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ACCUMULATION POINTS OF THE BOUNDARY OF A CAT(0) SPACE ON WHICH A GROUP ACTS GEOMETRICALLY

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ABSTRACT. In this paper, using a result of Ontaneda, we show that there is no isolated point in the boundary of a CAT(0) space on which a group acts geometrically, i.e., properly and cocompactly by isometries, if the cardinal number of the boundary is greater than two.

1. Introduction and preliminaries. The purpose of this paper is to study boundaries of CAT(0) groups, i.e., the boundary of a CAT(0) space on which a group acts geometrically.

We say that a metric space (X, d) is a *geodesic space* if, for each $x, y \in X$, there exists an isometry $\xi : [0, d(x, y)] \to X$ such that $\xi(0) = x$ and $\xi(d(x, y)) = y$ (such a ξ is called a *geodesic*). Also a metric space (X, d) is said to be *proper* if every closed metric ball is compact.

Let (X, d) be a geodesic space, and let T be a geodesic triangle in X. A *comparison triangle* for T is a geodesic triangle $\overline{T}$ in the Euclidean plane $\mathbf{R}^2$ with same edge lengths as T. Choose two points x and y in T. Let $\bar{x}$ and $\bar{y}$ denote the corresponding points in $\overline{T}$. Then the inequality

$$d(x, y) \leq d_{\mathbf{R}^2}(\bar{x}, \bar{y})$$

is called the CAT(0)-*inequality*, where $d_{\mathbf{R}^2}$ is the natural metric on $\mathbf{R}^2$. A geodesic space (X, d) is called a CAT(0) *space* if the CAT(0)-inequality holds for all geodesic triangles T and for all choices of two points x and y in T.

Let (X, d) be a proper CAT(0) space and $x_0 \in X$. The *boundary of X with respect to* x_0, denoted by $\partial_{x_0} X$, is defined as the set of all

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geodesic rays issuing from x_0. Then we define a topology on $X \cup \partial_{x_0} X$ by the following conditions:

(1) X is an open subspace of $X \cup \partial_{x_0} X$.

(2) For $\alpha \in \partial_{x_0} X$ and r, $\varepsilon > 0$, let

$$U_{x_0}(\alpha; r, \varepsilon) = \{x \in X \cup \partial_{x_0} X \mid x \notin B(x_0, r),\ d(\alpha(r), \xi_x(r)) < \varepsilon\},$$

where $\xi_x : [0, d(x_0, x)] \to X$ is the geodesic from x_0 to x ($\xi_x = x$ if $x \in \partial_{x_0} X$). Then, for each $\varepsilon_0 > 0$, the set

$$\{U_{x_0}(\alpha; r, \varepsilon_0) \mid r > 0\}$$

is a neighborhood basis for α in $X \cup \partial_{x_0} X$.

This is called the *cone topology* on $X \cup \partial_{x_0} X$. It is known that $X \cup \partial_{x_0} X$ is a metrizable compactification of X, [**1, 2**].

Let (X, d) be a geodesic space. Two geodesic rays $\xi, \zeta : [0, \infty) \to X$ are said to be *asymptotic* if there exists a constant N such that $d(\xi(t), \zeta(t)) \le N$ for each $t \ge 0$. It is known that, for each geodesic ray ξ in X and each point $x \in X$, there exists a unique geodesic ray ξ' issuing from x such that ξ and ξ' are asymptotic.

Let x_0 and x_1 be two points of a proper CAT(0) space X. Then there exists a unique bijection $\Phi : \partial_{x_0} X \to \partial_{x_1} X$ such that ξ and $\Phi(\xi)$ are asymptotic for each $\xi \in \partial_{x_0} X$. It is known that $\Phi : \partial_{x_0} X \to \partial_{x_1} X$ is a homeomorphism [**1, 3**].

Let (X, d) be a proper CAT(0) space. The asymptotic relation is an equivalence relation in the set of all geodesic rays in X. The *boundary of* X, denoted by ∂X, is defined as the set of asymptotic equivalence classes of geodesic rays. The equivalence class of a geodesic ray ξ is denoted by $\xi(\infty)$. For each $x_0 \in X$ and each $\alpha \in \partial X$, there exists a unique element $\xi \in \partial_{x_0} X$ with $\xi(\infty) = \alpha$. Thus we may identify ∂X with $\partial_{x_0} X$ for each $x_0 \in X$.

Let (X, d) be a proper CAT(0) space and G a group which acts on X by isometries. For each element $g \in G$ and each geodesic ray $\xi : [0, \infty) \to X$, a map $g\xi : [0, \infty) \to X$ defined by $(g\xi)(t) := g(\xi(t))$ is also a geodesic ray. If geodesic rays ξ and ξ' are asymptotic, then $g\xi$ and $g\xi'$ are also asymptotic. Thus g induces a homeomorphism of ∂X and G acts on ∂X.

A *geometric* action on a CAT(0) space is an action by isometries which is proper [**1**, p. 131] and cocompact. We note that every CAT(0) space on which a group acts geometrically is a proper space [**1**, p. 132]. A group which acts geometrically on some CAT(0) space is called a CAT(0) *group*.

Details of CAT(0) spaces and their boundaries are found in [**1, 3**], and we can see several studies of boundaries of CAT(0) groups in [**2, 5, 6**]. In recent years, Ontaneda proved that a CAT(0) space on which a group acts geometrically is almost extendible [**7**]. In this paper, using this result, we show the following theorem.

Main theorem. *Suppose that a group G acts geometrically on a* CAT(0) *space X. If $|\partial X| > 2$, then each point of ∂X is an accumulation point.*

Here a point x of a topological space X is called an *accumulation point*, if $\overline{X \setminus \{x\}} = X$, where $\overline{X \setminus \{x\}}$ is the closure of $X \setminus \{x\}$ in X.

2. Lemmas on CAT(0) spaces and their boundaries. In this section, we give some lemmas about CAT(0) spaces and their boundaries needed later.

We first introduce some basic properties of CAT(0) spaces.

Lemma 2.1 (cf. [**1**, pp. 159–160, 176], [**3**]. *Let (X, d) be a proper* CAT(0) *space.*

(i) *For each two points $x, y \in X$, there exists a unique geodesic segment between x and y in X.*

(ii) *For each three points $x_0, x_1, x_2 \in X$ and each $t \in [0, 1]$,*

$$d\big(\xi_1(td(x_0, x_1)), \xi_2(td(x_0, x_2))\big) \leq td(x_1, x_2),$$

where $\xi_i : [0, d_i] \to X$ is the geodesic segment from x_0 to x_i for each $i = 1, 2$.

(iii) *If geodesic rays $\xi, \xi' : [0, \infty) \to X$ are asymptotic, then $d(\xi(t), \xi'(t)) \leq d(\xi(0), \xi'(0))$ for each $t \geq 0$.*

We obtain the following lemmas from some basic properties of CAT(0) spaces. Details of the proofs of the following two lemmas are found in [**4**]. We give outlines of the proofs here.

Lemma 2.2. *Let x_0, x_1, x_2 be three points of a* CAT(0) *space (X, d), let $d_i = d(x_0, x_i)$ for $i = 1, 2$, and let $\xi_2 : [0, d_2] \to X$ be the geodesic segment from x_0 to x_2. If $d_1 < d_2$, then*

$$d(x_1, \xi_2([d_1, d_2])) = d(x_1, \xi_2(d_1)).$$

Proof. We can show that $d(x_1, \xi_2(d_1)) \leq d(x_1, \xi_2(t))$ for each $t \in [d_1, d_2]$ by considering a comparison triangle $\overline{x_0}, \overline{x_1}, \overline{\xi_2(t)}$ in $\mathbf{R}^2$ for the geodesic triangle $x_0, x_1, \xi_2(t)$. $\square$

Lemma 2.3. *Let (X, d) be a proper* CAT(0) *space, let $x_0 \in X$ and let $\alpha \in \partial_{x_0} X$. For each $r > 0$ and $\varepsilon > 0$, we define a subset $U'_{x_0}(\alpha; r, \varepsilon)$ of $X \cup \partial_{x_0} X$ as*

$$U'_{x_0}(\alpha; r, \varepsilon) = \{x \in X \cup \partial_{x_0} X \mid x \notin B(x_0, r),\ d(\alpha(r), \operatorname{Im} \xi_x) < \varepsilon\},$$

where $\xi_x : [0, d(x_0, x)] \to X$ is the geodesic (*segment or ray*) *from x_0 to x. Then for each $\varepsilon_0 > 0$, $\{U'_{x_0}(\alpha; r, \varepsilon_0) \mid r > 0\}$ is a neighborhood basis for α in $X \cup \partial_{x_0} X$.*

Proof. For each $r > 0$ and $\varepsilon > 0$, the subset $U_{x_0}(\alpha; r, \varepsilon)$ is defined as

$$U_{x_0}(\alpha; r, \varepsilon) = \{x \in X \cup \partial_{x_0} X \mid x \notin B(x_0, r),\ d(\alpha(r), \xi_x(r)) < \varepsilon\},$$

and, for each $\varepsilon_0 > 0$, $\{U_{x_0}(\alpha; r, \varepsilon_0) \mid r > 0\}$ is a neighborhood basis for α in $X \cup \partial_{x_0} X$ by the definition of the cone topology.

It is clear that

$$U_{x_0}(\alpha; r, \varepsilon) \subset U'_{x_0}(\alpha; r, \varepsilon)$$

for each $r > 0$ and $\varepsilon > 0$. Using Lemma 2.2, we also can show that

$$U'_{x_0}(\alpha; r + \varepsilon, \varepsilon) \subset U_{x_0}(\alpha; r, \varepsilon)$$

for each $r > 0$ and $\varepsilon > 0$. Thus

$$U'_{x_0}(\alpha; r+\varepsilon, \varepsilon) \subset U_{x_0}(\alpha; r, \varepsilon) \subset U'_{x_0}(\alpha; r, \varepsilon)$$

for each $r > 0$ and $\varepsilon > 0$. Since $\{U_{x_0}(\alpha; r, \varepsilon_0) \mid r > 0\}$ is a neighborhood basis for α in $X \cup \partial_{x_0} X$, $\{U'_{x_0}(\alpha; r, \varepsilon_0) \mid r > 0\}$ is also. $\square$

Definition 2.4. A CAT(0) space X is said to be *almost extendible*, if there exists a constant $E > 0$ such that for each pair of points $x, y \in X$, there is a geodesic ray $\zeta : [0, \infty) \to X$ such that $\zeta(0) = x$ and ζ passes within E of y.

Ontaneda proved the following.

Theorem 2.5 (Ontaneda [**7**]). *A* CAT(0) *space on which a group acts geometrically is almost extendible.*

3. Proof of the main theorem. In this section, we prove the following main theorem.

Theorem 3.1. *Suppose that a group G acts geometrically on a CAT(0) space (X, d). If $|\partial X| > 2$, then each point of ∂X is an accumulation point.*

Proof. Suppose that $|\partial X| > 2$. Let $x_0 \in X$. Since G acts cocompactly on X, there exists a constant $N > 0$ such that

$$GB(x_0, N) = X. \tag{1}$$

By Theorem 2.5, X is almost extendible. Hence, there exists a constant $E > 0$ such that

(2) for each pair of points $x, y \in X$, there is a geodesic ray $\zeta : [0, \infty) \to X$ such that $\zeta(0) = x$ and ζ passes within E of y.

To prove that every point of ∂X is an accumulation point, we show that, for each $\alpha \in \partial_{x_0} X$,

$$\overline{\partial_{x_0} X \setminus \{\alpha\}} = \partial_{x_0} X,$$

where $\overline{\partial_{x_0}X \setminus \{\alpha\}}$ is the closure of $\partial_{x_0}X \setminus \{\alpha\}$ in $\partial_{x_0}X$. Now we identify as $\partial X = \partial_{x_0}X$.

Let $\alpha \in \partial X$ (hence α is a geodesic ray with $\alpha(0) = x_0$). Then, by (1), there exists a sequence $\{g_i \mid i \in \mathbf{N}\} \subset G$ such that

(3) $\{g_i x_0\}_i$ converges to α and

(4) $d(g_i x_0, \operatorname{Im}\alpha) \leq N$ for each i.

Since $X \cup \partial X$ and ∂X are compact and the action of G on X is proper, we may suppose that

(5) $\{g_i^{-1}x_0\}_i$ converges to some point $\alpha' \in \partial X$ in $X \cup \partial X$ and

(6) $\{g_i^{-1}\alpha\}_i$ converges to some point $\beta \in \partial X$ in ∂X.

Now $|\partial X| > 2$. Hence there exists an element $\delta \in \partial X \setminus \{\alpha', \beta\}$. Since $\delta \neq \alpha'$ and $\delta \neq \beta$, there exists a number $R > 0$ such that

(7) $d(\delta(R), \operatorname{Im}\alpha') > E + N + 1$ and

(8) $d(\delta(R), \operatorname{Im}\beta) > E + N + 1$.

To prove $\overline{\partial X \setminus \{\alpha\}} = \partial X$, we show that

$$(\partial X \setminus \{\alpha\}) \cap U'_{x_0}(\alpha; r, \varepsilon) \neq \varnothing$$

for each $r > 0$ and $\varepsilon > 0$, where $U'_{x_0}(\alpha; r, \varepsilon)$ is the set defined in Lemma 2.3.

Let $r > 0$, $\varepsilon > 0$, and let

$$S > \max\left\{R,\ \frac{r(N+R+E)}{\varepsilon}\right\}. \tag{9}$$

By (3), (4), (5) and (6), there exists a large number n such that

(10) $d(g_n x_0, \alpha(T)) \leq N$ for some $T \geq S$,

(11) $d(\alpha'(R), \xi_n(R)) < 1$, where ξ_n is the geodesic from x_0 to $g_n^{-1}x_0$, and

(12) $d(\beta(R), (g_n^{-1}\alpha)_{x_0}(R)) < 1$, where $(g_n^{-1}\alpha)_{x_0} : [0, \infty) \to X$ is the geodesic ray with $(g_n^{-1}\alpha)_{x_0}(0) = x_0$ and $(g_n^{-1}\alpha)_{x_0}(\infty) = g_n^{-1}\alpha(\infty)$,

see Figure 1.

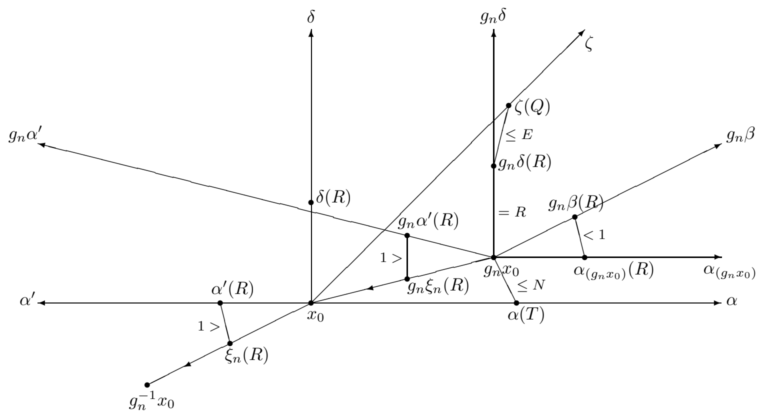

FIGURE 1.

Then we show that $d(g_n\delta(R), \operatorname{Im}\alpha) > E$.

We first have that

$$\begin{aligned} d(g_n\delta(R), \operatorname{Im}\alpha) \geq\ & d(g_n\delta(R), (\operatorname{Im} g_n\xi_n) \cup (\operatorname{Im}\alpha_{(g_n x_0)})) \\ & - d_H(\operatorname{Im}\alpha, (\operatorname{Im} g_n\xi_n) \cup (\operatorname{Im}\alpha_{(g_n x_0)})), \end{aligned}$$

where d_H is the Hausdorff distance and $\alpha_{(g_n x_0)} : [0, \infty) \to X$ is the geodesic ray with $\alpha_{(g_n x_0)}(0) = g_n x_0$ and $\alpha_{(g_n x_0)}(\infty) = \alpha(\infty)$. Since ξ_n is the geodesic from x_0 to $g_n^{-1}x_0$, $g_n\xi_n$ is the geodesic from $g_n x_0$ to x_0. By (10) and Lemma 2.1 (ii) and (iii),

(13) $d_H(\alpha([0, T]), \operatorname{Im} g_n\xi_n) \leq N$ and

(14) $d_H(\alpha([T, \infty)), \operatorname{Im}\alpha_{(g_n x_0)}) \leq N$,

because the geodesic rays $\alpha|_{[T,\infty)}$ and $\alpha_{(g_n x_0)}$ are asymptotic. Hence

(15) $d_H(\operatorname{Im}\alpha, (\operatorname{Im} g_n\xi_n) \cup (\operatorname{Im}\alpha_{(g_n x_0)})) \leq N$.

Thus

$$\begin{aligned} d(g_n\delta(R), \operatorname{Im}\alpha) \geq\ & d(g_n\delta(R), (\operatorname{Im} g_n\xi_n) \cup (\operatorname{Im}\alpha_{(g_n x_0)})) \\ & - d_H(\operatorname{Im}\alpha, (\operatorname{Im} g_n\xi_n) \cup (\operatorname{Im}\alpha_{(g_n x_0)})) \\ \geq\ & d(g_n\delta(R), (\operatorname{Im} g_n\xi_n) \cup (\operatorname{Im}\alpha_{(g_n x_0)})) - N. \end{aligned}$$

Here, by Lemma 2.2,

(16) $d(g_n\delta(R)\mathrm{Im}\, g_n\xi_n \setminus g_n\xi_n([0,R))) = d(g_n\delta(R), g_n\xi_n(R))$ and

(17) $d(g_n\delta(R), \alpha_{(g_nx_0)}([R,\infty))) = d(g_n\delta(R), \alpha_{(g_nx_0)}(R))$.

Hence,

$$\begin{aligned} d(g_n\delta(R), \mathrm{Im}\,\alpha) &\geq d(g_n\delta(R), (\mathrm{Im}\, g_n\xi_n) \cup (\mathrm{Im}\,\alpha_{(g_nx_0)})) - N \\ &= d(g_n\delta(R), g_n\xi_n([0,R]) \cup \alpha_{(g_nx_0)}([0,R])) - N \\ &\geq d(g_n\delta(R), g_n\alpha'([0,R]) \cup g_n\beta([0,R])) \\ &\quad - d_H(g_n\alpha'([0,R]) \cup g_n\beta([0,R]), g_n\xi_n([0,R]) \\ &\cup \alpha_{(g_nx_0)}([0,R])) - N. \end{aligned}$$

By (11) and Lemma 2.1 (ii),

$$\begin{aligned} d_H(g_n\alpha'([0,R]), g_n\xi_n([0,R])) &\leq d(g_n\alpha'(R), g_n\xi_n(R)) \\ &= d(\alpha'(R), \xi_n(R)) < 1. \end{aligned} \tag{18}$$

Also, by (12) and Lemma 2.1 (ii),

$$\begin{aligned} d_H(g_n\beta([0,R]), \alpha_{(g_nx_0)}([0,R])) &\leq d(g_n\beta(R), \alpha_{(g_nx_0)}(R)) \\ &= d(\beta(R), g_n^{-1}\alpha_{(g_nx_0)}(R)) \\ &< 1, \end{aligned} \tag{19}$$

because $g_n^{-1}\alpha_{(g_nx_0)} = (g_n^{-1}\alpha)_{x_0}$. Hence,

$$\begin{aligned} & d_H(g_n\alpha'([0,R]) \cup g_n\beta([0,R]), g_n\xi_n([0,R]) \cup \alpha_{(g_nx_0)}([0,R])) \\ & \quad < 1 \end{aligned} \tag{20}$$

by (18) and (19). Thus,

$$\begin{aligned} & d(g_n\delta(R), \mathrm{Im}\,\alpha) \\ & \quad \geq d(g_n\delta(R), g_n\alpha'([0,R]) \cup g_n\beta([0,R])) \\ & \qquad - d_H(g_n\alpha'([0,R]) \cup g_n\beta([0,R]), g_n\xi_n([0,R]) \\ & \qquad\qquad \cup \alpha_{(g_nx_0)}([0,R])) - N \\ & \quad > d(g_n\delta(R), g_n\alpha'([0,R]) \cup g_n\beta([0,R])) - 1 - N \qquad \text{by (20)} \\ & \quad = d(\delta(R), \alpha'([0,R]) \cup \beta([0,R])) - 1 - N \\ & \quad > (E+N+1) - 1 - N \qquad \text{by (7) and (8)} \\ & \quad = E. \end{aligned}$$

Hence, we obtain that

$$d(g_n\delta(R), \operatorname{Im}\alpha) > E. \tag{21}$$

By (2), there exists a geodesic ray $\zeta : [0, \infty) \to X$ such that
(22) $\zeta(0) = x_0$ and
(23) $\operatorname{Im}\zeta \cap B(g_n\delta(R), E) \neq \varnothing$.
Since $d(g_n\delta(R), \operatorname{Im}\alpha) > E$ by (21),

$$\operatorname{Im}\alpha \cap B(g_n\delta(R), E) = \varnothing.$$

Hence $\zeta \neq \alpha$ by (23), i.e.,

$$\zeta \in \partial X \setminus \{\alpha\}. \tag{24}$$

Finally we show that $\zeta \in U'_{x_0}(\alpha; r, \varepsilon)$.
By (23), there exists a number $Q > 0$ such that

$$d(\zeta(Q), g_n\delta(R)) \leq E. \tag{25}$$

Then, by (10) and (25),
(26)

$$\begin{aligned} d(\alpha(T), \zeta(Q)) &\leq d(\alpha(T), g_n x_0) + d(g_n x_0, g_n\delta(R)) + d(g_n\delta(R), \zeta(Q)) \\ &\leq N + R + E. \end{aligned}$$

Hence,

$$\begin{aligned} d(\alpha(r), \operatorname{Im}\zeta) &\leq d\left(\alpha(r), \zeta\left(\frac{rQ}{T}\right)\right) \\ &\leq \frac{r}{T}\, d(\alpha(T), \zeta(Q)) && \text{by Lemma 2.1 (ii)} \\ &\leq \frac{r(N+R+E)}{T} && \text{by (26)} \\ &\leq \frac{r(N+R+E)}{S} && \text{by (10)} \\ &< \varepsilon && \text{by (9).} \end{aligned}$$

This means that

$$\zeta \in U'_{x_0}(\alpha; r, \varepsilon). \tag{27}$$

Thus by (24) and (27),

$$\zeta \in (\partial X \setminus \{\alpha\}) \cap U'_{x_0}(\alpha; r, \varepsilon),$$

i.e.,

$$(\partial X \setminus \{\alpha\}) \cap U'_{x_0}(\alpha; r, \varepsilon) \neq \varnothing.$$

Here $r > 0$ and $\varepsilon > 0$ are arbitrary. Therefore by Lemma 2.3,

$$\overline{\partial X \setminus \{\alpha\}} = \partial X,$$

i.e., α is an accumulation point of ∂X. $\square$

Remark. Concerning the *ends* of finitely generated groups, Hopf's theorem states that every finitely generated group G has either $0, 1, 2$ or infinitely many ends; and in the case of infinitely many ends, the end space Ends (G) has no isolated points, cf. [**1**, p. 146]. It is pointed out by the referee that the main theorem also can be proved by this fact.

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