. A stability estimate of the difference approximations problem is established.

1. Introduction. Very recently the optimal control distributed parameter systems has received the attention of many control engineers. Many of the problems of control in air-frames design, shipbuilding, nuclear reactors, magnetohydrodynamics and other engineering fields [**4, 9**] are problems of control of systems with distributed parameters, and, therefore, are more difficult to optimize. The first serious work in this direction was introduced by Botkovsky and Lerner [**2, 3**] and Butkovsky [**1**]. Warng [**16**] and Rehbock [**10**] attempted to present a general discussion of various problems associated with the control of distributed parameter systems. Chaudhuri [**5, 11**] discussed the derivation of a maximum principle and obtained the optimal control function through the discretization schemes and via the method of gradients and quasilinearization techniques for a class of hyperbolic partial differential equations. Farag [**6**] discussed the existence and uniqueness theorem, the sufficient differentiability conditions of the cost functional and its gradient formulae based on solving the adjoint system and the necessary optimality conditions for a class of hyperbolic partial differential equations.

This paper presents an optimal control problem governed by a hyperbolic equation. The control may be act in the cost functional and in the right side of this equation. The difference approximations prob-

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lem for the considered problem is obtained. A stability estimate of the difference approximations problem is established.

2. The optimal control problem. Now we need to introduce some functional spaces:

1) $L_2(\Omega)$ is a Hilbert space consisting of all measurable functions on Ω with

$$\langle z_1, z_2\rangle_{L_2(\Omega)} = \int_0^l \int_0^T z_1(x,t)z_2(x,t)\,dx\,dt, \quad \|z\|_{L_2(\Omega)} = \sqrt{\langle z, z\rangle_{L_2(\Omega)}}.$$

2) $W_2^1(\Omega) = \{z \in L_2(\Omega) \text{ and } (\partial z/\partial x) \in L_2(\Omega),\ (\partial z/\partial t) \in L_2(\Omega)\}$ is a Hilbert space with

$$\langle z_1, z_2\rangle_{W_2^1(\Omega)} = \int_0^l \int_0^T \left[z_1(x,t)z_2(x,t) + \frac{\partial z_1(x,t)}{\partial x}\,\frac{\partial z_2(x,t)}{\partial x} + \frac{\partial z_1(x,t)}{\partial t}\,\frac{\partial z_2(x,t)}{\partial t}\right] dx\,dt$$

$$\|z\|_{W_2^1(\Omega)} = \sqrt{\langle z, z\rangle_{W_2^1(\Omega)}}.$$

3) $W_2^1(0,l) = \{z(x) \in L_2(0,l) \text{ and } (\partial z(x)/\partial x) \in L_2(0,l)\}$ is a Hilbert space with

$$\langle z_1, z_2\rangle_{W_2^1(0,l)} = \int_0^l \left[z_1(x)z_2(x) + \frac{\partial z_1(x)}{\partial x}\,\frac{\partial z_2(x)}{\partial x}\right] dx$$

$$\|z\|_{W_2^1(0,l)} = \sqrt{\langle z, z\rangle_{W_2^1(0,l)}}.$$

4) $L_2(0,l)$ is a Hilbert space which consisting of all measurable functions on $(0,l)$ with

$$\langle z_1, z_2\rangle_{L_2(0,l)} = \int_0^l z_1(x)z_2(x)\,dx, \|z\|_{L_2(0,l)} = \sqrt{\langle z, z\rangle_{L_2(0,l)}}.$$

Consider a distributed parameter system described by the equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial^2 u(x,t)}{\partial x^2} + B(x,t)u(x,t) = f(x,t), \quad (x,t) \in \Omega \tag{1}$$

subject to initial conditions

$$(2)\qquad u(x,0) = \phi_0(x), \quad \frac{\partial u(x,0)}{\partial t} = \phi_1(x), \quad x \in (0,l),$$

and boundary conditions

$$(3)\qquad \frac{\partial u(x,t)}{\partial x}\Big|_{x=0} = \frac{\partial u(x,t)}{\partial x}\Big|_{x=l} = 0, \quad t \in (0,T),$$

where $\phi_0 \in W_2^1(0,l)$, $\phi_1 \in L_2(0,l)$ and $B(x,t) \in C(\overline{\Omega})$, for all $(x,t) \in \Omega |B(x,t)| < \mu_1$ and $\mu_1 > 0$.

On the admissible control set

$$V = \{f = f(x,t) : f \in L_2(\Omega),\ \|f\|_{L_2(\Omega)} < C_0,\ C_0 > 0\}$$

it is desired to compute the optimal control $f(x,t)$ which minimizes the cost function $J(f)$ given by

(4)

$$J(f) = \frac{1}{2}\int_0^T \int_0^l \left\{\beta_0 \Big[\frac{\partial u(x,t)}{\partial t}\Big]^2 - \beta_1 \Big[\frac{\partial u(x,t)}{\partial x}\Big]^2 + \alpha f^2(x,t)\right\} dx\, dt,$$

where α, β_0, β_1 are constants which depend on the system and its performance. l is the final point on the spatial coordinate axis x and T is the final time. $u(x,t)$ is the state variable at any time and spatial coordinate axis x; $f(x,t)$ represents the control variable. The expression for the cost function of (4) yields the energy of the vibrating system of (1)–(3) where β_0 stands for constant mass density and β_1 stands for tension which is exerted upon the end points.

Definition 1. The problem of finding a function $u = u(x,t) \in W_2^1(\Omega)$ from conditions (1)–(3) at given $f \in V$ is called the reduced problem.

Definition 2. For each $f \in V$, a function $u(x,t)$ is called a weak solution of the reduced problem (1)–(3) belonging to the control $f \in L_2(\Omega)$ if and only if (i) $u(x,t) \in W_2^1(\Omega)$ and equals ϕ_0 at $t = 0$,

(ii) The integral identity

$$\int_0^l \int_0^T \Big[- \frac{\partial \eta}{\partial t} \frac{\partial u(x,t)}{\partial t} + \frac{\partial u}{\partial x} \frac{\partial \eta}{\partial x} + B(x,t)u(x,t)\eta(x,t) - f(x,t)\eta(x,t)\Big]\, dx\, dt = \int_0^l \phi_1(x)\eta(x,0)\, dx, \tag{5}$$

is valid for all $\eta = \eta(x,t) \in W_2^1(\Omega)$ and $\eta(x,T) = 0$.

The solution of the reduced problem (1)–(3) explicitly depends on the control f, therefore we shall also use the notation $u = u(x,t;f)$.

On the basis of adopted assumptions and the results of [**8**] it follows that for every $f \in V$ the solution of the problem (1)–(3) exists, is unique and

$$\|u\|_{W_2^1(\Omega)} \le C \left[\|\phi_0\|_{W_2^1(0,l)} + \|\phi_1\|_{L_2(0,l)} + \|f\|_{L_2(\Omega)} \right], \tag{6}$$

where the constant C is independent of the numerical difference scheme used to approximate the problem (which would appear to be the main result later on).

Optimal control problems of the coefficients of differential equations do not always have solution [**15**]. In [**7, 17**], the numerical solution of problem (1)–(4) and necessary conditions for minimization are presented.

3. The difference approximations problem. Now, we shall find the difference approximations problem for (1)–(4). For discretization the optimal control problem (1)–(4), we introduce the following net $\{(x_{i,n}, t_{j,n}\},\ n = 1, 2, \dots,$ in Ω where

$$x_{i,n} = ih_n, \quad i = \overline{0, N_n} \equiv 0, 1, 2, \dots, N_n,$$
$$h_n = \frac{l}{N_n}; \quad t_{j,n} = j\tau_n, \quad j = \overline{0, M_n},$$
$$\tau_n = \frac{T}{M_n}, \quad N = N_n, \quad M = M_n, \quad h = h_n, \quad \tau = \tau_n, \quad x_i = x_{i,n},$$
$$x_{i-(1/2)} = x_{i-(1/2),n}, \quad t_j = t_{j,n}.$$

Here and further for the functions u, $(\partial u/\partial x)(\partial u/\partial t)$, η, $(\partial\eta/\partial x)$, $(\partial\eta/\partial t)$, $B(x,t)$ in Ω take the net form u_i^j, $(u_i^j)_x(u_i^j)_{\bar{t}}$, η_i^j, $(\eta_i^j)_x$, $(\eta_i^j)_{\bar{t}}$, $B_i^j = B(x_i, t_j)$, $i = \overline{0,N}$, $j = \overline{0,M}$, $\eta_i^M = 0$, $i = \overline{0,N}$ and use the following notation

$$(u_i^j)_x = \frac{u_{i+1}^j - u_i^j}{h}, \quad (u_i^j)_{\bar{x}} = \frac{u_i^j - u_{i-1}^j}{h}, \quad (u_i^j)_{\bar{t}} = \frac{u_i^j - u_i^{j-1}}{\tau}$$
$$l_0 = [0, x_{1/2}), \quad l_i = [x_{i-(1/2)}, x_{i+(1/2)}),$$
$$i = \overline{1, N-1},$$
$$l_N = [x_{N-(1/2)}, l],$$
$$T_j = (t_{j-1}, t_j], \quad j = \overline{1, M-2},$$
$$T_{M-1} = (t_{M-2}, t_M],$$
$$\Omega_{ij} = l_i \times T_j,$$
$$i = \overline{0,N}, \quad j = \overline{1, M-1}.$$

The given functions in (5) approximate as follows:

$$(\phi_0)_i = \frac{1}{\operatorname{meas} l_i} \int_{l_i} \phi_0(x)\, dx,$$
$$(\phi_1)_i = \frac{1}{\operatorname{meas} l_i} \int_{l_i} \phi_1(x)\, dx,$$
$$f_i^j = \frac{1}{\operatorname{meas} \Omega_{ij}} \int_{\Omega_{ij}} f(x,t)\, dx\, dt,$$
$$i = \overline{0,N}, \quad j = \overline{1, M-1},$$

where $\operatorname{meas} l_i$ and $\operatorname{meas} \Omega_{ij}$ are the Lebesgue-measure of l_i, Ω_{ij}.

The discrete analogy of the integral identity (5) writes in the form

$$\sum_{j=1}^{M} \int_{t_{j-1}}^{t_j} \left\{ \frac{1}{2} \Big[\int_0^{x_{(1/2)}} -(u_0^j)_{\bar{t}} (\eta_0^j)_{\bar{t}}\, dx + \int_{x_{N-(1/2)}}^{l} (-(u_N^j)_{\bar{t}} (\eta_N^j)_{\bar{t}})\, dx \Big] \right.$$
$$\left. + \sum_{i=1}^{N-1} \int_{x_{i-(1/2)}}^{x_{i+(1/2)}} (-(u_i^j)_{\bar{t}} (\eta_N^j)_{\bar{t}})\, dx \right\} dt$$
$$+ \frac{1}{2} \sum_{i=0}^{N-1} \sum_{j=1}^{M-1} \int_{x_i}^{x_{i+1}} \int_{t_{j-1}}^{t_j} [(u_i^j)_x + (u_i^j)_x] (\eta_i^j)_x\, dx\, dt$$

(7)

$$+\sum_{j=1}^{M-1}\int_{t_{j-1}}^{t_j}\Big\{\frac{1}{2}\Big[\int_0^{x_{(1/2)}}(B_0^j u_0^j - f(x,t))\eta_0^j\,dx + \int_{x_{N-(1/2)}}^{l}(B_N^j u_N^j - f(x,t))\eta_N^j\,dx\Big] + \sum_{i=1}^{N-1}\int_{x_{i-(1/2)}}^{x_{i+(1/2)}}[B_i^j u_i^j - f(x,t)]\eta_i^j\,dx\Big\}\,dt = \int_0^l \phi_1(x)\eta(x,0)\,dx.$$

Then from (7) we have

$$h\tau\sum_{j=1}^{M}\Big\{\Big[-\frac{1}{2}(u_0^j)_{\bar t}(\eta_0^j)_{\bar t} - \frac{1}{2}\,(u_N^j)_{\bar t}(\eta_N^j)_{\bar t}\Big] - \sum_{i=1}^{N-1}(u_i^j)_{\bar t}(\eta_i^j)_{\bar t})\Big\} + \frac{h\tau}{2}\sum_{i=0}^{N-1}\sum_{j=1}^{M-1}[(u_i^{j+1})_x + (u_i^{j-1})_x](\eta_i^j)_x$$
$$+h\tau\sum_{j=1}^{M-1}\Big\{\frac{1}{2}\,(B_0^j u_0^j - f_0^j)\eta_0^j + \frac{1}{2}\,(B_N^j u_N^j - f_N^j)\eta_N^j + \sum_{i=1}^{N-1}(B_i^j u_i^j - f_i^j)\eta_i^j\Big\} = h\Big\{\sum_{i=1}^{N-1}(\phi_1)_i\eta_i^0 + \frac{1}{2}\,[(\phi_1)_0\eta_0^0 + (\phi_1)_N\eta_N^0]\Big\}. \tag{8}$$

From [**12**], we have

(9)

$$h\sum_{i=0}^{N-1}(u_i^{j+1})_x(\eta_i^j)_x = -h\sum_{i=1}^{N-1}(u_i^{j+1})_{x\bar x}\eta_i^j + (u_N^{j+1})_{\bar x}\eta_N^j - (u_0^{j+1})_x\eta_0^j,\quad j=\overline{1,M-1}$$

(10)

$$h\sum_{i=0}^{N-1}(u_i^{j-1})_x(\eta_i^j)_x = -h\sum_{i=1}^{N-1}(u_i^{j-1})_{x\bar x}\eta_i^j + (u_N^{j-1})_{\bar x}\eta_N^j - (u_0^{j-1})_x\eta_0^j,\quad j=\overline{1,M-1}$$

$$\tau\sum_{j=1}^{M}(-u_i^j)_{\bar t}(\eta_i^j)_t = \tau\sum_{j=1}^{M-1}(u_i^j)_{t\bar t}\eta_i^j + \frac{1}{\tau}\,[u_i^1 - u_i^0]\eta_i^0,\quad i=\overline{0,N}. \tag{11}$$

Using (9)–(11), from (8) we obtain

(12)

$$\tau \sum_{j=1}^{M-1} (u_i^j)_{t\bar{t}} \eta_i^j + \frac{h}{\tau} \sum_{i=1}^{N-1} [u_i^1 - u_i^0] \eta_i^0 - \frac{h\tau}{2} \sum_{i=1}^{N-1} \sum_{j=1}^{M-1} (u_i^{j+1})_{x\bar{x}} \eta_i^j$$
$$+ \frac{\tau}{2} \sum_{j=1}^{M-1} (u_N^{j+1})_{\bar{x}} \eta_N^j - \frac{\tau}{2} \sum_{j=1}^{M-1} (u_0^{j+1})_x \eta_0^j - \frac{h\tau}{2} \sum_{i=1}^{N-1} \sum_{j=1}^{M-1} (u_i^{j-1})_{x\bar{x}} \eta_i^j$$
$$+ \frac{\tau}{2} \sum_{j=1}^{M-1} (u_N^{j-1})_{\bar{x}} \eta_N^j - \frac{\tau}{2} \sum_{j=1}^{M-1} (u_0^{j-1})_x \eta_0^j$$
$$+ \frac{h\tau}{2} \sum_{j=1}^{M-1} \{(B_0^j u_0^j - f_0^j \eta_0^j) + (B_N^j u_N^j - f_N^j \eta_N^j)\} + h\tau \sum_{i=1}^{N-1} \sum_{j=1}^{M-1} [B_i^j u_i^j - f_i^j] \eta_i^j$$
$$+ \frac{h\tau}{2} \sum_{j=1}^{M-1} (u_0^j)_{t\bar{t}} \eta_0^j + \frac{h}{2\tau} [u_0^1 - u_0^0] \eta_0^0$$
$$+ \frac{h\tau}{2} \sum_{j=1}^{M-1} (u_N^j)_{t\bar{t}} \eta_N^j + \frac{h}{2\tau} [u_N^1 - u_N^0] \eta_N^0$$
$$= h \sum_{i=1}^{N-1} (\phi_1)_i \eta_i^0 + \frac{1}{2} [(\phi_1)_0 \eta_0^0 + (\phi_1)_N \eta_N^0].$$

Hence, equality to zero the coefficients of η_i^j, $i = \overline{0,N}$, $j = \overline{0,M}$ in (12), we obtain the difference approximations problem for (1)–(3):

$$(u_i^j)_{t\bar{t}} - \frac{1}{2} [(u_i^{j+1})_{x\bar{x}} + (u_i^{j-1})_{x\bar{x}}] + B_i^j u_i^j = f_i^j, \quad i = \overline{1, N-1}, \quad j = \overline{1, M-1}, \tag{13}$$

$$u_i^0 = (\phi_0)_i, \quad (u_i^0)_t = (\phi_1)_i, \quad i = \overline{0, N-1}, \tag{14}$$

$$(u_0^{j+1})_x + (u_0^{j-1})_x = h [(u_0^j)_{t\bar{t}} + B_0^j u_0^j - f_0^j], \quad j = \overline{1, M-1}, \tag{15}$$

$$(16)\qquad \begin{gathered}(u_N^{j+1})_{\bar{x}} + (u_N^{j-1})_{\bar{x}} = -h[(u_N^j)_{t\bar{t}} + B_N^j u_N^j - f_N^j],\\ j = \overline{1, M-1}.\end{gathered}$$

But the functional (4) is approximated by the following way [**13, 14**]:

$$(17)\qquad I([v]) = \frac{h\tau}{2} \sum_{j=0}^{N} \sum_{1}^{M} \{\beta_0[(u_i^j)_t]^2 - \beta_1[(u_i^j)_x]^2\} + \frac{\alpha}{2} \|[f]\|^2_{\overline{L}_2(\Omega)}.$$

The controls $[f]$ matrix with elements f_i^j, $i = \overline{0, N}$, $j = \overline{1, M}$ are chosen from the set

$$(18)\qquad F_n = \{[f] : [f] \in \overline{L}_2(\Omega),\ \|[f]\|_{\overline{L}_2(\Omega)} \le R,\ R > 0\}$$

where

$$(19)\qquad \begin{aligned}\|[f]\|^2_{\overline{L}_2(\Omega)} &= h\tau \sum_{j=1}^{M-2} \left\{ \sum_{i=1}^{N-1} (f_i^j)^2 + \frac{1}{2}\,[(f_0^j)^2 + (f_N^j)^2] \right\}\\ &\quad + h\tau \left\{ 2 \sum_{i=1}^{N-1} (f_i^{M-1})^2 + (f_0^{M-1})^2 + (f_N^{M-1})^2 \right\}.\end{aligned}$$

4. Stability theorem. In this section, we show a stability result with respect to the $\overline{W}_2^1(\Omega)$-norm, where $\overline{W}_2^1(\Omega)$ is the Hilbert space of network functions $u = u_i^j$, $i = \overline{0, N}$, $\overline{0, M}$ and here the usual notations for the norm and inner product are used, i.e.,

$$(20)\qquad \begin{gathered}\langle u, g\rangle_{\overline{W}_2^1(\Omega)} = \langle u, g\rangle_{\overline{L}_2(\Omega)} + \langle u_x, g_x\rangle_{\overline{L}_2(\Omega)} + \langle u_{\bar{t}}, g_{\bar{t}}\rangle_{\overline{L}_2(\Omega)}\\ \|u\|_{\overline{W}_2^1(\Omega)} = \sqrt{\langle u, u\rangle_{\overline{W}_2^1(\Omega)}}\end{gathered}$$

where

(21)

$$\langle u_{\bar{t}}, g_{\bar{t}}\rangle_{\overline{L}_2(\Omega)} = \sum_{j=1}^{M} \left\{ \sum_{i=1}^{N-1} (u_i^j)_{\bar{t}}(g_i^j)_{\bar{t}} + \frac{1}{2}\,[(u_0^j)_{\bar{t}}(g_0^j)_{\bar{t}} + (u_N^j)_{\bar{t}}(g_N^j)_{\bar{t}}] \right\},$$

$$\langle u_x, g_x\rangle_{\overline{L}_2(\Omega)} = h\tau \sum_{j=0}^{M} \sum_{i=0}^{N-1} (u_i^j)_x (g_i^j)_x.$$

Theorem 4.1. *Suppose that all the functions in the system* (1)–(3) *satisfy the above enumerated conditions. Then the estimation of stability for difference scheme* (13)–(16) *is*

$$\|u\|^2_{\overline{W}^1_2(\Omega)} \le C\left[\,\|\phi_0\|^2_{\overline{W}^1_2(0,l)} + \|\phi_1\|^2_{L_2(0,l)} + \|[f]\|^2_{L_2(\Omega)}\right]. \tag{22}$$

Proof. Multiplying (13) by $[(u_i^j)_t + (u_i^j)_{\bar t}]$ and summing on i from 1 to $N-1$, and on j from 1 to $p-1$, $p \le M$, we obtain

$$h\tau \sum_{j=1}^{p-1}\sum_{i=1}^{N-1}\left\{(u_i^j)_{t\bar t} - \frac{1}{2}\left[(u_i^{j+1})_{x\bar x} + (u_i^{j-1})_{x\bar x}\right] + B_i^j u_i^j - f_i^j\right\} \left[((u_i)^j)_t + ((u_i)^j)_{\bar t}\right] = 0. \tag{23}$$

But

$$u_{t\bar t}(u_t + u_{\bar t}) = (u_{\bar t})_t, \quad j = \overline{1, M-1}, \tag{24}$$

and, using (15)–(16), we have

$$\begin{aligned} -\frac{h}{2}\sum_{i=1}^{N-1}&[(u_i^{j+1})_{x\bar x} + (u_i^{j-1})_{x\bar x}][((u_i)^j)_t + ((u_i)^j)_{\bar t}] \\ &= \frac{h}{2}\sum_{i=0}^{N-1}[(u_i^{j+1})_x + (u_i^{j-1})_x][((u_i)^j)_{tx} + ((u_i)^j)_{\bar t x}] \\ &\quad + \frac{h}{2}[(u_0^j)_{t\bar t} + B_0^j u_0^j - f_0^j][(u_0^j)_t + (u_0^j)_{\bar t}] \\ &\quad + \frac{h}{2}[(u_N^j)_{t\bar t} + B_N^j u_N^j - f_N^j][(u_N^j)_t + (u_N^j)_{\bar t}]. \end{aligned} \tag{25}$$

If we take into account (24) and (25) in (23), then we obtain

$$\begin{aligned} &h\tau\sum_{j=1}^{p-1}\sum_{i=1}^{N-1}((u_i^j)_{\bar t})_t + \sum_{j=1}^{p-1}\sum_{i=0}^{N-1}[(u_i^{j+1})_x + (u_i^{j-1})_x][(u_i^j)_{tx} + (u_i^j)_{\bar t x}] \\ &\quad + \frac{h\tau}{2}\sum_{j=1}^{p-1}[((u_0^j)^2_{\bar t})_t + ((u_N^j)^2_{\bar t})_t] + \frac{h\tau}{2}\sum_{j=1}^{p-1}[(B_0^j u_0^j - f_0^j)((u_0^j)_t + (u_0^j)_{\bar t}) \\ &\quad + (B_N^j u_N^j - f_N^j)((u_N^j)_t + (u_N^j)_{\bar t})] + h\tau\sum_{j=1}^{p-1}\sum_{i=1}^{N-1}[B_i^j u_i^j - f_i^j][(u_i^j)_t + (u_i^j)_{\bar t}]. \end{aligned} \tag{26}$$

It is valid the identity

$$\sum_{i=0}^{N-1}[(u_i^{j+1})_x+(u_i^{j-1})_x][(u_i^j)_{tx}+(u_i^j)_{\bar{t}x}]=h\sum_{i=0}^{N-1}[(u_i^{j+1})_x^2+(u_i^j)_x^2], \tag{27}$$

$$\tau\sum_{j=1}^{p-1}((u_i^j)_{\bar{t}}^2)_t=(u_i^j)_{\bar{t}}^2\mid_{t=\tau}^{t=p\tau}, \tag{28}$$

$$\tau\sum_{j=1}^{p-1}((u_i^{j+1})_x^2+(u_i^j)_x^2)_{(\bar{t})}=[(u_i^j)_x^2+(u_i^{j-1})_x^2]\mid_{t=\tau}^{t=p\tau}\ . \tag{29}$$

Using the above relations and from (26), we have

$$\begin{aligned}
&h\sum_{i=1}^{N-1}[(u_i^j)_{\bar{t}}^2]\mid_{t=p\tau}+\frac{h}{2}\sum_{i=0}^{N-1}[(u_i^j)_x^2+(u_i^{j-1})_x^2]\mid_{t=p\tau}\\
&+\frac{h}{2}\,[(u_0^j)_{\bar{t}}^2]\mid_{t=p\tau}+(u_N^j)_{\bar{t}}^2]\mid_{t=p\tau}]\\
&\qquad=-h\tau\sum_{j=1}^{p-1}\sum_{i=1}^{N-1}[B_i^ju_i^j-f_i^j][(u_i^j)_t+(u_i^j)_{\bar{t}}]\\
&\qquad\quad-\frac{h\tau}{2}\sum_{j=1}^{p-1}[(B_0^ju_0^j-f_0^j)((u_0^j)_t+(u_0^j)_{\bar{t}})\\
&\qquad\quad+(B_N^ju_N^j-f_N^j)((u_N^j)_t+(u_N^j)_{\bar{t}})]\\
&\qquad\quad+\left\{h\sum_{i=1}^{N-1}(u_i^j)_t^2\mid_{t=0}+\frac{h}{2}\,[(u_0^j)_t^2]\mid_{t=0}+[(u_N^j)_t^2]\mid_{t=0}\right\}\\
&\qquad\quad+\frac{h}{2}\sum_{i=0}^{N-1}[(u_i^j)_x^2+(u_i^{j-1})_x^2]\mid_{t=\tau}\ .
\end{aligned} \tag{30}$$

But

$$
\begin{aligned}
&\left|h\tau\sum_{j=1}^{p-1}\sum_{i=1}^{N-1}[B_i^j u_i^j - f_i^j][(u_i^j)_t + (u_i^j)_{\bar{t}}]\right| \\
&\qquad \le \mu_1 h\sum_{i=1}^{N-1}\tau\sum_{j=1}^{p-1}(|u_i^j| + |f_i^j|)(|(u_i^j)_t| + |(u_i^j)_{\bar{t}}|) \\
&\qquad \le \mu_1 h\sum_{i=1}^{N-1}\tau\sum_{j=1}^{p-1}([|u_i^j| + |f_i^j|]^2 + [|(u_i^j)_t| + |(u_i^j)_{\bar{t}}|]^2) \qquad (31) \\
&\qquad \le \mu_1 h\sum_{i=1}^{N-1}\tau\sum_{j=1}^{p-1}(2|u_i^j|^2 + 2|f_i^j|^2 + 2|(u_i^j)_t|^2 + 2|(u_i^j)_{\bar{t}}|^2) \\
&\qquad \le 4\mu_1 h\sum_{i=1}^{N-1}\tau\sum_{j=1}^{p-1}[(u_i^j)^2 + (f_i^j)^2 + (u_i^j)_{\bar{t}}^2],
\end{aligned}
$$

$$
\begin{aligned}
&\left|\frac{h\tau}{2}\sum_{j=1}^{p-1}[(B_0^j u_0^j - f_0^j)((u_0^j)_t + (u_0^j)_{\bar{t}})]\right| \\
&\qquad \le 4\mu_1\frac{h\tau}{2}\sum_{j=1}^{p-1}[(u_0^j)^2 + (f_0^j)^2 + (u_0^j)_{\bar{t}}^2], \qquad (32)
\end{aligned}
$$

$$
\begin{aligned}
&\left|\frac{h\tau}{2}\sum_{j=1}^{p-1}[(B_N^j u_N^j - f_N^j)((u_N^j)_t + (u_N^j)_{\bar{t}})]\right| \\
&\qquad \le 4\mu_1\frac{h\tau}{2}\sum_{j=1}^{p-1}[(u_N^j)^2 + (f_N^j)^2 + (u_N^j)_{\bar{t}}^2]. \qquad (33)
\end{aligned}
$$

Using these inequalities, from (30) we have

(34)

$$\left\{h\sum_{i=1}^{N-1}(u_i^j)_{\bar t}^2\,\frac{h}{2}\,[(u_0^j)_{\bar t}^2+(u_N^j)_{\bar t}^2]\right\}\Big|_{t=p\tau}$$

$$+\frac{h}{2}\sum_{i=0}^{N-1}[(u_i^j)_x^2+(u_i^{j-1})_x^2]\,|_{t=p\tau}$$

$$\le C\,\tau\sum_{j=1}^{p}\left\{h\sum_{i=1}^{N-1}(u_i^j)^2+\frac{h}{2}\,[(u_0^j)^2+(u_N^j)^2]\right.$$

$$\left.+\,h\sum_{i=1}^{N-1}(u_i^j)_{\bar t}^2+\frac{h}{2}\,[(u_0^j)_{\bar t}^2+(u_N^j)_{\bar t}^2]+h\sum_{i=1}^{N-1}(f_i^j)^2\frac{h}{2}\,[(f_0^j)^2+(f_N^j)^2]\right\}$$

$$+\left\{h\sum_{i=1}^{N-1}(u_i^j)_t^2+\frac{h}{2}\,[(u_0^j)_t^2+(u_N^j)_t^2]\right\}\Big|_{t=0}$$

$$+\frac{h}{2}\sum_{i=0}^{N-1}[(u_i^j)_x^2+(u_i^{j-1})_x^2]\Big|_{t=\tau}.$$

But

$$\frac{h}{2}\sum_{i=0}^{N-1}[(u_i^j)_x^2+(u_i^{j-1})_x^2]\,|_{t=\tau}$$

$$=\frac{h}{2}\sum_{i=0}^{N-1}[\tau(\phi_1)_x+(\phi_0)_x]^2$$

$$\le\frac{3h}{2}\sum_{i=0}^{N-1}(\phi_0)_x^2+\frac{\tau^2}{h}\sum_{i=0}^{N-1}[(\phi_1)_{i+1}-(\phi_1)_i]^2 \tag{35}$$

$$\le\frac{3h}{2}\sum_{i=0}^{N-1}(\phi_0)_x^2+8\left\{h\sum_{i=0}^{N-1}(\phi_1)_i^2+\frac{h}{2}\,[(\phi_1)_0^2+(\phi_1)_N^2]\right\}$$

$$=\frac{3}{2}\|(\phi_0)_x\|_{\overline{L}_2(0,l)}^2+8\|(\phi_1)\|_{\overline{L}_2(0,l)}^2.$$

Using the following identity $u_i^j=u_0^j+\tau\sum_{k=1}^{j}(u_i^k)_{\bar t}$ and the Cauchy-

Bunyakovsky inequality for the sum, we have

$$h\sum_{i=1}^{N-1}(u_i^j)^2 \le 2h\sum_{i=1}^{N-1}(u_0^j)^2 + 2t_j\tau\sum_{k=1}^{j}h\sum_{i=1}^{N-1}(u_i^k)_{\bar t}, \tag{36}$$

$$\frac{h}{2}\,u_0^j \le \frac{2h}{2}\,(u_0^0)^2 + 2t_j\tau\sum_{k=1}^{j}\frac{h}{2}\,(u_0^k)_{\bar t}, \tag{37}$$

$$\frac{h}{2}\,u_N^j \le \frac{2h}{2}\,(u_N^0)^2 + 2t_j\tau\sum_{k=1}^{j}\frac{h}{2}\,(u_N^k)_{\bar t}. \tag{38}$$

Summing these inequalities, we obtain

$$\begin{aligned} &h\sum_{i=1}^{N-1}(u_i^j)^2 + \frac{h}{2}\,[(u_0^j)^2 + (u_N^j)^2] \\ &\quad\le 2\Bigg\{h\sum_{i=1}^{N-1}(u_i^0)^2 + \frac{h}{2}\,[(u_0^0)^2 + (u_N^0)^2]\Bigg\} \\ &\qquad + 2t_j\Bigg\{\tau\sum_{k=1} j\Big[h\sum_{i=1}^{N-1}(u_i^j)_{\bar t}^2 + \frac{h}{2}\,((u_0^j)_{\bar t}^2 + (u_N^j)_{\bar t}^2)\Big]\Bigg\} \\ &\quad\le \|\phi_0\|^2_{\bar L_2(0,l)} + 2T\Bigg\{\tau\sum_{j=1} p\Big[h\sum_{i=1}^{N-1}(u_i^j)_{\bar t}^2 + \frac{h}{2}\,((u_0^j)_{\bar t}^2 + (u_N^j)_{\bar t}^2)\Big]\Bigg\}. \end{aligned} \tag{39}$$

Substituting (35) and (39) into (34), we have

$$\begin{aligned} \Bigg\{h\sum_{i=1}^{N-1}(u_i^j)_{\bar t}^2 + \frac{h}{2}\,[(u_0^j)_{\bar t}^2 + (u_N^j)_{\bar t}^2]\Bigg\}\Bigg|_{t=p\tau} + \frac{h}{2}\sum_{i=0}^{N-1}(u_i^j)_x^2|_{t=p\tau} \\ \le C\,Z(p) + F(p), \end{aligned} \tag{40}$$

where

$$Z(p) = \tau\sum_{j=1} p\Bigg\{h\sum_{i=1}^{N-1}(u_i^j)_{\bar t}^2 + \frac{h}{2}\,[(u_0^j)_{\bar t}^2 + (u_N^j)_{\bar t}^2]\Bigg\} + h\sum_{i=0}^{N-1}(u_i^j)_x^2. \tag{41}$$

$$\begin{aligned} F(p) = C\Bigg\{&\|\phi_0\|^2_{\bar L_2(0,l)}\|(\phi_0)_x\|^2_{\bar L_2(0,l)}\|\phi_1\|_{\bar L_2(0,l)} \\ &+ \tau\sum_{j=1}^{p}\Big[h\sum_{i=1}^{N-1}(f_i^j)^2 + \frac{h}{2}\,((f_0^j)^2 + (f_N^j)^2)\Big]\Bigg\}. \end{aligned} \tag{42}$$

Then the inequality (40) may be written in the form

$$Z(p) \leq \frac{1}{1-\tau C}\, Z(p-1) + \frac{\tau}{1-\tau C}\, F(p). \tag{43}$$

Assume that $1-\tau C \geq 1/2$ and denote $E = 1/(1-\tau C)$, then

$$Z(p) \leq EZ(p-1) + \tau E F(p). \tag{44}$$

Now, applying the inequality (44) successively $p-1$ times, we obtain

$$Z(p) \leq E^{p-1} Z(1) + E\tau \sum_{s=2}^{p} E^{p-s} F(s), \tag{45}$$

$$\begin{aligned} Z(1) = \tau\Big\{ h \sum_{i=1}^{N-1} ((\phi_1)_i)^2 + \frac{h}{2}\,[((\phi_1)_0)^2 + ((\phi_1)_N)^2] \Big\} \\ + \tau h \sum_{i=0}^{N-1} [(\phi_0)_x + \tau(\phi_1)_x]^2. \end{aligned} \tag{46}$$

Besides, the following inequalities are valid

$$E^p = \left(1 + \frac{C\tau}{1-C\tau}\right)^p \leq e^{(Cp\tau)/(1-C\tau)} \leq e^{(TC/0.5)} = e^{2TC}, \tag{47}$$

(48)

$$E\tau \sum_{s=2} pE^{p-s} F(s) = E^{p-1}\tau \sum_{s=2} pF(s) \leq E^{p-1} F(p)\tau p \leq Te^{2TC} F(p).$$

Using the inequalities (47), (48) and (46), then from (45) we obtain

$$\begin{aligned} Z(p) &\leq e^{2TC} Z(1) + Te^{2TC} F(p) = e^{2TC}[Z(1) + TF(p)] \\ &= e^{2TC}\Big\{ \tau\Big[h \sum_{i=1}^{N-1} ((\phi_1)_i)^2 + \frac{h}{2}\,(((\phi_1)_0)^2 + ((\phi_1)_N)^2) \Big] \\ &\quad + \tau h \sum_{i=0}^{N-1} ((\phi_0)_x + \tau(\phi_1)_x)^2 + TF(p) \Big\}. \end{aligned} \tag{49}$$

Besides, we have

(50)

$$
\begin{aligned}
&h\sum_{i=1}^{N-1}((\phi_1)_i)^2+\frac{h}{2}\,[((\phi_1)_0)^2+((\phi_1)_N)^2]+h\sum_{i=0}^{N-1}[(\phi_0)_x+\tau(\phi_1)_x]^2\\
&\quad=h\sum_{i=1}^{N-1}((\phi_1)_i)^2+\frac{h}{2}\,[((\phi_1)_0)^2+((\phi_1)_N)^2]\\
&\qquad+h\sum_{i=0}^{N-1}[(\phi_0)_x+\frac{\tau}{h}\,((\phi_1)_{i+1}-(\phi_1)_i)]^2\\
&\quad\le h\sum_{i=1}^{N-1}((\phi_1)_i)^2+\frac{h}{2}\,[((\phi_1)_0)^2+((\phi_1)_N)^2]\\
&\qquad+3h\sum_{i=0}^{N-1}\Big\{(\phi_0)_x^2+\frac{\tau^2}{h^2}\,(\phi_1)_{i+1}^2-(\phi_1)_i^2\Big\}\\
&\quad\le C\,[\,\|(\phi_0)_x\|^2_{\overline{L}_2(0,l)}+\|\phi_1\|_{\overline{L}_2(0,l)}].
\end{aligned}
$$

Using (50) and expressions for $F(p)$, then from (49) we have

$$
\begin{aligned}
Z(p)\le C\Big\{&\|\phi_0\|^2_{\overline{L}_2(0,l)}\|(\phi_0)_x\|^2_{\overline{L}_2(0,l)}\|\phi_1\|^2_{\overline{L}_2(0,l)}\\
&+\tau\sum_{j=1}^{p}\Big[h\sum_{i=1}^{N-1}(f_i^j)^2+\frac{h}{2}\,((f_0^j)^2+(f_N^j)^2)\Big]\Big\}.
\end{aligned}
\tag{51}
$$

From (51) and (39), then the estimation of stability (22) is obtained. The theorem is proved. □

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