BOUNDEDNESS OF SOLUTIONS OF SECOND-ORDER FORCED NONLINEAR DYNAMIC EQUATIONS

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ABSTRACT. In this paper, we give some sufficient conditions which ensure that every solution of a certain class of forced nonlinear dynamic equations of the form

$$(*) \qquad x^{\Delta\Delta}(t) + q^{\sigma}(t)f(x(t)) = r(t),$$

on time scale $\mathbf{T}$ is bounded. To the best of our knowledge nothing is known regarding the qualitative behavior of solutions of the nonlinear dynamic equation $(*)$ on time scales until now. Our results not only unify the boundedness of differential and difference equations but are also new for the q-difference equations.

1. Introduction. The theory of time scales, which has recently received a lot of attention, was introduced by Stefan Hilger in his Ph.D. Thesis [**16**] in order to unify continuous and discrete analysis. Not only can this theory of the so-called "dynamic equations" *unify* the theories of differential equations and difference equations, but also it is able to *extend* these classical cases to cases "in between," e.g., to so-called q-difference equations. A time scale $\mathbf{T}$ is an arbitrary closed subset of the reals, and the cases when this time scale is equal to the reals or to the integers represent the classical theories of differential and of difference equations. Many other interesting time scales exist, and they give rise to plenty of applications, among them the study of population dynamic models, see [**5**]. A book on the subject of time scales by Bohner and Peterson [**5**] summarizes and organizes much of the time scale calculus, see also [**1**]. For the notions used below we refer to [**5**] and to the next section, where we recall some of the main tools used in the subsequent sections of this paper.

The problem of obtaining sufficient conditions to ensure that all solutions of certain classes of second order dynamic equations are oscillatory has been studied by a number of authors [**2–4, 6–15, 17, 18**]. A large portion of these results have been for the nonlinear dynamic equation of the form

$$(1.1) \qquad (\alpha(t)x^\Delta)^\Delta + q(t)(f \circ x^\sigma) = 0, \quad t \in \mathbf{T},$$

with $q(t) \geq 0$. Although much less is known regarding the oscillatory behavior of (1.1), when $q(t)$ is oscillatory, to the best of our knowledge there is only one paper in this direction [**13**].

Following this trend, in this paper we will consider the forced nonlinear dynamic equation

$$(1.2) \qquad x^{\Delta\Delta}(t) + q^\sigma f(x(t)) = r(t), \quad t \in \mathbf{T},$$

where $q^\sigma(t)$ and $r(t)$ are real-valued $rd-$continuous functions defined on the time scale $\mathbf{T}$ and $q^\sigma(t) > 0$ (throughout $a, b \in \mathbf{T}$ with $a < b$). Since we are interested in asymptotic behavior of solution of (1.2), we suppose that the time scale under consideration is not bounded above, i.e., it is a time scale interval of the form $[t_0, \infty)$. By a solution of (1.2) we mean a nontrivial real-valued function x satisfying equation (1.2) for $t \geq t_0$.

In this paper, in Section 3, we show that, under quite general conditions, solutions of (1.2) can be defined for all $t \geq t_0$. We will establish some sufficient conditions for the boundedness of all solutions of (1.2). To the best of our knowledge this approach to the study of (1.2) of asymptotic behavior has not been studied before.

Note that if $\mathbf{T} = \mathbf{R}$, we have $\sigma(t) = t$, $\mu(t) = 0$, $x^\Delta(t) = x'(t)$, and (1.2) becomes the second-order nonlinear differential equation

$$(1.3) \qquad x''(t) + q(t)f(x(t)) = r(t), \quad t \in [t_0, \infty).$$

If $\mathbf{T} = \mathbf{N}$, we have $\sigma(t) = t + 1$, $\mu(t) = 1$, $x^\Delta(t) = \Delta x(t) = x(t+1) - x(t)$, and (1.2) becomes the second order nonlinear difference equation

$$(1.4) \qquad \Delta^2 x_t + q_{t+1} f(x_t) = r_t, \quad t \in [t_0, \infty).$$

If $\mathbf{T} = h\mathbf{N}$, $h > 0$, we have $\sigma(t) = t + h$, $\mu(t) = h$, $x^\Delta(t) = \Delta_h x(t) = (x(t+h) - x(t))/h$, and (1.2) becomes

$$(1.5) \qquad \Delta_h^2 x_t + q_{t+h} f(x_t) = r_t, \quad t \in [t_0, \infty).$$

If $\mathbf{T} = \varrho^{\mathbf{N}} = \{\varrho^k, \ k \in \mathbf{N}, \ \varrho > 1\}$, we have $\sigma(t) = \varrho t$, $\mu(t) = (\varrho - 1)t$, $x_\varrho^\Delta(t) = (x(\varrho t) - x(t))/(\varrho - 1)t$, and (1.2) becomes

$$(1.6) \qquad \Delta_\varrho^2 x_t + q_{\varrho t} \, f(x_t) = r_t, \quad t \in [t_0, \infty).$$

Our results not only unify the boundedness of all solutions of (1.3) and (1.4) but also are essentially new for equations (1.5) and (1.6). Also, we note that our results can be applied on other time scales for which the forward jump operator and the step function are defined.

2. Some preliminaries on time scales. A time scale $\mathbf{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbf{R}$. On any time scale $\mathbf{T}$ we define the forward jump operator and the graininess function by:

$$\sigma(t) := \inf\{s \in \mathbf{T} : \ s > t\} \quad \text{and} \quad \mu(t) := \sigma(t) - t.$$

A point $t \in T$ with $\sigma(t) = t$ is called right-dense while t is referred to as being right-scattered if $\sigma(t) > t$. The backward jump operator $\rho(t)$ and left-dense and left-scattered points are defined in a similar way.

A function $f : \mathbf{T} \to \mathbf{R}$ is said to be *rd-continuous* if it is continuous at each *right-dense* point and if there exists a finite left limit in all left-dense points. The (delta) *derivative* is defined by

$$f^\Delta(t) = \lim_{\substack{s \to t \\ s \in U(t)}} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}, \quad \text{where} \quad U(t) = \mathbf{T} \setminus \{\sigma(t)\}.$$

The derivative and the shift operator σ are related by the useful formula

$$(2.1) \qquad f^\sigma = f + \mu f^\Delta, \quad \text{where} \quad f^\sigma := f(\sigma(t)).$$

We will make use of the following product and quotient rules for the derivative of the product fg and the quotient f/g (where $gg^\sigma \neq 0$) of two differentiable functions f and g:

$$(2.2) \qquad (fg)^\Delta = f^\Delta g + f^\sigma g^\Delta, \quad \text{and} \quad \left(\frac{f}{g}\right)^\Delta = \frac{f^\Delta g - f g^\Delta}{gg^\sigma}.$$

By using the product rule form (2.2), the derivative of $f(t) = (t - \alpha)^m$ for $m \in \mathbf{N}$, and $\alpha \in \mathbf{R}$ can be calculated, see [**5**, Theorem 1.24], as

$$f^\Delta(t) = \sum_{\nu=0}^{m-1} (\sigma(t) - \alpha)^\nu \, (t - \alpha)^{m-\nu-1}.$$

For $a, b \in \mathbf{T}$, and a differentiable function f, the Cauchy integral of f^{Δ} is defined by

$$\int_a^b f^{\Delta}(t)\Delta t = f(b) - f(a).$$

The integration by parts formula follows from (2.2) and reads

$$(2.5) \qquad \int_a^b f(t)g^{\Delta}(t)\Delta t = [f(t)g(t)]_a^b - \int_a^b f^{\Delta}(t)g(\sigma(t))\Delta t,$$

and infinite integrals are defined as

$$\int_a^{\infty} f(t)\Delta t = \lim_{b \to \infty} \int_a^b f(t)\Delta t.$$

Note that if $\mathbf{T} = \mathbf{R}$, we have $\sigma(t) = \rho(t) = t$, $\mu(t) = 0$, $f^{\Delta}(t) = f'(t)$, and

$$\int_a^b f(t)\Delta t = \int_a^b f(t)\, dt.$$

If $\mathbf{T} = \mathbf{N}$, we have $\sigma(t) = t + 1$, $\rho(t) = t - 1$, $\mu(t) = 1$,

$$f^{\Delta}(t) = \Delta f(t) = f(t+1) - f(t), \quad \text{and} \quad \int_a^b f(t)\Delta t = \sum_{i=a}^{b-1} f(i).$$

If $\mathbf{T} = h\mathbf{N}$, we have $\sigma(t) = t + h$, $\rho(t) = t - h$, $\mu(t) = h$,

$$f^{\Delta}(t) = \Delta f(t) = \frac{f(t+h) - f(t)}{h}, \quad \text{and} \quad \int_a^b f(t)\Delta t = \sum_{i=(a/h)}^{(b/h)-1} h f(ih).$$

If $\mathbf{T} = q^{\mathbf{N}} = \{t : t = q^k, \ k \in \mathbf{N}, \ q > 1\}$, we have $\sigma(t) = qt$, $\mu(t) = (q-1)t$,

$$x_q^{\Delta}(t) = \frac{x(qt) - x(t)}{(q-1)t} \quad \text{and} \quad \int_a^{\infty} f(t)\Delta t = \sum_{k=0}^{\infty} \mu(q^k) f(q^k).$$

On an arbitrary time scale $\mathbf{T}$, the usual chain rule from calculus is no longer valid, see Bohner and Peterson [**5**, p. 31]. One form of the extended chain rule is as follows, see [**5**, p. 32].

Lemma 2.1. *Assume that $g : \mathbf{T} \to \mathbf{R}$ is delta differentiable on $\mathbf{T}$. Assume that $f : \mathbf{R} \to \mathbf{R}$ is continuously differentiable. Then $f \circ g : \mathbf{T} \to \mathbf{R}$ is delta differentiable and satisfies*

$$(f \circ g)^{\Delta}(t) = \left\{ \int_{0}^{1} f'(g(t) + h g^{\Delta}(t)\mu(t))\, dh \right\} g^{\Delta}(t).$$

We denote the set of all $p : \mathbf{T} \to \mathbf{R}$ which are rd-continuous and regressive by $\mathcal{R}$. If $p \in \mathcal{R}$, we define the exponential function by

$$e_p(t, s) = \exp\left(\int_{s}^{t} \xi_{\mu(\tau)}(p(\tau))\Delta\tau \right),$$

for $t \in \mathbf{T}$, $s \in \mathbf{T}^k$, where $\xi_h(z)$ is the cylinder transformation, which is given by

$$\xi_h(z) = \begin{cases} \dfrac{\log(1 + hz)}{h} & h \neq 0, \\ z & h = 0. \end{cases}$$

Alternately, for $p \in \mathcal{R}$, one can define the exponential function $e_p(\cdot, t_0)$, to be the unique solution of the IVP

$$x^{\Delta} = p(t)x, \quad x(t_0) = 1.$$

We define

$$\mathcal{R}^+ := \{ f \in \mathcal{R} : 1 + \mu(t)p(t) > 0,\ t \in \mathbf{T} \}.$$

For properties of this exponential function, we refer to the book by Bohner and Peterson [**5**]. One such property that we will use is the formula

$$e_p^{\Delta}(t, t_0) = p(t)e_p(t, t_0).$$

Also if $p \in \mathcal{R}$, then $e_p(t, s)$ is real-valued and nonzero on $\mathbf{T}$. If $p \in \mathcal{R}^+$, then $e_p(t, t_0)$ always positive.

3. Global existence and boundedness of solutions. In this section, we show that under some conditions all solutions of (1.2) may be continued indefinitely to the right. Also, we determine sufficient conditions for boundedness of solutions of (1.2), and in the case when $r(t)$ is small, we are able to obtain necessary and sufficient conditions. Before stating our main results, it will be convenient to write equation (1.2) as the equivalent system

$$(3.1) \qquad \begin{aligned} x^\Delta &= y, \\ y^\Delta &= -q^\sigma(t)f(x(t)) + r(t). \end{aligned}$$

Let

$$p(t) = e_{-q^\Delta(t)/q(t)}(t, t_0),$$

and there exists a function F such that

$$F^\Delta(x) = x^\Delta f(x).$$

Theorem 3.1. *If $F(x)$ is bounded from below, $q^\Delta(t) \geq 0$,*

$$\int_{t_0}^t \mu(s) \frac{p^\sigma r^2(s)}{q^\sigma(s)} \Delta s + \frac{p(t)}{q(t)} \quad and \quad \frac{p^\sigma rq}{q^\sigma p},$$

are bounded. Then every solution of (3.1) *can be defined for all $t \geq t_0$.*

Proof. Suppose the theorem is false so that there exists a solution $(x(t), y(t))$ of (3.1) and a number $T > t_0$ such that $\lim_{t \to T}(x(t)+y(t)) = \infty$. Since $F(x)$ is bounded from below, there exists a constant $K > 0$ such that $F(x) > -K$ for all x. Let

$$V(x, y, t) = p(t)\left(\frac{y^2(t)}{q(t)} + F(x) + K\right).$$

Then, by using (2.2), we have

$$V^\Delta = p^\sigma\left(\frac{y^2(t)}{q(t)} + F(x) + K\right)^\Delta + p^\Delta(t)\left(\frac{y^2(t)}{q(t)} + F(x) + K\right)$$

$$= p^\sigma \left(\frac{q(y + y^\sigma)y^\Delta - y^2 q^\Delta}{qq^\sigma} + yf(x) \right)$$

$$- \frac{q^\Delta(t)}{q(t)} e_{-q^\Delta(t)/q(t)}(t, t_0) \left(\frac{y^2(t)}{q(t)} + F(x) + K \right)$$

$$\leq p^\sigma \left(\frac{(y + y^\sigma)y^\Delta}{q^\sigma} - \frac{y^2 q^\Delta}{qq^\sigma} + yf(x) \right)$$

$$\leq p^\sigma \left(\frac{-yq^\sigma(t)f(x(t)) + yr(t)}{q^\sigma} + \frac{-y^\sigma q^\sigma(t)f(x(t)) + y^\sigma r(t)}{q^\sigma} + yf(x) \right)$$

$$= \frac{p^\sigma r(t)}{q^\sigma} \left(y + y + \mu(t) \left(-q^\sigma(t)f(x(t)) + r(t) \right) \right)$$

$$\leq \frac{p^\sigma r(t)}{q^\sigma} \left(2y + \mu(t)r(t) \right)$$

$$= \frac{2p^\sigma r(t)}{q^\sigma} y(t) + \mu(t) \frac{p^\sigma r^2(t)}{q^\sigma}.$$

Integrating, we obtain

$$V \leq G(t) + \int_{t_0}^{t} \frac{2p^\sigma(s)r(s)}{q^\sigma(s)} y(s)\Delta s,$$

where $G(t) = \int_{t_0}^{t} \mu(s)(p^\sigma r^2(s))/(q^\sigma(s))\Delta s + V(t_0)$. Then

$$\frac{p(t)y^2(t)}{q(t)} \leq V(t) \leq G(t) + \int_{t_0}^{t} \frac{2p^\sigma(s)r(s)}{q^\sigma(s)} y(s)\Delta s.$$

By using the fact that $2|y| \leq y^2 + 1$, we obtain

$$\frac{p(t)y(t)}{q(t)} \leq G'(t) + \int_{t_0}^{t} \frac{p^\sigma(s)r(s)}{q^\sigma(s)} y(s)\Delta s,$$

where $G'(t) = G(t) + p(t)/2q(t)$. Now, by using the boundedness of $G'(t)$, we see that

$$\frac{p(t)y(t)}{q(t)} \leq K_1 + \int_{t_0}^{t} \frac{p^\sigma(s)r(s)q(s)}{q^\sigma(s)p(s)} \frac{p(s)y(s)}{q(s)} \Delta s,$$

for some $K_1 > 0$. Then by Gronwall's inequality [**5**], we have

$$\frac{p(t)y(t)}{q(t)} \le K_1 e_{(p^\sigma rq)/(q^\sigma p)}(t, t_0).$$

Then, $p(t)y(t)/q(t)$ is bounded on $[t_0, T]$, say $p(t)y(t)/q(t) < M$. Then, $y(t) < Mq(t)/p(t)$ is bounded. Say $y(t) < N$. But, since $x^\Delta(t) = y(t)$, we have

$$x(t) \le x(t_0) + N(t - t_0) < x(t_0) + N(T - t_0),$$

and so $x(t)$ is bounded on $[t_0, T]$ contradicting the assumption that $(x(t), y(t))$ was a solution of (3.1) with finite escape time. the proof is complete.   □

In the following theorems we establish some sufficient conditions for the boundedness of solutions of (1.2).

Theorem 3.2. *Suppose that $q(t)$, $p(t)$ and $r(t)$ are bounded,*

$$q^\Delta(t) \ge 0, \quad \int_{t_0}^{\infty} \frac{p^\sigma(s)r(s)q(s)}{q^\sigma(s)p(s)} \Delta s < \infty \quad \text{and} \quad e_{(p^\sigma rq)/(q^\sigma p)}(\infty, t_0) < \infty.$$

If $F(x) \to \infty$ as $|x| \to \infty$, then all solutions of (1.2) are bounded.

Proof. Since $F(x) \to \infty$ as $|x| \to \infty$, $F(x)$ is bounded from below, say $F(x) > -K$ for some $K > 0$. As in the proof of Theorem 3.1, we let $V(x, y, t) = p(t)[y^2(t)/q(t) + F(x) + K]$, and proceeding as in the proof of Theorem 3.1 to obtain

$$\frac{p(t)y(t)}{q(t)} \le G'(t) + \int_{t_0}^{t} \frac{p^\sigma(s)r(s)}{q^\sigma(s)} y(s)\Delta s,$$

where

$$G'(t) = \int_{t_0}^{t} \mu(s) \frac{p^\sigma r^2(s)}{q^\sigma(s)} \Delta s + V(t_0) + \frac{p(t)}{2q(t)}.$$

Now by using the boundedness of $G'(t)$, we see that

$$\frac{p(t)y(t)}{q(t)} \leq K_1 + \int\limits_{t_0}^{t} \frac{p^\sigma(s)r(s)q(s)}{q^\sigma(s)p(s)} \frac{p(s)y(s)}{q(s)} \Delta s,$$

for some $K_1 > 0$. Then by Gronwall's inequality [**5**], we have

$$\frac{p(t)y(t)}{q(t)} \leq K_1 e_{(p^\sigma r q)/(q^\sigma p)}(t, t_0) < K_1 e_{(p^\sigma r q)/(q^\sigma p)}(\infty, t_0) < A.$$

Hence,

$$V(t) \leq V(t_0) + G(t) + A \int\limits_{t_0}^{t} \frac{p^\sigma(s)r(s)q(s)}{q^\sigma(s)p(s)} \Delta s$$

$$< V(t_0) + G(t) + A \int\limits_{t_0}^{\infty} \frac{p^\sigma(s)r(s)q(s)}{q^\sigma(s)p(s)} \Delta s < B,$$

and so
$$p(t)F(x(t)) < B.$$

Since $p(t) > p > 0$, then $F(x(t))$ is bounded from which it follows that $x(t)$ is bounded. The proof is complete. $\square$

Theorem 3.3. *Suppose there exist positive constants c, B, and k such that*

$$q^\sigma(t) \leq c, \quad \left| \int\limits_{t_0}^{t} r(s)\Delta s \right| \leq B,$$

for all $t \geq t_0$ and $xf(x) > 0$ for $|x| \geq k$. Then $\int_{t_0}^{\pm\infty} f(x(s))\Delta s = \infty$ is a necessary condition for all solutions of (1.2) to be bounded.

Proof. Suppose to the contrary that $\int_{t_0}^{\infty} f(x(s))\Delta s \leq M < \infty$ (the case when $\int_{t_0}^{-\infty} f(x(s))\Delta s$ is similar and will be omitted). Integrating

the second equation in (3.1), we get

$$y(t) = y(t_0) - \int_{t_0}^{t} q^{\sigma}(s) f(x(s)) \Delta s + \int_{t_0}^{t} r(s) \Delta s \geq y(t_0)$$

$$- \int_{t_0}^{t} q^{\sigma}(s) f(x(s)) \Delta s - B$$

$$\geq y(t_0) - c \int_{t_0}^{t} f(x(s)) \Delta s - B \geq y(t_0) - cM - B.$$

Choose the point $(x(t_0), y(t_0))$ to be such that $x(t_0) > k$ and $y(t_0) > cM + B + \alpha$. Then we have $y(t) > \alpha$ and, by the first equation of (3.1), we have $x^{\Delta}(t) > \alpha$. Thus

$$x(t) > x(t_0) + \alpha(t - t_0) \longrightarrow \infty \quad \text{as} \quad t \to \infty,$$

which is an unbounded solution. The proof is complete.   □

Theorem 3.4. *If* $F(x) \to \infty$ *as* $|x| \to \infty$, *bounded from below,*

$$\frac{1}{p(t)} \int_{t_0}^{t} \mu(s) \frac{p^{\sigma} r^2(s)}{q^{\sigma}(s)} \Delta s \leq k_1, \quad \frac{1}{p(t)} \leq k_2,$$

and

$$\int_{t_0}^{t} \left(\frac{p^{\sigma}(s)|r(s)|}{(q^{\sigma}(s))^{1/2}} \right) \Delta s < \infty,$$

then every solution of (1.2) *is bounded.*

Proof. Since $F(x)$ is bounded from below, there exits a constant $K > 0$ such that $F(x) > -K$ for all x. Let V be as defined in Theorem 3.1. Differentiating and integrating V, as in Theorem 3.1,

we obtain

$$\frac{y^2(t)}{q(t)} \le V(t) \le \frac{1}{p(t)} \int_{t_0}^{t} \mu(s) \frac{p^\sigma r^2(s)}{q^\sigma(s)} \Delta s + \frac{V(t_0)}{p(t)}$$

$$+ \frac{2}{p(t)} \int_{t_0}^{t} \frac{p^\sigma(s)r(s)}{q^\sigma(s)} y(s)\Delta s.$$

Then,

$$\frac{y^2(t)}{q(t)} \le K_1 + K_2 \int_{t_0}^{t} \frac{p^\sigma(s)|r(s)|}{q^\sigma(s)} y(s)\Delta s.$$

for some $K_1 > 1$ and $K_2 > 0$. Thus,

$$\frac{|y(t)|}{(q(t))^{1/2}} \le K_1 + K_2 \int_{t_0}^{t} \frac{p^\sigma(s)|r(s)|}{(q^\sigma(s))^{1/2}} \left(\frac{y(s)}{(q^\sigma(s))^{1/2}} \right) \Delta s,$$

and so, by Gronwall's inequality, we get

$$\frac{|y(t)|}{(q(t))^{1/2}} \le K_1 e_{K_2(p^\sigma|r|)/((q^\sigma)^{1/2})} (t,t_0) < L < \infty.$$

Then, we have

$$p(t)F(x) \le V \le G(t) + \int_{t_0}^{t} \frac{2p^\sigma(s)r(s)}{q^\sigma(s)} y(s)\Delta s$$

$$\le K + 2L \int_{t_0}^{t} \frac{p^\sigma(s)|r(s)|}{(q^\sigma(s))^{1/2}} \Delta s < M < \infty,$$

and so $p(t)F(x) < M/p(t) < M/k_2 < \infty$; then $x(t)$ is bounded. The proof is complete. □

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