ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 36, Number 4, 2006

NONEXISTENCE OF POSITIVE SOLUTIONS TO A QUASI-LINEAR ELLIPTIC EQUATION AND BLOW-UP ESTIMATES FOR A NONLINEAR HEAT EQUATION

YANG ZUODONG

ABSTRACT. In this paper we prove blow-up estimates for a class of quasi-linear heat equations (non-Newtonian filtration equations). These estimates extend results for semi-linear heat equations (Newtonian filtration equations). Our method of proof is to first establish a nonexistence result for quasi-linear elliptic equations and then established to blow-up estimates for a class of quasi-linear heat equations.

1. Introduction. The purpose of this paper is to derive a bound for the rate of blow-up of solutions to the quasi-linear heat equation

$$u_t = \operatorname{div}(|\bigtriangledown u|^{p-2} \bigtriangledown u) + f(u), \tag{1}$$

where $u \geq 0$, $p \geq 2$. Throughout this paper we assume that $f \in C[0,\infty)$ is positive and nondecreasing on $(0,\infty)$. This problem appears in the study of non-Newtonian fluids [**1, 8**] and in nonlinear filtration theory [**2**]. In the non-Newtonian fluids theory, the quantity p is a characteristic of the medium. Media with $p > 2$ are called dilatant fluids and those with $p < 2$ are called pseudo-plastics. If $p = 2$, they are Newtonian fluids.

The blow-up rate estimates of positive radial solutions were established by Weissler in [**13**] for the (1) with $p = 2$, $f(u) = u^m$ $(m > 1)$, and Yang and Lu in [**16**] for the (1) with $p \geq 2$, $f(u) = u^m$ $(m > p-1)$. In this paper we get the same result for the (1) with $p \geq 2$. Then we extend and complement the results in [**13, 16**].

2000 AMS *Mathematics Subject Classification.* Primary 35K05, 35K60.
Key words and phrases. Blow up estimate, non-existence, quasi-linear heat equation.
The project was supported by the National Natural Science Foundation of China (No. 10571022); the Science Foundation of Nanjing Normal University (No. 2003SXXXGQ2B37) and the Natural Science Foundation of Jiangsu Province Educational Dept. (No. 04KJB110062).
Received by the editors on August 1, 2003, and in revised form on Jan. 13, 2004.

This paper is arranged as follows. In Section 2 some sufficient conditions for the nonexistence of positive solutions of the elliptic equation (steady state equation of the (1)) in $\mathbf{R}^{\mathbf{N}}$ are given. By using this nonexistence result, the blow-up estimates for equation (1) are obtained in Section 3.

2. Nonexistence for the steady equation of (1). We first consider quasi-linear elliptic inequalities of the form

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) \geq q(x)f(u), \quad x \in \mathbf{R}^{\mathbf{N}}\ (N \geq 2), \tag{2}$$

where $p > 1$, $\nabla u = (\nabla_1 u, \dots, \nabla_N u)$, $q(x) : \mathbf{R}^{\mathbf{N}} \to (0,\infty)$ and $f : (0,\infty) \to (0,\infty)$ are continuous functions. A positive entire solution of the inequality (2) is defined to be a positive function $u \in C^1(\mathbf{R}^{\mathbf{N}})$ satisfying (2) at every point of $\mathbf{R}^{\mathbf{N}}$.

Define $q_1, m \in C[0,\infty)$ to be the functions satisfying

$$0 < q_1(r) \leq \min_{|x|=r} q(x),$$
$$0 < m(r) \leq \min_{r/2 \leq |x| \leq 3r/2} q(x) \quad \text{for} \quad r \geq 0.$$

Throughout this section we make the following assumptions without further mention.

(H_1) $f : (0,\infty) \to (0,\infty)$ is locally Lipschitz continuous and strictly increasing.

(H_2) f is super-linear in the sense that

$$\int_1^\infty \left(\int_0^u f(s)\,ds\right)^{-1/p} du < \infty \quad \text{and} \quad \int_{0+}^1 \left(\int_0^u f(s)\,ds\right)^{-1/p} du = \infty.$$

An important special case of (2) satisfying the above hypotheses is the inequality

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) \geq q(x)u^\sigma, \quad x \in \mathbf{R}^{\mathbf{N}}\ (N \geq 2),$$

where $\sigma > p-1$.

Under our conditions we find that the function

$$G(s) = \int_s^\infty \left(\int_s^u f(\xi)\, d\xi \right)^{-1/p} du, \quad s > 0,$$

is well-defined in $(0,\infty)$. It is not hard to see that G is strictly decreasing, $G(0) = +\infty$ and $G(+\infty) = 0$. Therefore, its inverse function $G^{-1} : (0,\infty) \to (0,\infty)$ exists. We use H for G^{-1} below. Note that H is also strictly decreasing, $H(0) = +\infty$ and $H(+\infty) = 0$. If $f(u) = u^\sigma$, $\sigma > p-1$, then a simple computation gives

$$H(s) = C(\sigma) s^{-p/(\sigma-(p-1))}, \quad \text{for} \quad s > 0,$$

where $C(\sigma) > 0$ is a constant.

From reference [**5, 7**], we give the following lemma.

Lemma 2.1 (Weak comparison principle). *Let Ω be a bounded domain in $\mathbf{R}^\mathbf{N}$ $(N \geq 2)$ with smooth boundary $\partial\Omega$ and $\theta : (0,\infty) \to (0,\infty)$ is continuous and nondecreasing. Let $u_1, u_2 \in W^{1,p}(\Omega)$ satisfy*

$$\int_\Omega |\nabla u_1|^{p-2} \nabla u_1 \nabla \psi\, dx + \int_\Omega \theta(u_1)\psi\, dx \leq \int_\Omega |\nabla u_2|^{p-2} \nabla u_2 \nabla \psi\, dx + \int_\Omega \theta(u_2)\psi\, dx$$

for all nonnegative $\psi \in W_0^{1,p}(\Omega)$. Then the inequality

$$u_1 \leq u_2 \quad \text{on} \quad \partial\Omega$$

implies that

$$u_1 \leq u_2 \quad \text{in} \quad \Omega.$$

Lemma 2.2 *Let $x^0 \in \mathbf{R}^\mathbf{N}$ and $k, R > 0$. If a positive C^1-function u satisfies*

$$\operatorname{div}(|\nabla u|^{p-2} \nabla u) \geq k f(u), \quad |x - x^0| \leq R,$$

then

$$u(x^0) \leq H((pk/(p-1))^{1/p} R).$$

Proof. If we can construct a positive C^1-function u with properties

$$\operatorname{div}(|\nabla v|^{p-2}\nabla v) = kf(v), \quad |x-x^0| \le R,$$

and $v \to \infty$ as $|x-x^0| \to R$, then Lemma 2.1 implies that $u(x) \le v(x)$, $|x-x^0| < R$ (especially $u(x^0) \le v(x^0)$). By the argument as in Lemma 2.3 of [**7**], there is a positive C^1-function $v(r)$, $r = |x-x^0|$, satisfying

$$(\phi_p(v'))' + \frac{N-1}{r}\phi_p(v') = kf(v(r)), \quad 0 \le r < R, \tag{3}$$

$$v'(0) = 0, \quad v(r) \longrightarrow \infty \quad \text{as} \quad r \to R. \tag{4}$$

where $\phi_p(v) = |v|^{p-2}v$. From (3), we obtain

$$(\phi_p(v'))'v' \le (\phi_p(v'))'v' + \frac{N-1}{r}\phi_p(v')v' = kf(v)v',$$

and

$$\int_0^r (\phi_p(v'))'v'(s)\,ds \le k\int_0^r f(v)v'\,ds.$$

Then

$$\frac{v'}{\sqrt[p]{F(v(r)) - F(v(0))}} \le \left(\frac{pk}{p-1}\right)^{1/p},$$

it follows that

$$\begin{aligned} G(v(0)) &= \int_{v(0)}^{\infty} (F(z) - F(v(0)))^{-1/p}\,dz \\ &= \int_0^R (F(v(r)) - F(v(0)))^{-1/p} v'\,dr \\ &\le \left(\frac{pk}{p-1}\right)^{1/p} R, \\ &\Longrightarrow\ v(0) < H\left(\left(\frac{pk}{p-1}\right)^{1/p} R\right). \end{aligned}$$

Thus, we conclude that

$$u(x^0) \le v(0) < H\left(\left(\frac{pk}{p-1}\right)^{1/p} R\right).$$

This completes the proof. □

Theorem 2.3. *Let $p > 1$. If*

$$\liminf_{r\to\infty} \frac{H(r(pm(r)/(p-1))^{1/p})}{\int_0^r (\int_0^r (s/t)^{N-1} q_1(s)\,ds)^{1/(p-1)}\,dt} = 0, \tag{5}$$

then inequality (2) *has no positive entire solutions.*

Proof. Suppose to the contrary that there exists a positive entire solution u of (2). First, we see that u satisfies

$$0 < u(x) \leq H((m(|x|)p/(p-1))^{1/p}|x|/2), \quad x \neq 0. \tag{6}$$

In fact, let $x^0 \neq 0$ and $|x^0| = r$. Then in view of the definition of $m(r), u$ satisfies

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) \geq m(r)f(u), \quad |x - x^0| \leq r/2.$$

Hence Lemma 2.2 gives

$$u(x^0) \leq H((pm(r)/(p-1))^{1/p} r/2),$$

which is equivalent to (6).

Next, let $r_0 > 0$ be fixed arbitrarily and then choose a sufficiently small number $\delta > 0$ so that $f(\max_{|x|=r_0} u) \geq \delta > 0$. Define $v(r)$ by

$$v(r) = 1/\delta \int_{r_0}^r \phi_p^{-1}\Big(\delta/2 \int_{r_0}^s (t/s)^{N-1} q_1(t)\,dt\Big)\,ds, \quad r \geq r_0. \tag{7}$$

Then it is easily seen that

$$\begin{gathered} v(r_0) = v'(r_0) = 0, \\ v(r) > 0, \quad v'(r) > 0, \quad r > r_0, \\ \operatorname{div}(|\nabla(\delta v)|^{p-2}\delta\nabla v(|x|)) = \delta/2 q_1(|x|) < \delta q_1(|x|), \quad |x| \geq r_0, \end{gathered} \tag{8}$$

and

$$\begin{aligned} v(r) &\geq \delta^{(2-p)/(p-1)} \int_{r_0}^r \Big(\int_{r_0}^r (t/s)^{N-1} q_1(t)\,dt\Big)^{1/(p-1)}\,ds \\ &= \delta^{(2-p)/(p-1)}\theta(r), \quad r \geq r_0. \end{aligned} \tag{9}$$

Now, we consider the function $\omega(x) = u(x) - \delta v(|x|)$, $|x| \geq r_0$. Since $\omega > 0$ on $|x| = r_0$, from (5),(6) and (9) we see that

$$\lim_{|x|\to\infty} \inf \omega(x) = \lim_{|x|\to\infty} \inf v(|x|)(u(x)/v(|x|) - \delta) < 0,$$

(since $\lim_{|x|\to\infty}((u(x)/\theta(|x|)) - \delta^{1/(p-1)}) < 0$ by the assumption of this theorem) and so ω becomes negative on some sphere $|x| = r_1 > r_0$, sufficiently large. Hence ω takes a maximum for region $r_0 \leq |x| < r_1$, at some point $\tilde{x}$ which belongs to $r_0 < |x| < r_1$. In fact, suppose to the contrary that $|\tilde{x}| = r_0$. Then $u(\tilde{x}) = \max_{|x|=r_0} u(x)$, because $v(|x|)$ is radial. Moreover, we shall conclude that $\tilde{x}$ is also the maximum point of u in $B_{r_0} = \{x;\ |x| < r_0\}$. In fact, we know that u has no maximum point in B_{r_0} unless $u \equiv$ constant (this implies that u can only attain its maximum on $|x| = r_0$). Suppose not, if there exists $\hat{x} \in B_{r_0}$ at which u attains its maximum $u(\hat{x}) = \beta$, then $\nabla u(\hat{x}) = 0$. On the other hand, choose a small ball $B \subset\subset B_{r_0}$ such that $\hat{x} \in \partial B$, let $\omega(x) = \beta - u$, then $\omega > 0$ in B and $\omega = 0$ at $\hat{x}$. Now, $-\mathrm{div}\,(|\nabla\omega|^{p-2}\nabla\omega) = \mathrm{div}\,(|\nabla u|^{p-2}\nabla u) > 0$ in B, so Lemma 2.2 of [**5**] implies that $\nabla u(\hat{x}) \neq 0$. This contradicts the definition of $\hat{x}$. Therefore, $u(\hat{x}) = \max_{\overline{B}_{r_0}} u$. Now, choose a small ball $B_1 \subset B_{r_0}$ such that $\tilde{x} \in \partial B_1$ and $u(\tilde{x}) - u > 0$ for $x \in B_1$. Then $\omega_1 = u(\tilde{x}) - u$ has the same properties of the ω above. Lemma 2.2 of [**5**] implies that $(\partial\omega_1/\partial n)(\tilde{x}) < 0$. Thus, $(\partial\omega/\partial n)(\tilde{x}) = (\partial u/\partial n)(\tilde{x}) > 0$, where n is the outward normal vector to $|x| = r_0$, ω becomes greater than $\omega(\tilde{x})$ at some x. This contradiction shows that $r_0 < |\tilde{x}| < r_1$, as stated above, thus $\nabla u(\tilde{x}) = 0$. On the other hand, we also conclude that $\nabla u(\tilde{x}) \neq 0$. Otherwise, we have that $\nabla v(\tilde{x}) = 0$ and thus $v_r'(|\tilde{x}|) = 0$. But we see that it is impossible from (7) and $|\tilde{x}| > r_0$. This contradiction proves our theorem. □

Remark 1. When $p = 2$, the related results have been obtained by [**11**]. Our theorem for nonexistence extends the results of [**11**].

Corollary 2.4. *Let* $N \geq p + 1$. *If*

$$\lim_{|x|\to\infty} \inf |x|^p q(x) > 0, \tag{10}$$

then inequality (2) *has no positive entire solutions.*

Proof. Put

$$q_1(r) = C\,(r+1)^{-p}, \quad r \ge 0$$

where $C > 0$ is a constant. Because of (10), $C > 0$ can be chosen so that $q_1 \le \min_{|x|=r} q(x)$. Since

$$\int_0^r \left(\int_0^r (s/t)^{N-1} q_1(s)\,ds \right)^{1/(p-1)} dt > C_1 > 0 \quad \text{for} \quad r \ge 1,$$

condition (5) is satisfied. The conclusion then follows immediately from Theorem 2.3.

Corollary 2.5. *Let* $N \ge p+1$. *Consider the elliptic equation*

$$\operatorname{div}(|\nabla u|^{p-2}\,\nabla u) = q(x)f(u), \quad x \in \mathbf{R^N} \tag{11}$$

where q *is positive and continuous in* $\mathbf{R^N}$ *and* f *satisfies conditions* $(H_1), (H_2)$. *Corollary* 2.4 *implies that if*

$$\lim_{|x|\to\infty} \inf |x|^p q(x) > 0,$$

then equation (11) *has no positive entire solutions.*

Corollary 2.6. *Let* $N \ge p+1$. *Consider the elliptic equation*

$$\operatorname{div}(|\nabla u|^{p-2}\,\nabla u) = q(x)u^{\sigma}, \quad x \in \mathbf{R^N} \tag{12}$$

where $\sigma > p-1$ *and* $q(x)$ *are continuous in* $\mathbf{R^N}$. *If* $q(x) > 0$ *in* $\mathbf{R^N}$ *and*

$$\lim_{|x|\to\infty} \inf |x|^p q(x) > 0,$$

then equation (12) *has no positive entire solutions.*

Theorem 2.7. *Let* $m > p-1$ *and* $N \ge 1$, *and suppose* $N/p < (m+1)/(m-p+1)$. *Then there does not exist a positive* C^1 *function* $v(r) : [0,\infty) \to \mathbf{R}$ *with* $v'(0) = 0$ *and*

$$(|v'|^{p-2}\,v')' + \frac{N-1}{r}\,|v'|^{p-2}\,v' + v^m(r) = 0, \quad r > 0. \tag{13}$$

Proof. Suppose there exists such a function v. Then

$$(r^{N-1}\,\phi_p(v'))' + r^{N-1}v^m(r) = 0,$$

and

$$r^{n-1}\,\phi_p(v')(r) = -\int_0^r s^{n-1}v^m(s)\,ds, \tag{14}$$

where $\phi_p(v) = |v|^{p-2}v$. We first dispense with the case $N \le p$. Using (14), we see that if $r \ge 1$, then $v'(r) \le -C^{1/(p-1)}r^{(1-N)/(p-1)}$ for some $C > 0$. Integrating, we get

$$v(r) \le v(1) + C^{1/(p-1)}(p-1)/(N-p)(r^{(p-N)/(p-1)} - 1),$$

and so $v(r) \to -\infty$ as $r \to \infty$. This contradicts $v(r) > 0$ and proves the lemma for $N \le p$.

Now suppose $N > p$. Formula (14) implies that $v(r)$ is decreasing and therefore that

$$-r^{N-1}\,\phi_p(v') = \int_0^r s^{N-1}\,v^m(s)\,ds \ge r^N\,v^m(r)/N,$$

or $v'(r) \le -(1/N)^{1/(p-1)}r^{1/(p-1)}v^{m/(p-1)}(r)$. This inequality is easily integrated to give

$$v^{(m-p+1)/(p-1)} \le p/(m-p+1)\,N^{1/(p-1)}\,r^{-p/(p-1)}.$$

In particular,

$$\lim_{r\to+\infty}\sup r^{p/(m-p+1)}\,v(r) < +\infty. \tag{15}$$

At this point we use the hypothesis that $N/p < (m+1)/(m-p+1)$. This, along with (15), implies that

$$\int_0^{+\infty} r^{N-1}\,v^{m+1}(r)\,dr < +\infty. \tag{16}$$

We multiply (13) by $r^{N-1}v(r)$ and use the identity

$$(r^{N-1}\,\phi_p(v')v)' = (N-1)r^{N-2}\,\phi_p(v')v + r^{N-1}(\phi_p(v'))'v + r^{N-1}|v'|^p.$$

This gives

$$(r^{N-1}\,\phi_p(v')v)' - r^{N-1}|v'|^p + r^{N-1}\,v^{m+1} = 0.$$

Integrating from 0 to r we get

$$-r^{N-1}\phi_p(v')v(r) + \int_0^r s^{N-1}|v'(s)|^p\,ds = \int_0^r s^{N-1}v^{m+1}(s)\,ds. \tag{17}$$

Since $v(r) > 0$ and $v'(r) < 0$, formulas (16) and (17) imply

$$\int_0^{+\infty} s^{N-1}|v'|^p\,ds \le \int_0^{+\infty} s^{N-1}v^{m+1}(s)\,ds < +\infty. \tag{18}$$

We multiply (13) by $r^N v'(r)$ and use the identities

$$\begin{gathered}(r^N|v'|^p)' = Nr^{N-1}|v'|^p + pr^N|v'|^{p-1}v'',\\ (r^N v^{m+p-1})' = Nr^{N-1}v^{m+p-1} + (m+p-1)r^N v^{m+p-2}v'.\end{gathered}$$

This gives

$$\begin{aligned}\frac{d}{dr}\left(r^N|v'|^p/p + \frac{r^N v^{m+p-1}}{m+p-1}\right) &= \frac{N}{(m+p-1)}\,r^{N-1}v^{m+p-1}\\ &\quad+ \frac{N}{p}\,r^{N-1}|v'|^p + r^N v^{m+p-2}v'\\ &\quad+ \frac{1}{p-1}\,(-(N-1)r^{N-1}|v'|^p - r^N v^m v').\end{aligned}$$

Integrating from 0 to x we get

$$\begin{aligned}\frac{r^N|v'|^p}{p} + \frac{r^N v^{m+p-1}(r)}{m+p-1} &= \frac{N}{(m+p-1)}\int_0^r s^{N-1}v^{m+p-1}\,ds\\ &\quad+ \left(\frac{N}{p} - \frac{N-1}{p-1}\right)\int_0^s s^{N-1}|v'|^p\,ds\\ &\quad+ \int_0^r s^N v^{m+p-2}v'\,ds - \frac{1}{p-1}\int_0^r s^N v^m v'\,ds,\end{aligned}$$

then

$$\begin{aligned}&\frac{r^N|v'|^p}{p} + \frac{r^N}{(p-1)(m+1)}\,v^{m+1}(r)\\ &= \left(\frac{N}{p} - \frac{N-1}{p-1}\right)\int_0^r s^{p-1}|v'|^p\,ds + \frac{N}{(p-1)(m+1)}\int_0^r s^{N-1}v^{m+1}(s)\,ds.\end{aligned} \tag{19}$$

Let $h(r) = r^N|v'|^p/p + ((r^N)/((p-1)(m+1)))v^{m+1}(r)$. By (18)and (19) we see that $\lim_{x\to\infty} h(x) = l$ exists. Furthermore, again by virtue of (18), we have that $\int_0^\infty t^{-1}h(t)\,ds < +\infty$; and so $l = 0$. Thus, letting $r \to +\infty$ in (18), yields

$$\frac{N}{(p-1)(m+1)}\int_0^{+\infty} s^{N-1}v^{m+1}(s)\,ds = \left(\frac{N-1}{p-1} - \frac{N}{p}\right)\int_0^{+\infty} s^{N-1}|v'|^p\,ds.$$

Finally, (16) and (18) together imply

$$N/p \geq \frac{m+1}{m-p+1}.$$

This contradicts the hypothesis that $N/p < (m+1)/(m-p+1)$ and thereby proves the theorem. □

3. Blow-up estimates for the equation (1). Motivated by Weissler [**13**] and Yang and Lu [**16**], we use the nonexistence result of the elliptic equation obtained in Section 2 to establish the blow-up estimates for equation (1).

Let $B(\rho)$ denote the open ball in $\mathbf{R^N}(N \geq p,\ p \geq 2)$ of radius ρ, center at 0. Also, for $T > 0$, let $\Gamma = \Gamma(\rho, T) = B(\rho) \times (0, T) \subset \mathbf{R^{N+1}}$. A typical point in Γ is denoted by (x, t), with $x \in B(\rho)$ and $t \in (0, T)$.

Theorem 3.1. *Suppose for* $\rho > 0$ *and* $T > 0$ *the function* $u : \Gamma(\rho, T) \to \mathbf{R}$ *satisfies*:

(a) $u \in C^1(\Gamma)$ *and* u *has continuous second order* x*-derivatives throughout* Γ;

(b) $u \geq 0$ *and* $u_t \geq 0$ *in* Γ;

(c) *for each* $t \in (0, T)$, $u(\cdot, t)$ *is radially symmetric and non-increasing as a function of* $r = |x|$;

(d) *for each* $t \in (0, T)$, $u_t(\cdot, t)$ *achieves its maximum at* $x = 0$;

(e) u *satisfies* (1) *throughout* Γ;

(f) $u(0, t) \to \infty$ *as* $t \to T$.

(g) *there are constants* $\beta > 0$ *and* $m > p-1$ $(p \geq 2)$ *such that*

$$s^{-m} f(s) \to \beta \quad as \quad s \to \infty.$$

Then there exists a constant $C > 0$ *such that*

$$u(x,t) \leq C_1 (T-t)^{-1/(m-1)} \tag{19}$$

for all $(x,t) \in \Gamma$.

Proof. We consider equation (1). For $0 < t < T$, let $\alpha(t) = u(0,t)^{(m-p+1)/p}$; then $\alpha(t) \to \infty$ as $t \to T$. For $t \in (0,T)$ and $y \in B(\rho\alpha(t))$, let

$$v(y,t) = \frac{u(y/\alpha(t),t)}{u(0,t)}.$$

Since $0 \leq u(x,t) \leq u(0,t)$, it follows that

$$0 \leq v(y,t) \leq 1. \tag{20}$$

Furthermore, a routine calculation shows that

$$\operatorname{div}(|\nabla v|^{p-2}\, \nabla v) = \frac{[u_t(y/\alpha(t),t) - f(u(y/\alpha(t),t))]}{u^m(0,t)}.$$

Hypotheses (b) and (d) therefore imply that

$$0 \leq \operatorname{div}(|\nabla v|^{p-2}\, \nabla v) + \frac{f(v(y,t)u(0,t))}{u^m(0,t)} \leq \frac{u_t(0,t)}{u^m(0,t)}. \tag{21}$$

Since $u(\cdot,t)$ is radially symmetric, the same is true for $v(\cdot,t)$; and thus we may set

$$v(y,t) = w(r,t),$$

where $|y| = r$ and $0 \leq r < \rho\alpha(t)$. Note that for each $t \in (0,T)$, $w(\cdot,t)$ is a C^1 function on $[0, \rho\alpha(t)]$ with $w(0,t) = 1$ and $w_r(0,t) = 0$. Rewriting (20) and (21) in terms of w, we get

$$0 \leq w(r,t) \leq 1, \tag{22}$$

(23)

$$0 \leq (\Phi_p(w_r))_r + (N-1)/r\, \Phi_p(w_r) + \frac{f(w(r,t)u(0,t))}{u^m(0,t)} \leq \frac{u_t(0,t)}{u^m(0,t)},$$

where $\Phi_p(w) = |w|^{p-2}w$ and w_r denote the derivative of w with respect to r. Furthermore, $w_r \le 0$ by hypothesis (c), and so (23) implies

$$(\Phi_p(w_r))_r w_r + (N-1)/r\,|w_r|^p + \frac{f(w(r,t)u(0,t))}{u^m(0,t)}\,w_r \le 0,$$

which in turn says that

$$\frac{\partial}{\partial r}\,((p-1)/p\,|w_r|^p) + \frac{f(w(r,t)u(0,t))}{u^m(0,t)}\,w_r \le -(N-1)/r\,|w_r|^p \le 0.$$

Integrating this last inequality from 0 to r shows that

$$\frac{(p-1)}{p}\,|w_r|^p + \int_0^r \frac{f(w(r,t)u(0,t))}{u^m(0,t)}\,w_r\,dr \le 0,$$

and thus

$$\frac{(p-1)}{p}\,|w_r(r,t)|^p \le \frac{1}{u^{m+1}(0,t)}\int_{w(r,t)u(0,t)}^{u(0,t)} f(z)\,dz.$$

From $\lim_{t\to T} u(0,t) = +\infty$ and (g) we see that there exists an $\varepsilon > 0$, for $t \in (T-\varepsilon, T)$, $\rho \in [w(r,t)u(0,t), u(0,t))$ such that $f(\rho) \le c_1\rho^m$. Then

$$\begin{aligned}
&\frac{1}{u^{m+1}(0,t)}\int_{w(r,t)u(0,t)}^{u(0,t)} f(\rho)\,d\rho\\
&\qquad\le \frac{c_1}{u^{m+1}(0,t)}\int_{w(r,t)u(0,t)}^{u(0,t)} \rho^m\,d\rho\\
&\qquad\le \frac{c_1}{(m+1)u^{m+1}(0,t)}\,(u^{m+1}(0,t) - w^{m+1}(r,t)u^{m+1}(0,t))\\
&\qquad= \frac{c_1}{m+1}\,(1 - w^{m+1}(r,t)) \le \frac{c_1}{m+1}.
\end{aligned}$$

For $t \in [0, T-\varepsilon]$, we have $|f(u(0,t)w(r,t))| \le M$, which implies that

$$\begin{aligned}
\Big|\frac{1}{u^{m+1}(0,t)}\int_{w(r,t)u(0,t)}^{u(0,t)} f(\rho)\,d\rho\Big| &\le \frac{M}{u^m(0,t)}\,(1 - w(r,t))\\
&\le \frac{M}{u^m(0,t)} \le M_1,
\end{aligned}$$

and thus

$$|w_r(r,t)| \le c_2, \tag{24}$$

for $t \in [0,T)$. We now claim that

$$\lim_{t\to T}\inf \frac{u_t(0,t)}{u^m(0,t)} > 0. \tag{25}$$

We proceed by contradiction as in [**13, 16**]. Suppose t_n is a sequence in $(0,T)$ with $t_n \to T$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{u_t(0,t_n)}{u^m(0,t_n)} = 0. \tag{26}$$

By using the Ascoli-Alzela theorem, we know that there is a subsequence, which we still call t_n, and a function $\overline{w} \in C([0,\infty))$ such that $w(\cdot,t_n) \to \overline{w}$ uniformly on compact subsets of $[0,\infty)$. In particular, because of the properties of each $w(\cdot,t_n)$, we know that $\overline{w} \ge 0$, $\overline{w}(0) = 1$, and $\overline{w}$ is nonincreasing on $[0,\infty)$. Moreover, formula (24) implies that each $w(\cdot,t_n)$ is Lipschitz with a Lipschitz constant of c_2. The same is therefore true of $\overline{w}$, and so $\overline{w}$ is absolutely continuous on $[0,\infty)$. Next we consider $w(\cdot,t_n)$ and $\overline{w}$ as distributions on $(0,\infty)$. (Let $w(r,t_n) = 0$ for $r \ge \rho\alpha(t_n)$.) Clearly, $w(\cdot,t_n) \to \overline{w}$ in the sense of distributions; and hence

$$w_r(\cdot,t_n) \longrightarrow \overline{w}_r, \quad (\Phi_p(w_r))_r(\cdot,t_n) \longrightarrow (\Phi_p(\overline{w}_r))_r,$$

in the sense of distributions. Thus, formulas (23) and (26) imply that

$$(\Phi_p(\overline{w}_r))_r + (N-1)/r\ \Phi_p(\overline{w}_r) + \beta\,\overline{w}^m = 0, \tag{27}$$

as distributions on $(0,\infty)$. This can be rewritten as

$$(r^{N-1}\Phi_p(\overline{w}_r))_r + r^{N-1}\,\beta\,\overline{w}^m = 0. \tag{28}$$

Since $\overline{w}$ is absolutely continuous, it follows immediately from (28) that $\overline{w}$ is C^1 on $(0,\infty)$. In particular, since $\overline{w} \ge 0$, the local existence and uniqueness of C^1 solutions of (28) on $(0,\infty)$ guarantees that $\overline{w} > 0$ on $(0,\infty)$.

If $N=2$, $p>2$, we proceed as follows. From equation (28), we infer that $r\Phi_p(\overline{w}_r)$ are decreasing and that there exist $M<0$ and $r_0>0$ such that

$$r\,\Phi_p(\overline{w}_r)<M \quad \text{for} \quad r\in(r_0,+\infty).$$

The last inequality implies that

$$\begin{aligned}\overline{w}(s)>\overline{w}(s)-\overline{w}(t)&=(-M)^{1/(p-1)}\int_s^t r^{-1/(p-1)}\,dr\\ &=(-M)^{1/(p-1)}\big(t^{(p-2)/(p-1)}-s^{(p-2)/(p-1)}\big)\end{aligned}\tag{29}$$

for $r_0\le s\le t$. Letting $t\to+\infty$ in (23), we obtain a contradiction.

If $N=2, p=2$, a similar argument to the one above shows that

$$\overline{w}(s)>\overline{w}(s)-\overline{w}(t)>(-M)\,[\ln(t)-\ln(s)]$$

for $r_0\le s\le t$. Letting $t\to+\infty$ in the last inequality, we obtain a contradiction.

In the case $N>p$, it follows from Theorem 2.7 (or from Theorem 3.2 of [**17**]) that equation (28) has no positive solution. It may be concluded that equation (20) also cannot hold. Hence, there exist a $c>0$ such that, for all $t\in(0,T)$ close enough to T,

$$\frac{u_t(0,t)}{u^m(0,t)}\ge c>0.$$

This can be rewritten as

$$(u^{1-m}(0,t))_t\;\le-\,(m-1)\,c.\tag{30}$$

Since $\lim_{t\to T}u^{1-m}(0,t)=0$, integrating (30) from t to T yields

$$u^{1-m}\ge c_1(T-t)\tag{31}$$

for t close to T. Finally, hypotheses (b) and (c) in the Theorem 3.1, along with formula (31), show that

$$u(x,t)\le C_1(T-t)^{-1/(m-1)}$$

for all $(x,t)\in\Gamma$. This completes the proof of the theorem. □

Finally, we give lower bounds for the blow-up rates.

Theorem 3.2. *Assume that the conditions* (a)–(g) *in Theorem* 3.1 *hold. Then there are positive constants* C_2, δ *such that*

$$u(0,t) \geq C_2(T-t)^{-1/(m-1)}$$

for $t \in (\delta, T)$.

Proof. From (1) and condition (c), we get

$$(p-1)(-u')^{p-2}\, u'' + (N-1)/r\, |u'|^{p-2}\, u' + f(u) = u_t. \tag{32}$$

Since $u'' \leq 0$ at $r = 0$ with $t \in (0,T)$, we see from (32) and (g) of Theorem 3.1 that

$$u_t(0,t) \leq f(u(0,t)) \leq c_1 + c_2\, u^m(0,t),$$

hence for $t \in (\delta, s) \subset (\delta, T)$, we have

$$\frac{u_t(0,t)}{u^m(0,t)} \leq \frac{f(u(0,t))}{u^m(0,t)} \leq c_2 + \frac{c_1}{u^m(0,t)} \leq c_3. \tag{32}$$

Integrating (32) over $(t,s) \subset (\delta, T)$ and letting $s \to T$, we get by condition (f):

$$u(0,t) \geq C_2(T-t)^{-1/(m-1)}. \qquad \square$$

Remark 2. Combining Theorem 3.1 and Theorem 3.2, we conclude that the blow-up rates of radial positive solutions of (1) under the conditions of the theorems are

$$u(0,t) = O((T-t)^{-1/(m-1)}),$$

as t tends to T.

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School of Mathematics and Computer Science, Nanjing Normal University, Jiangsu Nanjing, 210097, P.R. China
E-mail address: zdyang_jin@263.net