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ON CR-STRUCTURES AND F-STRUCTURE SATISFYING $F^K + (-)^{K+1}F = 0$

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ABSTRACT. CR-submanifolds of a Kahlerian manifold have been defined by Bejancu [**1**], and are now being studied by various authors, see [**2, 3, 9**]. The theory of f-structure was developed by Yano [**7**], Yano and Ishihara [**8**]. Goldberg [**5**] and others. The purpose of this paper is to show a relationship between CR-structures and F-structure satisfying $F^K + (-)^{K+1}F = 0$, $F^W + (-)^{W+1}F \neq 0$, for $1 < W < K$, where K is a fixed positive integer greater than 2. The case when k is odd (≥ 3) has been considered in this paper.

1. Introduction. Let F be a nonzero tensor field of the type $(1,1)$ and of class c^∞ on an n-dimensional manifold M such that [**6**].

$$F^K + (-)^{K+1}F = 0 \quad \text{and} \quad F^W + (-)^{W+1}F \neq 0 \quad \text{for} \quad 1 < W < K \tag{1.1}$$

where K is a fixed positive integer greater than 2. Such a structure on M is called an F-structure of rank r and of degree K. If the rank of F is constant and $r = r(F)$, then M is called an F-structure manifold of degree $K(\geq 3)$.

Let the operators on M be defined as follows [**6**]

$$l = (-)^K F^{K-1}, \quad m = I + (-)^{K-1}F^{K-1}, \tag{1.2}$$

where I denotes the identity operator on M.

We will state the following two theorems [**6**].

Theorem (1.1). *Let M be an F-structure manifold. Then*

$$l + m = I, \quad l^2 = l \quad \text{and} \quad m^2 = m. \tag{1.3}$$

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For $F \neq 0$ satisfying (1.1)*, there exist complementary distributions D_l and D_m corresponding to the projection operators l and m respectively.*

If the rank $(F) =$ constant and $r = r(F)$ on M, then $\dim D_l = r$ and $\dim D_m = (n - r)$.

Theorem (1.2). *We have*

$$Fl = lF = F, \quad Fm = mF = 0 \tag{1.4a}$$

$$F^{K-1} = (-)^K l, \quad F^{K-1} l = -l, \quad F^{K-1} m = 0. \tag{1.4b}$$

Thus $F^{(K-1)/2}$ acts on D_l as an almost complex structure and on D_m as a null operator.

2. Nijenhuis tensor. The Nijenhuis tensor $N(X, Y)$ of F satisfying (1.1) in M is expressed as follows for every vector field X, Y, on M.

$$N(X, Y) = [FX, FY] - F[FX, Y] - F[X, FY] + F^2[X, Y]. \tag{2.1}$$

We state the following theorem without proof [**4**].

Theorem (2.1). *A necessary and sufficient condition for the f-structure 'f' to be integrable is that $N(X, Y) = 0$ for any two vector fields X and Y on M.*

Definition 2.1. If X, Y are two vector fields in M, then their Lie bracket $[X, Y]$ is defined by

$$[X, Y] = XY - YX. \tag{2.2}$$

3. CR-structure. Let M be a differentiable manifold and T_cM its complexified tangent bundle. A CR-structure on M is a complex subbundle H of T_cM such that $H_p \cap \overline{H}_p = 0$ and H is involutive, i.e., for complex vector fields X and Y in $H, [X, Y]$ is in H. In this case we say M is a CR-manifold. Let F be an integrable F-structure satisfying

(1.1) of rank $r = 2m$ on M. We define complex subbundle H of T_cM by $H_p = \{X - \sqrt{-1}\,FX, X \in \mathcal{X}(D_l)\}$, where $\mathcal{X}(D_l)$ is the $\mathcal{F}(D_m)$ module of all differentiable sections of D_l. Then $\mathrm{Re}\,(H) = D_l$ and $H \cap \overline{H}_p = 0$, where $\overline{H}_p$ denotes the complex conjugate of H.

Theorem (3.1). *If P and Q are two elements of H, then the following relations hold*

$$[P, Q] = [X, Y] - [FX, FY] - \sqrt{-1}\,([X, FY] + [FX, Y]). \tag{3.1}$$

Proof. Let us define $P = X - \sqrt{-1}\,FX$ and $Q = Y - \sqrt{-1}\,FX$. Then by direct calculation and on simplifying, we obtain

$$\begin{aligned} [P, Q] &= [X - \sqrt{-1}\ FX, Y - \sqrt{-1}\ FY] \\ &= [X, Y] - [FX, FY] - \sqrt{-1}\,([X, FY] + [FX, Y]). \qquad \square \end{aligned}$$

Theorem (3.2). *If the F-structure satisfying* (1.1) *is integrable, then we have*

$$(-)^K F^{K-2}([FX, FY] + F^2[X, Y]) = l([FX, Y] + [X, FY]). \tag{3.2}$$

Proof. From equation (2.1) we have

$$N(X, Y) = [FX, FY] - F[FX, Y] - F[X, FY] + F^2[X, Y].$$

Since $N(X, Y) = 0$ we obtain

$$[FX, FY] + F^2[X, Y] = F([FX, Y] + [X, FY]) \tag{3.2a}$$

operating (3.2a) by $(-)^K F^{K-2}$ we get

(3.2b)

$$\begin{aligned} (-)^K F^{K-2}([FX, FY] + F^2[X, Y]) &= (-)^K F^{K-2} F([FX, Y] + [X, FY]) \\ &= (-)^K F^{K-1}([FX, Y] + [X, FY]). \end{aligned}$$

On making use of equation (1.2) we obtain (3.2), which proves Theorem (3.2). □

Theorem (3.3). *The following identities hold*

$$mN(X,Y) = m[FX,FY] \tag{3.3}$$

$$mN(F^{K-2}X,Y) = m[F^{K-1}X,FY]. \tag{3.4}$$

Proof. The proof of (3.3) and (3.4) follows by virtue of Theorems (1.1), (1.2) and equations (1.2) and (2.1). □

Theorem (3.4). *For any two vector fields X and Y the following conditions are equivalent.*

a. $mN(X,Y) = 0$

b. $m[FX,FY] = 0$

c. $mN(F^{K-2}X,Y) = 0$

d. $m[F^{K-1}X,FY] = 0$

e. $m[F^{K-1}lX,FY] = 0.$

Proof. In consequence of equations (1.1), (1,2), (2.1) and Theorems (1.2) and (3.3), the above conditions can be proved to be equivalent. □

Theorem (3.5). *If $F^{(K-1)/2}$ acts on L as an almost complex structure, then*

$$m[F^{K-1}lX,FY] = M[-X,FY] = 0. \tag{3.5}$$

Proof. In view of equation (1.4), we see that $F^{(K-1)/2}$ acts on L as an almost complex structure then (3.5) follows in an obvious manner. To show that $m[F^{K-1}lX,FY] = 0$ we use Definition (2.1), i.e., $[X,Y] = XY - YX$ where X,Y are c^{∞} vector fields and in view of equation (1.4a), the result follows directly. □

Theorem (3.6). *For $X, Y \in \mathcal{X}(D_l)$ we have*

$$l([X, FY] + [FX, Y]) = [X, FY] + [FX, Y].$$

Proof. Since $[X, FY]$ and $[FX, Y] \in \mathcal{X} - (D_l)$. On making use of (1.4a) and Definition (2.1), we obtain the result. □

Theorem (3.7). *The integrable F-structure satisfying* (1.1) *on M defines a CR-structure H on it such that* $\operatorname{Re} H \equiv D_l$.

Proof. In view of the fact that $[X, FY]$ and $[FX, Y] \in \mathcal{X}(D_l)$ and on using equations (3.1), (3.2) and Theorem (3.6) we have $[P, Q] \in \mathcal{X}(D_l)$. Thus every F structure satisfying (1.1) on M defines a CR-structure. □

Definition (3.1). Let $\widetilde{K}$ be the complementary distribution of $\operatorname{Re}(H)$ to TM. We define a morphism of vector bundles $F : TM \to TM$ given by $F(X) = 0$ for all $X \in \mathcal{X}(\widetilde{K})$, such that

$$F(X) = \frac{1}{2}\sqrt{-1}\,(P - \overline{P}) \tag{3.6}$$

where $P = X + \sqrt{-1}\,Y \in \mathcal{X}(Hp)$ and $\overline{P}$ is a complex conjugate of P.

Corollary 3.1. *If $P = X + \sqrt{-1}\,Y$ and $\overline{P} = X - \sqrt{-1}\,Y$ belong to Hp and $F(X) = (1/2)\sqrt{-1}\,(P - \overline{P})$, $F(Y) = (1/2)(P + \overline{P})$ and $F(-Y) = -(1/2)(P + \overline{P})$, then $F(X) = -Y$, $F^2(X) = -X$ and $F(-Y) = -X$.*

Proof. On using Definition (3.1) we have

$$\begin{aligned} F(X) &= \frac{1}{2}\sqrt{-1}\,(X + \sqrt{-1}\,Y - X - \sqrt{-1}\,Y) \\ &= \frac{1}{2}\sqrt{-1}\,(2\sqrt{-1}\,Y) = -Y. \end{aligned}$$

Thus, $F(X) = -Y$, which on operating by F yields

$$F(F(X)) = F(-Y). \tag{3.7}$$

But

$$F(Y) = \frac{1}{2}\,(X + \sqrt{-1}\;Y + X - \sqrt{-1}\;Y),$$

which on simplifying gives

$$F(Y) = X.$$

Also,

$$F(-Y) = -\frac{1}{2}\,(X + \sqrt{-1}\;Y + X - \sqrt{-1}\;Y) \\ = -X. \tag{3.8}$$

Combining (3.7) and (3.8) we get

$$F^2(X) = -X.$$

Theorem (3.8). *If M has a CR-structure H, then we have $F^K + (-)^{K+1}F = 0$ and consequently an F-structure is defined on M such that the distributions D_l and D_m coincide with $\mathrm{Re}\,(H)$ and K respectively.*

Proof. Suppose M has a CR-structure on M. Then in view of Definition (3.1) and Corollary (3.1) we can write

$$F(X) = -Y; \tag{3.9}$$

operating (3.9) by $(-)^{K+1}F^{K-1}$ we get

$$(-)^{K+1}F^{K-1}(F(X)) = (-)^{K+1}F^{K-1}(-Y). \tag{3.10}$$

We can write the right-hand side $r \cdot h \cdot s$ of (3.10) as follows:

$$(-)^{K+1}F^{K}(X) = (-)^{K+1}F^{K-2}(F(-Y)). \tag{3.11}$$

On making use of Corollary (3.1), the $(r \cdot h \cdot s)$ of the above equation (3.11) becomes

$$\begin{aligned}
(-)^{K+1}F^K(X) &= (-)^{K+1}F^{K-2}(-X) \\
&= (-)(-)^{K+1}F^{K-2}(X), \quad \text{which can be written as} \\
&= (-)(-)^{K+1}F^{K-3}(F(X)), \\
&\qquad \text{which in view of Corollary (3.1) becomes} \\
&= (-)(-)^{K+1}F^{K-3}(-Y) \\
&= (-)^{K+1}F^{K-3}(Y) \\
&\quad - - - - - \\
&\quad - - - - - \\
&= (-)^{K+4}F^{K-5}(F(Y)).
\end{aligned}$$

We continue on simplifying in this manner K times. We get

$$\begin{aligned}
(-)^{K+1}F^K(X) &= (-)^{K+K}F^{K-(K+1)}(F(Y)) \\
&= Y \\
(-)^{K+1}F^K(X) &= -F(X).
\end{aligned}$$

On simplifying the above equation we get

$$F^K(X) + (-)^{K+1}F(X) = 0.$$

REFERENCES

1. A. Bejancu, *CR submanifolds of a Kaehler manifold*, 1, Proc. Amer. Math. Soc. **69** (1978), 135–142.

2. D.E. Blair and B.Y. Chen, *On CR-submanifolds of Hermitian manifolds*, Israel J. Math. **34** (1979), 353–363.

3. B.Y. Chen, *Geometry of submanifolds*, Marcel Dekker, New York, 1973.

4. Lovejoy S. Das, *Submanifolds of F-structure satisfying $F^K + (-)^{K+1}F = 0$*, Internat. J. Math. Math. Sci. **26** (2001), 167–172.

5. S.I. Goldberg, *On the existence of manifolds with an f-structure*, Tensor, N.S. **26** (1972), 323–329.

6. J.B. Kim, *Notes on f-manifolds*, Tensor, N.S. **3** (1975), 299–302.

7. K. Yano, *On a structure defined by a tensor field of type* $(1,1)$ *satisfying* $f^3 + f = 0$, Tensor, N.S. **14** (1963), 99–109.

8. K. Yano and S. Ishihara, *On integrability of a structure F satisfying* $f^3+f=0$, Quart. J. Math., Oxford, **25** (1964), 217–222.

9. K. Yano and Masahiro Kon, *Differential geometry of CR-submanifolds*, Geom. Dedicata **10** (1981), 369–391.

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