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ON A CLASS OF TWO-POINT BOUNDARY VALUE PROBLEMS WITH SINGULAR BOUNDARY CONDITIONS

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ABSTRACT. A new existence theory for a class of second order two-point boundary value problems with nonlinear boundary conditions which can blow up in finite intervals is established. The proofs are based on the dual variational principle and the critical point theory.

1. Introduction. In this paper we consider second order mixed boundary value problems of the type

$$\begin{cases} (1/p)(py')' + f(t,y) = 0 & 0 < t < 1, \\ \lim_{t\to 0^+} p(t)y'(t) = 0, \\ \lim_{t\to 1^-} p(t)y'(t) + g(y(1)) = 0, \end{cases} \tag{1.1}$$

where $p > 0$ and g is defined only on a bounded open interval, with singularities at the extremities. A typical situation is when $g : (a,b) \to \mathbf{R}$, is increasing, where

$$-\infty < a < 0 < b < \infty$$

and

$$\lim_{y\to a^+} g(y) = -\infty \quad \text{and} \quad \lim_{y\to b^-} g(y) = +\infty. \tag{1.2}$$

In recent years, the problem (1.1) when $f(t,y)$ is singular at $y = 0$ and the boundary conditions are linear has been studied extensively. We refer the reader to the books [**2, 14**] for up-to-date information on the subject. Current research on this type of problem has been mainly

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based on fixed point theory and monotone arguments of sub- and super-solutions, cf. [**3–6, 9, 10**]. On the other hand, when problem (1.1) has no singularities at $y = 0$, but, eventually singular at $t = 0$ or $t = 1$, it admits a direct variational framework. In fact, in this case, the weak solutions of the problem (1.1) are critical points of $\Phi : E \to \mathbf{R}$, defined by

$$\Phi(y) = \frac{1}{2}\int_0^1 p\,|y'|^2\,dt - \int_0^1 pF(t,y)\,dt - G(y(1)), \tag{1.3}$$

where F, G are the primitives of f, g, respectively, and $E = H_p^1[0,1]$ is the p-weighted Sobolev space modeled from $H^1[0,1]$. Then, adding suitable hypotheses on f, g, the existence of solutions of (1.1) follows immediately by standard critical point theory. However, in the presence of singularities like (1.2), the functional (1.3) is not well defined, and then we need to adopt the dual variational principle [**8**], which is based on convexity arguments and Legendre transforms. This approach is somewhat similar to that of employed in earlier work in [**11, 12**], for the fourth order beam equations.

We shall present four existence criteria in Section 3. Under appropriate growth conditions on $f(t,y)$, as $y \to 0$ and $|y| \to \infty$, we shall show that the problem (1.1) has nonzero, sign changing or multiple solutions. Both sublinear and superlinear nonlinearities are considered. Section 2 provides the dual variational formulation of the problem (1.1) and Section 4 illustrates two examples.

2. Dual variational method. The main aim of this section is to provide a variational framework for the problem (1.1). We begin by stating some basic hypotheses on the functions p, f, and g which will be needed throughout this paper.

Basic hypotheses. We assume that $p \in C^0[0,1] \cap C^1(0,1)$ is positive in $(0,1)$ and satisfies

$$\theta_1 = \int_0^1 p(s)\,ds < \infty \quad \text{and} \quad \theta_2 = \int_0^1 \frac{1}{p(s)}\,ds < \infty. \tag{2.1}$$

The functions $f : [0,1] \times \mathbf{R} \to \mathbf{R}$ and $g : (a,b) \to \mathbf{R}$, $-\infty < a < 0 < b < \infty$, are continuous and invertible in $\mathbf{R}$ and (a,b) respectively. In addition, we suppose that

$$f(t,0) = g(0) = 0, \quad \text{for all} \quad t \in [0,1] \tag{2.2}$$

and

$$\lim_{u \to a^+, u \to b^-} |g(u)| = \infty, \tag{2.3}$$

i.e., g is singular. $\square$

To deal with the singularities at $t = 0$ and/or $t = 1$, when $p(0)$ or $p(1)$ vanish, we will set our study on weighted functions spaces, see, e.g., [**1**]. For this, for a given $\sigma \geq 1$, recall that

$$L_p^\sigma[0,1] = \left\{ u \text{ is measurable}\,;\ \int_0^1 p(s)|u(s)|^\sigma \, ds < \infty \right\}$$

is a real Banach space equipped with the norm

$$\|u\|_{L_p^\sigma} = \left(\int_0^1 p\,|u|^\sigma \, ds \right)^{1/\sigma}.$$

Clearly, if $u \in L_p^\sigma[0,1]$ and $v \in L_p^{\sigma'}[0,1]$, $1/\sigma + 1/\sigma' = 1$, one has the Hölder inequality $\|uv\|_{L_p^1} \leq \|u\|_{L_p^\sigma} \|v\|_{L_p^{\sigma'}}$ and

$$\|u\|_{L_p^1} \leq \theta_1^{1/\sigma'} \|u\|_{L_p^\sigma} \quad (\theta_1 \quad \text{defined in (2.1)}).$$

We will also consider the "orthogonal" decomposition

$$L_p^\sigma[0,1] = \mathbf{R} \oplus W, \tag{2.4}$$

where

$$W = \left\{ w \in L_p^\sigma[0,1]\,;\ \int_0^1 p(t) w(t) \, dt = 0 \right\}.$$

Now we recall some properties of the Legendre transform which are available in, e.g., [**13**]. Given a continuous convex function $H : (a,b) \to \mathbf{R}$, the Legendre transform of H is defined by

$$H^*(v) = \sup_{u \in (a,b)} \{vu - H(u)\}.$$

In addition, if H is differentiable, then

$$H^*(v) = vu - H(u), \quad \text{where} \quad u = (H')^{-1}(v),$$

which means that $H^{*\prime}$ is precisely the inverse of H'. The following two properties hold.

(i) If $C > 0$ and $H(u) = C/\sigma|u|^\sigma + D$, then $H^*(v) = C^*|v|^{\sigma'} - D$, where

$$C^* = \frac{C^{-\sigma'/\sigma}}{\sigma'}. \tag{2.5}$$

(ii) If $H_1 \leq H_2$ then $H_1^* \geq H_2^*$.

With respect to our problem, if we suppose that $f(t,y)$ and $g(y)$ are increasing and invertible functions with respect to y, then

$$F(t,u) = \int_0^u f(t,s)\,ds \quad \text{and} \quad G(u) = \int_0^u g(s)\,ds \tag{2.6}$$

are convex functions with respect to u, and the corresponding Legendre transforms can be given by

$$F^*(t,v) = \int_0^u f^*(t,s)\,ds \quad \text{and} \quad G^*(u) = \int_0^u g^*(s)\,ds, \tag{2.7}$$

where $f^* = f^{-1}$ and $g^* = g^{-1}$.

An important step in applying the dual variational principle to problem (1.1) is the unique solvability of the associated linear problem.

Lemma 1. *Let $p > 0$ satisfy (2.1) and $v \in L_p^\sigma[0,1]$ with $\sigma \geq 1$. Then, the problem*

$$\begin{cases} -1/p\,(py')' = v & 0 < t < 1, \\ \lim_{t\to 0+}(py')(t) = 0, \\ \lim_{t\to 1-}(py')(t) + \gamma = 0, \end{cases} \tag{2.8}$$

has a unique solution satisfying $y(1) = 0$, if and only if $\gamma = \int_0^1 p(s)v(s)\,ds$.

Proof. It suffices to note that the function

$$y(t) = \int_t^1 \frac{1}{p(s)} \int_0^s p(r)v(r)\,dr\,ds \tag{2.9}$$

is the desired solution of (2.8). □

Using Lemma 1 we can define an operator $K : L^\sigma[0,1] \to C[0,1]$ such that $y = Kv$ is the unique solution of the linear problem (2.8).

Lemma 2. *The operator* $K : L^\sigma[0,1] \to C[0,1]$ *is completely continuous and satisfies*

$$0 \le \int_0^1 p(Kv)v\,dt \le \theta_2\,\theta_1^{2/\sigma'} \|v\|^2_{L^\sigma_p}. \tag{2.10}$$

Proof. The compactness of K follows from the Arzela-Ascoli theorem. Now, since $y = Kv$ is a solution of (2.8), we have

$$\int_0^1 p(Kv)v\,dt = \int_0^1 -(Kv)(p(Kv)')'\,dt = \int_0^1 p(Kv)'^2\,dt \ge 0.$$

On the other hand, using (2.9) we have $|Kv| \le \theta_2 \|v\|_{L^1_p}$, and hence

$$\int_0^1 p(Kv)v\,dt \le \theta_2 \|v\|^2_{L^1_p} \le \theta_2\,\theta_1^{2/\sigma'} \|v\|^2_{L^\sigma_p}. \qquad \square$$

We are now ready to establish the dual variational setting for (1.1). We shall show that the existence of critical points of a suitably chosen functional in $L^\sigma_p[0,1]$ implies the existence of solutions of (1.1).

We define $\Psi : L^\sigma_p[0,1] \to \mathbf{R}$ as follows

$$\begin{aligned}\Psi(v) = &-\frac{1}{2}\int_0^1 p(t)(Kv)(t)v(t)\,dt + \int_0^1 p(t)F^*(t,v(t))\,dt \\ &- G^*\Big(\int_0^1 p(t)v(t)\,dt\Big),\end{aligned} \tag{2.11}$$

where F^*, G^* are assumed to be convex. Then, Ψ is of class C^1 and weakly lower semi-continuous. Indeed, since $\int p(Kv)\phi\,dt = \int p(K\phi)v\,dt$, the Gateaux derivative of Ψ is given by

$$\begin{aligned}\langle \Psi'(v), \phi\rangle = -\int_0^1 p(Kv)\phi\,dt + \int_0^1 pf^*(t,v)\phi\,dt \\ - g^*\bigg(\int_0^1 pv\,dt\bigg)\int_0^1 p\phi\,dt.\end{aligned}$$

We shall show that Ψ' is continuous in $L_p^\sigma[0,1]$. For this, first note that

$$\begin{aligned}\langle \Psi'(v_n) - \Psi'(v), \phi\rangle = &-\int_0^1 pK(v_n - v)\phi\,dt \\ &+ \int_0^1 p(f^*(t,v_n) - f^*(t,v))\phi\,dt \\ &- \bigg(g^*\bigg(\int_0^1 pv_n\,dt\bigg) - g^*\bigg(\int_0^1 pv\,dt\bigg)\bigg)\int_0^1 p\phi\,dt,\end{aligned}$$

and now, from the compactness of K, continuity of g^*, and since $f^*(t,\cdot)$ is continuous from $L_p^\sigma[0,1] \to L_p^{\sigma'}[0,1]$ as a Nemytskii operator, it follows that $\langle \Psi'(v_n) - \Psi'(v), \phi\rangle \to 0$ if $\|v_n - v\|_{L_p^\sigma} \to 0$. This shows that Ψ' is continuous in $\mathcal{L}(L_p^\sigma[0,1], \mathbf{R})$ and therefore Ψ is continuously Fréchet differentiable in $L_p^\sigma[0,1]$. Furthermore, since $L_p^\sigma[0,1]$, $\sigma > 1$, is reflexive, the lower semi-continuity of Ψ is a consequence of the fact that Ψ is a sum of the weakly continuous function $v \mapsto 1/2\int_0^1 p(Kv)v\,dt - G^*(\int_0^1 pv\,dt)$ and the convex function $v \mapsto \int_0^1 p(t)F^*(t,v)\,dt$.

Lemma 3. *Suppose the basic hypotheses on p, f and g hold. Suppose in addition that f, g are increasing. Then, for each critical point v of Ψ, there exists a constant $c_v \in \mathbf{R}$ such that $y = Kv - c_v$ is a solution of the problem* (1.1).

Proof. Let $v \in L_p^\sigma[0,1]$ be a critical point of Ψ. Then,

(2.12)
$$-\int_0^1 p(Kv)\phi\,dt + \int_0^1 pf^*(t,v)\phi\,dt - g^*\bigg(\int_0^1 pv\,dt\bigg)\int_0^1 p\phi\,dt = 0,$$

for all $\phi \in C[0,1]$. Choosing test functions ϕ such that $\int_0^1 p\phi\,dt = 0$, we have

$$\int_0^1 p(Kv - f^*(t,v))\phi\,dt = 0, \quad \text{for all} \quad \phi \in W,$$

with W as in (2.4). Then, from the orthogonality, there exists $c_v \in \mathbf{R}$ (complement of W) such that

$$Kv - f^*(\cdot, v) = c_v.$$

Let us put

$$y = Kv - c_v.$$

Then, $f^*(\cdot, v) = y$ and therefore $f(\cdot, y) = v$, since $f^* = f^{-1}$, which combined with the definition of Kv gives,

$$-\frac{1}{p}\,(py')' = f(\cdot, y). \tag{2.13}$$

We also note that

$$\lim_{t\to 0^+} p(t)y'(t) = \lim_{t\to 0^+} p(t)(Kv)'(t) = 0, \tag{2.14}$$

and

$$y(1) = (Kv - c_v)(1) = -\,c_v. \tag{2.15}$$

Now we take test functions such that $\int_0^1 p\phi\,dt \neq 0$. Then, from (2.12) it follows that

$$c_v + g^*\biggl(\int_0^1 pv\,dt\biggr) = 0$$

so that (2.15) implies $g(y(1)) = \int_0^1 pv\,dt$, i.e.,

$$\lim_{t\to 1^-} (py')(t) + g(y(1)) = 0. \tag{2.16}$$

Equations (2.13)–(2.16) confirm that $y = Kv - c_v$ is indeed a solution of the original problem (1.1). $\square$

3. Existence results.

Theorem 1. *Assume that the basic hypotheses on p, f, g hold. If $f(t,y)$ and $g(y)$ are increasing with respect to y and*

$$|f(t,u)| \le C_f(1+|u|), \quad \forall\, u \in \mathbf{R},\ \forall\, t \in [0,1], \tag{3.1}$$

where

$$0 < C_f < \frac{1}{\theta_1\,\theta_2}, \tag{3.2}$$

then the problem (1.1) *has a solution. If, in addition,*

$$\lim_{u\to 0} \frac{u}{f(t,u)} = 0, \quad \textit{uniformly in} \quad t \in [0,1], \tag{3.3}$$

then problem (1.1) *has a nonzero solution.*

Proof. We shall show that the functional $\Psi : L_p^2[0,1] \to \mathbf{R}$ defined in (2.11) has a critical point. Then, from Lemma 3 it follows that the problem (1.1) has a corresponding solution $y \in C^2(0,1)$. Note that, if v is a nonzero critical point of Ψ, then Kv is nonconstant, and hence $y = Kv - c_v$ is a nonconstant solution of (1.1).

From (3.1) and (2.5) it follows that, for some $C_1 > 0$,

$$F^*(t,v) \ge C_f^*|v|^2 - C_1, \quad \forall\, u \in \mathbf{R},\ \forall\, t \in [0,1],$$

where

$$C_f^* > \frac{1}{2}\,\theta_1\,\theta_2. \tag{3.4}$$

Since g^* is bounded, $|G^*(v)| \le C_2|v| + C_3$, where C_2, C_3 are positive constants. Then,

$$\Psi(v) \ge -\frac{1}{2}\,\theta_1\,\theta_2\|v\|^2_{L^2_p} + C_f^*\|v\|^2_{L^2_p} - C_1\theta_1 - C_2\theta_1^{1/2}\|v\|_{L^2_p} - C_3.$$

Now, from (3.4) we see that Ψ is coercive, and since it is weakly lower semi-continuous, it has a critical point.

Next, we show the existence of a nonzero critical point. From (3.3),

$$\lim_{v\to 0}\frac{f^*(t,v)}{v}=0, \quad \text{uniformly in} \quad t\in[0,1],$$

and therefore given $\varepsilon>0$ small, there is a $\lambda>0$ such that

$$F^*(t,v)\le \varepsilon|v|^2, \quad \forall\, t\in[0,1] \quad \text{and} \quad |v|\le\lambda.$$

Then, since $G^*\ge 0$, for sufficiently small $\epsilon>0$ it follows that

$$\begin{aligned}\Psi(\lambda) &= -\frac{1}{2}\int_0^1 p(K\lambda)\lambda\,dt+\int_0^1 pF^*(t,\lambda)\,dt\\ &\quad - G^*\bigg(\int_0^1 p(t)\lambda\,dt\bigg)\\ &\le -\frac{\lambda^2}{2}\,\theta_K+\varepsilon\,\lambda^2\theta_1<0,\end{aligned}$$

where $\theta_K=\int_0^1 pK(1)\,dt>0$. Finally, since $\Psi(0)=0$, it follows that Ψ has a nonzero global minimum in $L_p^2[0,1]$. □

Theorem 2. *Assume that* p *and* f *satisfy the hypotheses of Theorem* 1. *If* $g:(a,b)\to\mathbf{R}$ *is a decreasing singular function satisfying* (2.3)*, then the problem* (1.1) *has a sign changing solution if*

$$\lim_{u\to 0}\frac{u}{g(u)}=0. \tag{3.5}$$

Proof. Let us put $\bar g=-g$ so that $\bar g$ is increasing and $\overline{G}(u)=\int_0^u \bar g(s)\,ds$ is convex. Then, we can infer, as in Lemma 3, that if $v\in L_p^2[0,1]$ is a critical point of the modified functional

$$\Psi_1(v)=-\frac{1}{2}\int_0^1 p(Kv)v\,dt+\int_0^1 pF^*(t,v)\,dt+\overline{G}^*\bigg(-\int_0^1 pv\,dt\bigg), \tag{3.6}$$

then there exists $c_v\in\mathbf{R}$ such that $y=Kv-c_v$ is a solution of the problem (1.1). Besides, Ψ_1 is also C^1 and weakly lower semi-continuous in $L^2_p[0,1]$. Since $\overline{G}^*$ has linear growth, as before, Ψ_1 has a global minimum in $L^2_p[0,1]$.

Now, from conditions (3.3) and (3.5) there exist $\varepsilon > 0$ and $\lambda > 0$, small, such that

$$\Psi(\lambda) \leq -\frac{\lambda^2}{2}\,\theta_K + \varepsilon\,\lambda^2\theta_1 + \varepsilon\,\lambda^2\theta_1^2 < 0,$$

which shows that the global minimum of Ψ_1 is nonzero.

Finally, we note that since

$$y \neq 0 \quad \text{and} \quad \int_0^1 pf(t, y(t))\,dt - g(y(1)) = 0,$$

and $f, -g$ are strictly increasing, it follows from (2.2) that y must change signs in $[0,1]$. $\square$

Until now we have assumed that $f(t,y)$ has sublinear growth in y. Now we shall suppose that f is increasing and has superlinear growth. Then, clearly $\int_0^1 F^*(t,v)\,dt$ has subquadratic growth and cannot dominate $\int_0^1 pKvv\,dt$, and therefore Ψ is not bound from below anymore. To deal with this kind of indefinite functional, we will apply the well-known mountain pass theorem of Ambrosetti and Rabinowitz [**7**]. For this, suppose that Ψ satisfies the geometrical conditions

($mp1$) $\Psi(0) = 0$,

($mp2$) there exist $\rho, r > 0$ such that $\Psi(v) \geq \rho$ if $\|v\|_{L_p^\sigma} = r$,

($mp3$) there exists e such that $\|e\|_{L_p^\sigma} > r$ and $\Psi(e) < 0$.

Then, there exists a sequence v_n such that $\Psi'(v_n) \to 0$ and $\Psi(v_n) \to c$, with $c \geq \rho$. In addition, if the Palais-Smale compactness condition holds, then c is a critical value of Ψ.

Theorem 3. *Assume that the basic hypotheses on p, f, g hold. Assume further that f is increasing and satisfies*

$$Au^{\sigma-1} \leq f(t,u) \leq Bu^{\sigma-1}, \quad \forall\, u \in \mathbf{R},\ \forall\, t \in [0,1], \tag{3.7}$$

with

$$A > 0, \quad \sigma > 2, \quad \frac{B}{A} < \left(\frac{2}{\sigma'}\right)^{\sigma-1}. \tag{3.8}$$

Then, if $g : (a,b) \to \mathbf{R}$ *is a decreasing singular function satisfying* (3.5)*, the problem* (1.1) *has a sign changing solution.*

Proof. Let us put $\bar{g} = -g$. Then, as in Theorem 2, if $v \in L_p^{\sigma'}[0,1]$ is a critical point of $\Psi_1 : L_p^{\sigma'}[0,1] \to \mathbf{R}$ defined by (3.6), then there exists $c_v \in \mathbf{R}$ such that $y = Kv - c_v$ is a solution of problem (1.1).

From (3.7) and (2.5), we have

$$B^*|v|^{\sigma'} \le F^*(t,v) \le A^*|v|^{\sigma'}, \quad \forall\, v \in \mathbf{R},\ \forall\, t \in [0,1].$$

Now, since $\sigma' < 2$ and $\overline{G}^* \ge 0$,

$$\Psi_1(v) \ge -\frac{1}{2}\,\theta_2\,\theta_1^{2/\sigma}\|v\|^2_{L_p^{\sigma'}} + B^*\|v\|^{\sigma'}_{L_p^{\sigma'}} \ge \rho > 0,$$

if $\|v\|_{L_p^{\sigma'}} = r$ is sufficiently small. This shows that condition $(mp2)$ holds. Next, since $\overline{G}^*$ has linear growth, there exists a $C > 0$ such that

$$\begin{aligned}\Psi_1(\lambda) = &-\frac{1}{2}\int_0^1 p(K\lambda)\lambda\,dt + \int_0^1 pF^*(t,\lambda)\,dt \\ &+ \overline{G}^*\left(-\int_0^1 p\lambda\,dt\right) \\ &\le -\frac{\lambda^2}{2}\,\theta_K + \lambda^{\sigma'}\theta_1 A^* + C(\lambda\theta_1 + 1) \longrightarrow -\infty,\end{aligned}$$

as $\lambda \to \infty$, and so $(mp3)$ holds. Therefore, from the Mountain Pass theorem, there exists v_n such that $\Psi_1(v_n) \to c > 0$ and $\Psi_1'(v_n) \to 0$.

Next, we shall show that there is $v \in L_p^{\sigma'}[0,1]$ such that $\Psi_1(v_n) \to \Psi_1(v) = c$ and $\Psi_1'(v) = 0$, i.e., v is a nonzero critical point of Ψ_1. It means that Ψ_1 enjoys a weaker form of Palais-Smale condition. This kind of argument has been used earlier in, e.g., [**11, 12**].

By computing $2\Psi_1(v_n) - \langle \Psi_1(v_n), v_n\rangle$, we infer that

$$(3.9) \qquad \int_0^1 p[2F^*(t,v_n) - f^*(t,v_n)v_n]\,dt + h(v_n) \le c + o(1)\|v_n\|_{L_p^{\sigma'}},$$

where

$$h(v_n) = 2\overline{G}^*\left(-\int_0^1 pv_n\,dt\right) - \overline{g}^*\left(-\int_0^1 pv_n\,dt\right)\int_0^1 pv_n\,dt.$$

Using (3.8) we conclude that

$$2F^*(t, v_n) - f^*(t, v_n)v_n \geq \left(\frac{2}{\sigma'} B^{-\sigma'/\sigma} - A^{-\sigma'/\sigma}\right)|v_n|^{\sigma'} = C|v_n|^{\sigma'},$$

where $C > 0$ stands for several different constants. Then, since $|h(v_n)| \leq C\|v_n\|_{L_p^{\sigma'}}$, we conclude from (3.9) that

$$\|v_n\|_{L_p^{\sigma'}}^{\sigma'} \leq C + C\|v_n\|_{L_p^{\sigma'}}.$$

Noting that $\sigma' > 1$, this shows that $\|v_n\|_{L_p^{\sigma'}}$ is bounded. Hence there is a v such that $v_n \to v$ weakly in $L_p^{\sigma'}[0, 1]$.

Since K is compact and $\bar{g}^*$ is weakly continuous, from $\langle \Psi_1'(v_n), v_n - v\rangle \to 0$, we see that

$$\int_0^1 pf^*(t, v_n)(v - v_n)\, dt \to 0 \quad \text{as} \quad n \to \infty,$$

and now, the convexity of F^* implies that

$$\int_0^1 pF^*(t, v)\, dt = \lim_{n\to\infty} \int_0^1 pF^*(t, v_n)\, dt.$$

Hence, the compactness of K and the weak continuity of $\overline{G}^*$ show that $\Psi_1(v_n) \to \Psi_1(v) = c$.

It remains to show that $\Psi_1'(v) = 0$. For this, given $\phi \in L_p^{\sigma'}[0, 1]$, we have

$$\begin{aligned}\langle \Psi_1'(v_n), v_n - \phi\rangle = &- \int_0^1 p(Kv_n)(v_n - \phi)\, dt \\ &+ \int_0^1 pf^*(t, \phi)(v_n - \phi)\, dt \\ &- \bar{g}^*\left(-\int_0^1 pv_n\, dt\right) \int_0^1 p(v_n - \phi)\, dt.\end{aligned}$$

From the monotonicity of f^* and letting $n \to \infty$, we get

$$0 \geq \int_0^1 p[(Kv) + f^*(t, \phi)](v - \phi)\, dt - \bar{g}^*\left(-\int_0^1 pv\, dt\right) \int_0^1 p(v - \phi)\, dt.$$

Taking $\phi = v + \lambda w$, $\lambda > 0$ and $w \in C[0,1]$, we obtain

$$0 \geq -\lambda \int_0^1 p[(Kv) + f^*(t, v+\lambda w)]w\,dt - \lambda \bar{g}^*\left(-\int_0^1 pv\,dt\right)\int_0^1 pw\,dt.$$

Dividing by λ and then letting $\lambda \to 0$, we find

$$0 \leq \int_0^1 p[(Kv) + f^*(t, v)]w\,dt - \lambda \bar{g}^*\left(-\int_0^1 pv\,dt\right)\int_0^1 pw\,dt.$$

Finally, changing λ by $-\lambda$, we get the reverse inequality. Then,

$$\int_0^1 p[(Kv) + f^*(t, v)]w\,dt - \lambda \bar{g}^*\left(-\int_0^1 pv\,dt\right)\int_0^1 pw\,dt = 0, \\ w \in C[0,1]$$

i.e., $\Psi_1'(v) = 0$.

We note that this solution y changes sign as discussed in the proof of Theorem 3. □

To conclude this paper, we consider the existence of multiple solutions. The result follows from minimization arguments on open manifolds.

Theorem 4. *Assume that the basic hypotheses on p, f, g hold with $f(t,y)$ and $g(y)$ both decreasing with respect to y. Assume further that* (3.3) *holds and*

$$|f(t,u)| \leq C_f(1 + |u|^{\sigma-1}), \quad \sigma > 1, \tag{3.10}$$

for some $C_f > 0$. Then, if there exists a $\nu > 2$ such that

$$\lim_{u\to 0} \frac{g(u)}{u^{\nu-1}} = 0, \tag{3.11}$$

the problem (1.1) *has at least two solutions.*

Proof. Let us put $\bar{f} = -f$. Then, $\bar{f}(t,u)$ is increasing and $\overline{F}(t,u)$ is convex, with respect to u. We seek critical points for the new functional $\Psi_2 : L_p^{\sigma'}[0,1] \to \mathbf{R}$ given by

(3.12)
$$\Psi_2(v) = \frac{1}{2}\int_0^1 p(Kv)v\,dt + \int_0^1 p(t)\overline{F}^*(t,-v)\,dt - \overline{G}^*\bigg(-\int_0^1 pv\,dt\bigg),$$

which is C^1 and weakly lower semi-continuous. As in Lemma 3, we conclude that, to each critical point v of Ψ_2, there exists $c_v \in \mathbf{R}$ such that $y = Kv - c_v$ is a solution of problem (1.1).

The assumption (3.10) implies that $\overline{F}^*(t,v) \geq C_f^*|v|^{\sigma'} - C_1$, for some $C_1 > 0$, and therefore Ψ_2 is coercive in $L_p^{\sigma'}[0,1]$.

Next we write $L_p^{\sigma'}[0,1] = \mathbf{R} \oplus W$ as defined in (2.4) and put

$$\mathcal{O}^{\pm} = \Big\{\lambda + w \in L_p^{\sigma'}[0,1] \mid \pm\lambda > 0 \text{ and } w \in W\Big\}.$$

We show that Ψ_2 has a critical point in each one of the open sets $\mathcal{O}^{\pm}$. Since for $w \in W$ one has $\overline{G}^*(-\int_0^1 pw) = 0$, we see that

$$\Psi_2(w) = \frac{1}{2}\int_0^1 p(Kw)w\,dt + \int_0^1 p\overline{F}^*(t,-w)\,dt \geq 0, \quad \forall\, w \in W.$$

On the other hand, as in Theorem 1, conditions (3.3) and (3.11) imply the existence of $\varepsilon, \gamma, \delta > 0$ such that

$$\overline{F}^*(t,v) \leq \varepsilon|v|^2 \quad \text{and} \quad \overline{G}^*(v) \geq \gamma\,|v|^{\nu'}, \quad \text{if} \quad |v| < \delta, \quad \forall\, t \in [0,1].$$

Therefore, since $\nu' < 2$, for $\lambda \in \mathbf{R}$ sufficiently small, we have

$$\Psi_2(\lambda) \leq \frac{\lambda^2}{2}\,\theta_K + \varepsilon\,\theta_1|\lambda|^2 - \gamma\,\theta_1^{\nu'}|\lambda|^{\nu'} < 0.$$

Hence, we conclude that

$$m^{\pm} = \inf_{\mathcal{O}^{\pm}} \Psi_2 < \inf_W \Psi_2 = 0.$$

Since $W = \partial\mathcal{O}^{\pm}$, there exist two minimizing sequences, $v_n^+ \in \mathcal{O}^+$ and $v_n^- \in \mathcal{O}^-$ such that $\Psi_2(v_n^{\pm}) \to m^{\pm}$. These sequences are bounded because Ψ_2 is coercive. Then, the minima are attained because the weakly lower semi-continuity of Ψ_2, or alternatively, from the Palais-Smale condition. □

4. Examples. The existence theory established in Section 3 can be applied to concrete problems rather easily. For this, we illustrate the following two examples. Our first example involves a sublinear increasing f, whereas in the second example f is superlinear decreasing.

Example 1. Suppose that $\alpha, \beta \in (0,1)$ and $q \in C[0,1]$ is a positive function. Then, the problem

$$\begin{cases} t^{-\alpha}(t^{\alpha}y')' + q(t)|y|^{\beta-1}y = 0 \quad 0 < t < 1, \\ \lim_{t\to 0^+} t^{\alpha}y'(t) = 0, \\ y'(1) + \tan y(1) = 0, \end{cases} \tag{4.1}$$

has a nonzero solution. In fact, letting $p(t) = t^{\alpha}$, $g(y) = \tan y$ and $f(t,y) = q(t)|y|^{\beta-1}y$, we see that all the assumptions of Theorem 1 are satisfied. □

Example 2. Suppose that $\alpha \in (0,1)$ and $q \in C[0,1]$ is a positive function. Then, the problem

$$\begin{cases} t^{-\alpha}(t^{\alpha}y')' - q(t)(y^3 + |y|^{-1/2}y) = 0 \quad 0 < t < 1, \\ \lim_{t\to 0^+} t^{\alpha}y'(t) = 0, \\ y'(1) + g(y(1)) = 0, \end{cases} \tag{4.2}$$

where

$$g(y) = \begin{cases} (-y^{\mu-1})/(y+1) & -1 < y \le 0 \\ (y^{\mu-1})/(y-1) & 0 \le y < 1, \end{cases}$$

has at least two nonzero solutions if $\mu > 2$. In fact, letting $f(t,y) = -q(t)(y^3 + |y|^{-1/2}y)$, we see that

$$\lim_{u\to 0} \frac{u}{f(t,u)} = 0,$$

and taking $\mu > \nu > 2$, we have

$$\lim_{u\to 0} \frac{g(u)}{u^{\nu-1}} = 0.$$

Now, it is straightforward to verify the hypotheses of Theorem 4 with $p(t) = t^{\alpha}$. $\square$

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