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BELLMAN FUNCTIONS AND MRA WAVELETS

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ABSTRACT. In this paper, we discuss how far the method of Bellman functions can be generalized from use with Haar functions to use with general MRA wavelets.

1. Introduction. The method of Bellman functions is a very powerful method in harmonic analysis which has been used to prove a lot of interesting results, see [**3**–**7**] for just a few. In its original form, it is intimately connected with the Haar function system, as it is used to estimate sums involving Haar coefficients. A natural question arises: Can the method be generalized to work with coefficients of general multi-resolution-analysis wavelets? This paper strives to answer this question. The answer is yes, but the Bellman function will be much more difficult to find in this general case.

The method itself, namely the proof of the result given the Bellman function, generalizes with only one change: The inequality conditions we need the Bellman function to satisfy are summed versions of the conditions in the Haar wavelet case. The big difference comes in the application of the method, namely when we are looking for the Bellman function. In the Haar case, the scaling sequence has only positive terms. In general, the terms can be negative. This means that we cannot use the Cauchy-Schwarz inequality to define the domain of the Bellman function, and it also makes the usual differential inequality difficult to work with for general wavelets.

In this paper, we will first present a simple Bellman proof of the bound of the discrete square function, which is based on the Haar function system. We then discuss which aspects of a Haar-based proof need to be adjusted when we are working with wavelets. To illustrate the Bellman method when used with wavelets, we then give a Bellman proof of the bound of the wavelet square function.

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2. Notation. In this paper, I, J, M and N denote dyadic intervals, i.e., intervals of the form $[k2^i, (k+1)2^i)$ where k and i are integers. The halves of a dyadic interval I are again dyadic intervals and are denoted by I_l (left half) and I_r (right half). The set of all dyadic intervals will be denoted by D. Let ϕ be a scaling function with corresponding wavelet ψ.

For $I = [k2^i, (k+1)2^i)$, let

$$\phi_I(x) = \phi_{i,k} = \phi\left(\frac{x - 2^i k}{2^i}\right)\frac{1}{2^i}$$
$$\psi_I = \psi_{i,k} = \psi\left(\frac{x - 2^i k}{2^i}\right)\frac{1}{2^i}.$$

The simplest example of a scaling function is $H(x) = \chi_{[0,1)}(x)$. The corresponding wavelet is the well-known Haar wavelet, $h(x) = (1/2)H_{[1/2,1)}(x) - (1/2)H_{[0,1/2)}(x)$.

Every MRA wavelet has a scaling sequence α_k, for which

$$\phi(x) = \sum_k \alpha_k \phi_{-1,k}(x),$$

and a corresponding sequence for the wavelet for which

$$\psi(x) = \sum_k \gamma_k \phi_{-1,k}(x).$$

In this paper we will work only with wavelets that have real scaling sequences. For these, $\gamma_k = (-1)^{1-k}\alpha_{1-k}$. In the case of the Haar wavelet, the scaling sequence is $\alpha_0 = 1/2$, $\alpha_1 = 1/2$, $\alpha_i = 0$ for all $i \neq 0, 1$.

Scaling sequences α satisfy the following properties, see, for example, [**1, 2**]:

(1) $\sum_k \alpha_k = 1$

(2) $\sum_k (\alpha_k)^2 = 1/2$

(3) $\sum_k (-1)^{1-k}\alpha_{1-k} = \sum_k \gamma_k = 0$.

(4) The scaling sequence is orthogonal to even translates of itself, i.e., $\sum_k \alpha_k \alpha_{k+2s} = 0$ for $s > 0$ an integer.

(5) $\sum_{k\,\text{odd}} \alpha_k = \sum_{k\,\text{even}} \alpha_k = 1/2$.

Note. I am normalizing my scaling function and its dilates in L^1, rather than in L^2 which might be more customary.

We will use the following notation:

$$x_I = x_{i,k} = \langle f, \phi_{i,k} \rangle = \langle f, \phi_I \rangle$$

and

$$y_I = y_{i,k} = \langle f, \psi_{i,k} \rangle = \langle f, \psi_I \rangle.$$

We will switch between these two styles of subscripts depending on what is more intuitive in a given situation.

By the scaling equalities, we have

$$x_I = x_{i,k} = \sum_s \alpha_{s-2k} x_{i-1,s} = \sum_{J:|J|=|I|/2} \alpha_{J,I} x_J$$

and

$$y_I = y_{i,k} = \sum_s \gamma_{s-2k} x_{i-1,s}, = \sum_{J:|J|=|I|/2} \gamma_{J,I} x_J,$$

where $\alpha_{s-2k} = \alpha_{J,I}$ for

$$\begin{aligned} I &= [k2^i, (k+1)2^i), \\ J &= [s2^{i-1}, (s+1)2^{i-1}) \end{aligned}$$

and similarly for $\gamma_{J,I}$.

For $g \in L^2(dx)$, we define the wavelet square function as follows:

$$S_\psi g(x) = \left(\sum_{I \in D : x \in I} \langle g, \psi_I \rangle^2 \right)^{1/2}.$$

Then

$$\|S_\psi g(x)\|_{L^2}^2 = \sum_{I \in D} \langle g, \psi_I \rangle^2 |I|.$$

In the case where $\psi(x) = h(x)$, the Haar wavelet, we get the dyadic square function which is used frequently in harmonic analysis:

$$S_d g(x) = \left(\sum_{I \in D : x \in I} \langle g, h_I \rangle^2 \right)^{1/2}.$$

3. A traditional Bellman function proof. We will prove the following theorem using the method of Bellman functions.

Theorem 3.1.

$$\|S_d g\|_{L^2} \leq \|g\|_{L^2}.$$

Note that this is well known and follows from Bessel's inequality. We present this result here only for the purpose of illustration.

Proof. By the scaling equality, we can write

$$\|S_d g\|_{L^2}^2 = \frac{1}{4} \sum_{I \in D} \langle g, H_{I_r} - H_{I_l} \rangle^2 |I|.$$

We will prove

$$\frac{1}{4} \frac{1}{|J|} \sum_{I \subseteq J} \left(\langle g, H_{I_r} \rangle - \langle g, H_{I_l} \rangle \right)^2 |I| \leq \langle g^2, H_J \rangle = \frac{1}{|J|} \int_J g^2(x)\, dx$$

for any dyadic interval J. Then the desired result will follow by letting $|J| \to \infty$.

To simplify some of the formulas, we will use the following notation: $g_I = \langle g, H_I \rangle$ and $G_I = \langle g^2, H_I \rangle$.

Let $B(x, u) = u - x^2$ in the domain $x^2 \leq u$. This is what we refer to as the Bellman function of the problem. It satisfies the following two crucial properties:

(1) $u \geq B(x, u) \geq 0$.

(2) $B(x, u) \geq 1/4(x_- - x_+)^2 + (B(x_-, u_-) + B(x_+, u_+))/2$ for $(x_-, u_-), (x_+, u_+)$ and (x, u) in the domain such that

$$\frac{(x_-, u_-) + (x_+, u_+)}{2} = (x, u).$$

The second inequality is actually an equality. However, as inequality is sufficient for the proof, and we can not in general expect to have equality, we shall write it as an inequality in order to make it more like the general case.

The very existence of a function with these properties implies the bound, as we will now see.

Let $x_- = g_{J_l}$, $x_+ = g_{J_r}$, $x = g_J$ and $u_- = G_{J_l}$, $u_+ = G_{J_r}$ and $u = G_J$. These pairs (x_-, u_-), (x_+, u_+) and (x, u) are all in the domain of B, since by the Cauchy-Schwarz inequality, $(1/|I| \int_I g)^2 \leq 1/|I| \int_I g^2$ for any interval I. Furthermore, for these variables, we have $((x_-, u_-) + (x_+, u_+))/2 = (x, u)$. Thus we can apply the two inequalities to $B(g_J, G_J)$:

$$G_J \geq B(g_J, G_J) \geq \frac{1}{4}\left(g_{J_l} - g_{J_r}\right)^2 + \frac{B(g_{J_l}, G_{J_l}) + B(g_{J_r}, G_{J_r})}{2}$$

Apply the second inequality again, this time to $B(g_{J_l}, G_{J_l})$, with $x_- = g_{J_{l_l}}$, $x_+ = g_{J_{l_r}}$ and $x = g_{J_l}$, $u_- = G_{J_{l_l}}$, $u_+ = G_{J_{l_r}}$ and $u = G_{J_l}$ and similarly to $B(g_{J_r}, G_{J_r})$. Then

$$\begin{aligned}
G_J &\geq \frac{1}{4}\left(g_{J_l} - g_{J_r}\right)^2 + \frac{B(g_{J_l}, G_{J_l}) + B(g_{J_r}, G_{J_r})}{2} \\
&\geq \frac{1}{4}\left(g_{J_l} - g_{J_r}\right)^2 + \frac{1}{8}\left(g_{(J_l)_r} - g_{(J_l)_l}\right)^2 + \frac{1}{8}\left(g_{(J_r)_l} - g_{(J_r)_r}\right)^2 \\
&\quad + \frac{B(g_{(J_l)_l}, G_{(J_l)_l}) + B(g_{(J_l)_r}, G_{(J_l)_r})}{4} \\
&\quad + \frac{B(g_{(J_r)_l}, G_{(J_r)_l}) + B(g_{(J_r)_r}, G_{(J_r)_r})}{4}.
\end{aligned}$$

Repeat this procedure n times, each time applying the inequality to the B's on the right, expressing $B(g_M, G_M)$ in terms of $B(g_{M_l}, G_{M_l}), B(g_{M_r}, G_{M_r})$ and $(g_{M_l} - g_{M_r})^2$.

We get

$$\begin{aligned}
G_J \geq \frac{1}{4} &\sum_{I \subseteq J, |I| \geq 2^{-n}|J|} \left(g_{I_l} - g_{I_r}\right)^2 \frac{|I|}{|J|} \\
&+ \sum_{K \subseteq J, |K| = 2^{-n-1}|J|} B(g_K, G_K) \left(\frac{1}{2}\right)^{n+1}.
\end{aligned}$$

The $|I|/|J|$ stems from the factors $1/2$ that we get with each repetition.

Since $B(g_K, G_K)$ is always positive, we can simply omit the second sum without changing the inequality, i.e.,

$$\frac{1}{|J|}\int_J g^2(x)\,dx = G_J \geq \frac{1}{4} \sum_{\substack{I\subseteq J\\ |I|/|J|\geq 2^{-n}}} (g_{I_l} - g_{I_r})^2 \frac{|I|}{|J|}.$$

Multiplying both sides by $|J|$ and letting $n \to \infty$ establishes

$$\int_J g^2(x)\,dx \geq \frac{1}{4}\sum_{I\subseteq J} (g_{I_l} - g_{I_r})^2 \,|I|.$$

Since this is true for any dyadic interval J, it follows that

$$\int_{\mathbf{R}} g^2(x)\,dx \geq \frac{1}{4} \sum_{I \text{ dyadic}} (g_{I_l} - g_{I_r})^2 \,|I| = \|S_d g\|^2_{L^2}. \qquad \square$$

4. The extended Bellman function method. In this section we will investigate how the conditions for the Bellman function need to change in the general wavelet case. A Bellman function can have any number of variables, but in this discussion, we will use a Bellman function of one variable for simplicity.

In the Haar wavelet situation, where $x_I = (x_{I_l} + x_{I_r})/2$, we require our Bellman function to satisfy a condition of the form

$$B(x_I) \geq \text{ term for sum } + \frac{1}{2}\,(B(x_{I_l}) + B(x_{I_r})),$$

where the "term for sum" depends on the summand of the quantity to be estimated.

For general wavelets, $x_I = \sum_{J:|J|=|I|/2} \alpha_{J,I} x_J$. Therefore, a possible guess at a generalization to other wavelets might be to require the condition

$$B(x_I) \geq \text{ term for sum } + \sum_{J:|J|=|I|/2} \alpha_{J,I} B(x_J).$$

However, even in a simple situation, such as estimating the norm of the wavelet square function, such a Bellman function can fail to exist, as we will show in the following example.

In the case of the wavelet square function, the above condition would be

$$B(x_I, u_I) \geq \Bigg(\sum_{J:|J|=|I|/2} \gamma_{J,I} x_J \Bigg)^2 + \sum_{J:|J|=|I|/2} \alpha_{J,I} B(x_J, u_J).$$

Writing this in integer subscript notation, we see that we would need to find B satisfying

$$B(x_{i,k}, u_{i,k}) - \sum_s \alpha_{s-2k} B(x_{i-1,s}, u_{i-1,s}) \geq \Bigg(\sum_s (-1)^{1-(s-2k)} \alpha_{1-(s-2k)} x_{i-1,s} \Bigg)^2.$$

But say the wavelet is the sine wavelet (see, for example [**1**]), whose scaling sequence is $\alpha_n = \sin(\pi n/2)/(\pi n/2)$. Then $\alpha_4 = 0$, but $\alpha_{-3} = c \neq 0$. Let $x_{i-1,4} = a > 0$, $x_{i-1,s} = 0$, $s \neq 4$. Then $x_{i,0} = 0$, and

$$B(x_{i,0}) - \sum_s \alpha_s B(x_{i-1,s}) = B(0) - \sum_s \alpha_s B(0) = 0,$$

since $\sum_s \alpha_s = 1$. Thus the left-hand side is 0, but the right-hand side, $(\sum_s (-1)^{1-s} \alpha_{1-s} x_{i-1,s})^2 = (ac)^2$, is positive. Thus, our naive guess for the Bellman inequality fails, since even in the case of a wavelet as simple as the sine wavelet, the condition can not be satisfied. However, a small modification will solve this problem.

Consider the summed version of the previous condition,

$$\sum_{|I|=2^i} B(x_I) \geq \sum_{|I|=2^i} \sum_{J:|J|=|I|/2} \alpha_{J,I} B(x_J) + \sum_{|I|=2^i} \text{term in sum}.$$

The double sum can be simplified

$$\sum_{|I|=2^i} \sum_{J:|J|=|I|/2} \alpha_{J,I} B(x_J) = \sum_{J:|J|=2^{i-1}} \frac{1}{2} B(x_J),$$

since
$$\sum_k \sum_s \alpha_{s-2k} B(x_{i-1,s}) = \sum_s B(x_{i-1,s}) \sum_k \alpha_{s-2k} = \sum_s \frac{1}{2} B(x_{i-1,s}).$$
In the last equality, we used property 5 of scaling sequences.

Thus the convexity condition of Bellman function method now reads
$$\sum_{|I|=2^i} B(x_I) \geq \frac{1}{2} \sum_{J:|J|=|I|/2} B(x_J) + \sum_{|I|=2^i} \text{term in sum.}$$
which looks remarkably like the condition in the Haar wavelet case. We will see that this condition can be fulfilled and is the right one to work with.

The conditions for the bounds of the Bellman function become the summed versions of those in the Haar function case as well. Again it is easy to see that it not possible to use the original version. Terms like $u_I = \langle g^2, \phi_I \rangle$ no longer need to be positive, and so a condition like
$$0 \leq B(x_I, u_I) \leq C u_I,$$
as we had in the discrete square function case, becomes meaningless. The condition
$$0 \leq \sum_{J:|J|=2^i} B(x_J, u_J) \leq C \sum_{J:|J|=2^i} u_J,$$
however does make sense. This follows from the following claim.

Claim. *For continuous, compactly supported scaling functions ϕ,*
$$\sum_k \phi(2^{-i}x - k) = 1.$$

Proof. We can see this by using the scaling inequality in two different ways:
$$\begin{aligned}\sum_k \phi(x-k) &= \sum_k \sum_s \alpha_s \phi_{-1,s}(x-k) = 2 \sum_k \sum_s \alpha_s \phi(2x - (2k+s)) \\ &= 2 \sum_{s,k} \alpha_{s-2k} \phi(2x - s) = \sum_s \phi(2x-s) 2 \sum_k \alpha_{s-2k} \\ &= \sum_s \phi(2x-s),\end{aligned}$$
by property 5 of scaling sequences.

Note that the interchange of sums is justified because ϕ is compactly supported, and thus all sums are finite.

By induction,

$$\sum_k \phi(2^{-i}x - k) = \sum_k \phi(x - k)$$

for all $i \in \mathbf{Z}$. Thus, letting $\sum_k \phi(x-k) = F(x)$, we have $F(x) = F(2^{-i}x)$. Since F is continuous, $F(x) = \lim_{i\to\infty} F(x/2^i) = F(0)$. Thus, the sum is indeed constant.

To show that $F(0) = 1$, observe that by the scaling equality $\sum_s \phi(s/2^i) = \sum_k 2\alpha_k \sum_s \phi((s/2^{i-1}) - k) = \sum_k 2\alpha_k \sum_s \phi(s/2^{i-1}) = 2\sum_s \phi(s/2^{i-1})$.

By induction, $\sum_s \phi(s/2^i) = 2^i \sum_s \phi(s)$ for $i \geq 0$.

But $\lim_{i\to\infty} \sum_s 2^{-i}\phi(s/2^i) = \int \phi(x)\,dx = 1$ (Riemann sum), so $\sum_s \phi(s) = 1$. $\square$

Thus, for any positive function f,

$$\sum_k \langle f, \phi_I \rangle = \frac{1}{|I|} \int f \geq 0,$$

and thus a condition like

$$0 \leq \sum_{J:|J|=2^i} B(x_J) \leq C \sum_{J:|J|=2^i} \langle f, \phi_J \rangle$$

is meaningful. It is also the natural condition to ask for in light of the fact that the difference inequality must also be summed.

In the conventional Bellman function method, we often can translate the difference inequality into a differential condition, which helps greatly in finding the Bellman function. The following shows that we can still do this in the wavelet case. A word of warning, however: Since the α's are not all positive, this condition may be too complicated to be useful in most cases.

For clarity of exposition, we show how to convert the inequality for $B(x)$ a function of one variable only, but the same can be done for a function of any number of variables. In that case, B'' is replaced by D^2B, the matrix of second partial derivatives.

We will use Taylor's theorem with second degree remainder. $\xi_{k,s}$ is a point between $x_{i-1,s}$ and $x_{i,k}$.

$$\begin{aligned}
&\sum_k \left(B(x_{i,k}) - \sum_s \alpha_{s-2k} B(x_{i-1,s}) \right) \\
&\quad = -\sum_k \sum_s \alpha_{s-2k}(B(x_{i-1,s}) - B(x_{i,k})) \\
&\quad = -\sum_{s,k} \alpha_{s-2k}(B'(x_{i,k})(x_{i-1,s} - x_{i,k}) + B''(\xi_{k,s})(x_{i-1,s} - x_{i,k})^2) \\
&\quad = -\sum_{s,k} \alpha_{s-2k} B''(\xi_{k,s})(x_{i-1,s} - x_{i,k})^2
\end{aligned}$$

where the last equality follows because

$$\sum_s \alpha_{s-2k}(x_{i-1,s} - x_{i,k}) = 0$$

by the definition of the $x_{i-1,s}$.

Thus, in terms of the interval subscript, the Bellman inequality can be expressed as

$$-\sum_{|I|=2^i} \sum_{J:|J|=|I|/2} \alpha_{J,I} B''(\xi_{J,I})(x_J - x_I)^2 \leq \sum_{|I|=2^i} \text{term in sum.}$$

If the α's are simple enough, this may be a useful condition. In the Haar wavelet case, where $\alpha_{J,I} = 1/2$ for $J \subsetneq I$ and 0 otherwise, this condition is much easier to work with than the difference inequality.

5. An example of the extended Bellman function method: Estimating the unweighted wavelet square function. We will apply the Bellman function method to a simple estimate: Bounding the wavelet square function.

Theorem 5.1.

$$\sum_I \langle f, \psi_I \rangle^2 |I| \leq \int f^2(x)\, dx$$

for ψ a continuous, compactly supported MRI wavelet, $f \in L^2$.

Proof. Let $u_J = \langle f^2, \phi_J\rangle$ and $x_J = \langle f, \phi_J\rangle$ where ϕ is the scaling function corresponding to the wavelet ψ.

Note that neither of these variables is necessarily positive. However, $\sum_{J:|J|=2^n} u_J = \langle f^2, \sum_{J:|J|=2^n} \phi_J\rangle = \langle f^2, 1/|J|\rangle = 2^{-n}\int f^2 \geq 0$ by the preceding section.

Lemma 5.2. *Assume there exists $B = B(x,u)$ such that*

$$0 \leq \sum_{I:|I|=2^n} B(x_I, u_I) \leq \sum_{I:|I|=2^n} u_I$$

and

$$\sum_{I:|I|=2^n} B(x_I, u_I) \geq \sum_{J:|J|=2^{n-1}} \frac{1}{2}\, B(x_J, u_J) + \sum_{I:|I|=2^n} \Bigg(\sum_{J:|J|=2^{n-1}} \gamma_{J,I} x_J \Bigg)^2$$

for all $n \in \mathbf{Z}$.

Then

$$\sum_{I\in D} \langle f, \psi_I\rangle^2 |I| = \sum_{I\in D} \Bigg(\sum_{J:|J|=|I|/2} \gamma_{J,I} x_J \Bigg)^2 |I| \leq \int f^2(x)\, dx.$$

Proof. Fix n.

$$\begin{aligned}
\sum_{I:|I|=2^n} u_I &\geq \sum_{I:|I|=2^n} B(x_I, u_I) \\
&\geq \sum_{J:|J|=2^{n-1}} \frac{1}{2}\, B(x_J, u_J) + \sum_{I:|I|=2^n} \Bigg(\sum_{J:|J|=2^{n-1}} \gamma_{J,I} x_J \Bigg)^2 \\
&\geq \sum_{K:|K|=2^{n-2}} \frac{1}{2^2}\, B(x_K, u_K) \\
&\quad + \frac{1}{2} \sum_{J:|J|=2^{n-1}} \Bigg(\sum_{K:|K|=2^{n-2}} \gamma_{K,J} x_K \Bigg)^2 \\
&\quad + \sum_{I:|I|=2^n} \Bigg(\sum_{J:|J|=2^{n-1}} \gamma_{J,I} x_J \Bigg)^2.
\end{aligned}$$

After l applications of the inequality, the right-hand side is

$$\sum_{M:|M|=2^{n-l}} \frac{|M|}{2^n} B(x_M, u_M) + \sum_{M:2^{n-l}<|M|\le 2^n} \frac{|M|}{2^n}\Bigg(\sum_{N:|N|=|M|/2} \gamma_{N,M} x_N\Bigg)^2 \ge \sum_{M:2^{n-l}<|M|\le 2^n} \frac{|M|}{2^n}\Bigg(\sum_{N:|N|=|M|/2} \gamma_{N,M} x_N\Bigg)^2$$

where the last inequality follows from the positivity of

$$\sum_{M:|M|=2^{n-l}} B(x_M, u_M).$$

Let $l \to \infty$. Then we have

$$\sum_{I:|I|=2^n} u_I|I| \ge \sum_{M:|M|\le 2^n} |M|\Bigg(\sum_{N:|N|=|M|/2} \gamma_{N,M} x_N\Bigg)^2.$$

Recall that

$$\sum_{I:|I|=2^n} u_I|I| = \int f^2,$$

and

$$\sum_{M:|M|\le 2^n} |M|\Bigg(\sum_{N:|N|=|M|/2} \gamma_{N,M} x_N\Bigg)^2 = \sum_{M:|M|\le 2^n} |M|\langle f, \psi_M\rangle^2.$$

Thus, taking limits as $n \to \infty$, we have proven that

$$\sum_{M\in D} \langle f, \psi_M\rangle^2 |M| \le \int f^2(x)\, dx. \qquad \square$$

By this lemma, what we now need to do to prove the theorem is to find a Bellman function which satisfies the inequalities. A good starting point is the Bellman function that proved the analogous result for Haar functions, $B(x,u) = u - x^2$. The challenge in the wavelet case is to prove that this Bellman function indeed satisfies the inequalities we need. As it will turn out, we will be able to show that this function not only satisfies the needed inequalities, but that equality is attained in the second one.

Claim.

$$B(x,u) = u - x^2$$

satisfies the inequalities of Lemma 5.2.

First, show that

$$0 \leq \sum_{J:|J|=2^n} u_J - {x_J}^2 \leq \sum_{J:|J|=2^n} u_J.$$

The upper bound is obvious.

As mentioned above, $\sum_{J:|J|=2^n} u_J = 2^{-n} \int f^2$, so to prove the lower bound, we need to show $\sum_{J:|J|=2^n} \langle f, \phi_J \rangle^2 \leq 2^{-n} \int f^2$. (A fact which we get for free in the Haar function case, because we can simply use the Cauchy-Schwarz inequality there.) This is true since the ϕ_J form an orthogonal set for each n with $\|\phi_J\|_2^2 = 1/|J|$.

What remains to be proven is that

$$\sum_{I:|I|=2^n} B(x_I, u_I) - \sum_{J:|J|=2^{n-1}} \frac{1}{2} B(x_J, u_J) \geq \sum_{I:|I|=2^n} \Bigg(\sum_{J:|J|=2^{n-1}} \gamma_{J,I} x_J \Bigg)^2.$$

Note that, as the function is linear in u, it will cancel out of the left-hand side. Thus, what we need to prove is

$$-\sum_{I:|I|=2^n} (x_I)^2 + \sum_{J:|J|=2^{n-1}} \frac{1}{2} (x_J)^2 \geq \sum_{I:|I|=2^n} \Bigg(\sum_{J:|J|=2^{n-1}} \gamma_{J,I} x_J \Bigg)^2,$$

which follows from

Lemma 5.3.

$$\sum_{J:|J|=2^{n-1}} \frac{1}{2}(x_J)^2 - \sum_{I:|I|=2^n}\left(\sum_{J:|J|=2^{n-1}} \alpha_{J,I} x_J\right)^2 - \sum_{I:|I|=2^n}\left(\sum_{J:|J|=2^{n-1}} \gamma_{J,I} x_J\right)^2 = 0.$$

Proof. We will look at the coefficients of the various combinations of x's.

Recall that

$$\gamma_{J,I} = \gamma_{s-2k} = (-1)^{1-s+2k}\alpha_{1-s+2k},$$

for $I = [k2^i, (k+1)2^i)$ and $J = [s2^{i-1}, (s+1)2^{i-1})$.

For the purpose of this proof, we will switch to integer subscripts for the scaling and wavelet sequences. Thus we need to prove

$$\sum_s \frac{1}{2}(x_{i-1,s})^2 - \sum_k\left(\sum_s \alpha_{s-2k}x_{i-1,s}\right)^2 - \sum_k\left(\sum_s (-1)^{1-s+2k}\alpha_{1-s+2k}x_{i-1,s}\right)^2 = 0.$$

(1) The coefficients for $(x_{i-1,l})^2$ for fixed l are

$$\frac{1}{2} - \sum_k(\alpha_{l-2k})^2 - \sum_k(\alpha_{1-l+2k})^2 = \left(\frac{1}{2} - \sum_k(\alpha_k)^2\right) = 0.$$

(2) Coefficients of $x_{i-1,l}x_{i-1,r}$ where $l-r$ is odd.

$$\begin{aligned}
2\Big(-\sum_k \alpha_{l-2k}\alpha_{r-2k} - \sum_k (-1)^{1-l+2k}\alpha_{1-l+2k}(-1)^{1-r+2k}\alpha_{1-r+2k}\Big) \\
= 2\Big(-\sum_k \alpha_{l-2k}\alpha_{r-2k} + \sum_k \alpha_{1-l+2k}\alpha_{1-r+2k}\Big) \\
= 2\Big(-\sum_\sigma \alpha_{2\sigma}\alpha_{2\sigma+r-l} + \sum_\sigma \alpha_{2\sigma+r-l}\alpha_{2\sigma}\Big) = 0
\end{aligned}$$

where the last step is a change of variables in both sums. If r is odd (and thus l even), we let $\sigma = (l-2k)/2$ in the first sum and $\sigma = (1-r+2k)/2$ in the second sum, and the other way around if l is the odd one.

(3) Coefficients of $x_l x_r$ where $l-r$ is even, $r \neq l$.

$$
\begin{aligned}
2\Big(-\sum_k \alpha_{l-2k}\alpha_{r-2k} - \sum_k (-1)^{1-l+2k}\alpha_{1-l+2k}(-1)^{1-r+2k}\alpha_{1-r+2k}\Big) \\
= 2\Big(-\sum_k \alpha_{l-2k}\alpha_{r-2k} - \sum_k \alpha_{1-l+2k}\alpha_{1-r+2k}\Big) \\
= 2\Big(-\sum_\sigma \alpha_{2\sigma}\alpha_{2\sigma+l-r} - \sum_\sigma \alpha_{2\sigma+l-r-1}\alpha_{2\sigma-1}\Big) = 0
\end{aligned}
$$

where the terms are 0 because they add up to $\sum_k \alpha_k \alpha_{k+l-r}$, i.e., an inner product of the scaling sequence with an even translate of itself. The changes of variables from k to σ need a bit of care; if l, r are even, then we let $\sigma = (r-2k)/2$ in the first sum and $\sigma = (2-l+2k)/2$ in the second sum. If l, r are odd, then $\sigma = (1+r-2k)/2$ in the first sum and $\sigma = (1-l+2k)/2$ in the second sum. $\square$

Thus we have established our second Bellman condition, and thus the estimate holds. $\square$

REFERENCES

1. C.S. Burrus, R.A. Gopinath and H. Guo, *Introduction to wavelets and wavelet transforms, A primer*, Prentice Hall, Upper Saddle River, New Jersey, 1998.

2. Michael W. Frazier, *An introduction to wavelets through linear algebra*, Springer-Verlag, New York, 1999.

3. F.L. Nazarov and S.R. Treil, *The hunt for a Bellman function*: *Applications to estimates for singular integral operators and to other classical problems of harmonic analysis*, Algebra Anal. **8** (1996), 32–162.

4. F. Nazarov, S. Treil and A. Volberg, *The Bellman functions and two-weight inequalities for Haar multipliers*, J. Amer. Math. Soc. **12** (1999), 909–928.

5. Stefanie Petermichl and Alexander Volberg, *Heating of the Ahlfors-Beurling operator*: *Weakly quasiregular maps on the plane are quasiregular*, Duke Math. J. **112** (2002), 281–305.

6. S. Petermichl and J. Wittwer, *A sharp estimate for weighted Hilbert transform via Bellman functions*, Michigan Math. J. **50** (2002), 71–88.

7. J. Wittwer, *A sharp bound for the martingale transform*, Math. Res. Lett. **7** (2000), 1–12.

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