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# A CENTRAL LIMIT THEOREM FOR GENERAL WEIGHTED SUMS OF LNQD RANDOM VARIABLES AND ITS APPLICATION

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ABSTRACT. In this paper we derive the central limit theorem for $\sum_{i=1}^{n} a_{ni}\,\xi_i$, where $\{a_{ni},\ 1 \le i \le n\}$ is a triangular array of nonnegative numbers such that $\sup_n \sum_{i=1}^{n} a_{ni}^2 < \infty$, $\max_{1\le i\le n} a_{ni} \to 0$ as $n \to \infty$ and $\xi_i$'s are a linearly negative quadrant dependent sequence. We also apply this result to consider a central limit theorem for a partial sum of a generalized linear process of the form $X_n = \sum_{j=-\infty}^{\infty} a_{k+j}\,\xi_j$.

**1. Introduction and results.** Lehmann [**8**] introduced a simple and natural definition of positive (negative) dependence: A sequence $\{\xi_i,\ 1 \le i \le n\}$ of random variables is said to be pairwise positive (negative) quadrant dependent (pairwise PQD (NQD)) if, for any real $\alpha_i, \alpha_j$ and $i \ne j$ $P(\xi_i > \alpha_i,\ \xi_j > \alpha_j) \ge (\le) P(\xi_i > \alpha_i)P(\xi_j > \alpha_j)$. Much stronger dependent concepts than PQD and NQD were considered by Esary, Proschan and Walkup [**4**] and Joag-Dev and Proschan [**6**], respectively. A sequence $\{\xi_i,\ 1 \le i \le n\}$ of random variables is said to be associated if, for any real coordinatewise increasing functions $f, g$ on $\mathbf{R}^n$, $\mathrm{Cov}\,(f(\xi_1, \dots, \xi_n),\ g(\xi_1, \dots, \xi_n)) \ge 0$ and $\{\xi_i,\ 1 \le i \le n\}$ is said to be negatively associated if, for any disjoint subsets, $A, B \subset \{1, 2, \dots, n\}$ and any real coordinatewise increasing functions $f$ on $\mathbf{R}^A$ and $g$ on $\mathbf{R}^B$, $\mathrm{Cov}\,(f(\xi_i,\ i \in A),\ g(\xi_i \in B)) \le 0$.

Instead of association (negative association) Newman's [**10**] central limit theorem requires only that positive linear combinations of the random variables are PQD (NQD). The definition of positive (negative) dependence introduced by Newman [**10**] is the following: A sequence $\{\xi_i,\ 1 \le i \le n\}$ of random variables is said to be linearly positive

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(negative) quadrant dependent (LPQD (LNQD)) if, for every pair of disjoint subsets $A, B \subset \{1, 2, \dots n\}$ and positive $r_j$'s

$$\sum_{i \in A} r_i\, \xi_i \quad \text{and} \quad \sum_{j \in B} r_j\, \xi_j \quad \text{are PQD(NQD).} \tag{1.1}$$

Let us remark that LPQD (LNQD) is between pairwise PQD (NQD) and association (negative association) and it is well known, see, for example, [**10**, p. 131] that association (negative association) implies LPQD (LNQD) and LPQD (LNQD) implies PQD (NQD).

Newman [**10**] established the central limit theorem for a strictly stationary LPQD (LNQD) process and Birkel [**2**] also obtained a functional central limit theorem for LPQD processes which can be used to obtain the functional central limit theorem for LNQD processes. Kim and Baek [**7**] extended this result to a stationary linear process of the form $X_k = \sum_{j=0}^{\infty} a_j\, \xi_{k-j}$, where $\{a_j\}$ is a sequence of real numbers with $\sum_{j=0}^{\infty} |a_j| < \infty$ and $\{\xi_k\}$ is a strict stationary LPQD process with $E\,\xi_i = 0$, $0 < E\,\xi_i^2 < \infty$; this result can be extended to the LNQD case by a similar method.

In this paper we derive a central limit theorem for a linearly negative quadrant dependent sequence in a double array, replacing the strict stationarity assumption with uniform integrability, see Theorem 1.1 below. We apply this result to obtain a central limit theorem for a partial sum of a linear process of the form $X_n = \sum_{j=-\infty}^{\infty} a_{k+j}\, \xi_j$ generated by linearly negative quadrant dependent sequence $\{\xi_j\}$, see Theorem 1.2 below.

**Theorem 1.1.** *Let $\{a_{ni},\ 1 \le i \le n\}$ be a triangular array of nonnegative numbers such that*

$$\sup_n \sum_{i=1}^{n} a_{ni}^2 < \infty \tag{1.2}$$

*and*

$$\max_{1 \le i \le n} a_{ni} \longrightarrow 0 \quad \text{as} \quad n \to \infty \tag{1.3}$$

*Let* $\{\xi_i\}$ *be a centered sequence of linearly negative quadrant dependent random variables such that*

$$\{\xi_i^2\} \text{ is a uniformly integrable family,} \tag{1.4}$$

$$\mathrm{Var}\left(\sum_{i=1}^{n} a_{ni}\,\xi_i\right) = 1 \tag{1.5}$$

*and*

$$\sum_{j:|i-j|\geq u} \mathrm{Cov}\,(\xi_i, \xi_j)^- \longrightarrow 0 \quad \textit{as} \quad u \to \infty \quad \textit{uniformly in} \quad i \geq 1, \tag{1.6}$$

*see* [**3**]. *Then*

$$\sum_{i=1}^{n} a_{ni}\,\xi_i \xrightarrow{\mathcal{D}} N(0,1) \quad \textit{as} \quad n \to \infty.$$

*Remark.* Theorem 1.1 extends Newman's [**10**] central limit theorem for strictly stationary LNQD sequences from equal weights to general weights, while at the same time weakening the assumption of stationarity.

**Corollary 1.1.** *Let* $\{\xi_i\}$ *be a centered sequence of linearly negative quadrant dependent random variables such that* $\{\xi_i^2\}$ *is a uniformly integrable family, and let* $\{a_{ni}, 1 \leq i \leq n\}$ *be a triangular array of nonnegative numbers such that*

$$\sup_n \sum_{i=1}^{n} \frac{a_{ni}^2}{\sigma_n^2} < \infty, \tag{1.7}$$

$$\max_{1\leq i\leq n} \frac{a_{ni}}{\sigma_n} \longrightarrow 0 \quad \textit{as} \quad n \to \infty, \tag{1.8}$$

*where* $\sigma_n^2 = \mathrm{Var}\,(\sum_{i=1}^{n} a_{ni}\,\xi_i)$. *If* (1.6) *holds, then as* $n \to \infty$,

$$\frac{1}{\sigma_n} \sum_{i=1}^{n} a_{ni}\,\xi_i \xrightarrow{\mathcal{D}} N(0,1). \tag{1.9}$$

**Theorem 1.2.** *Let $\{a_j,\ j \in Z\}$ be a sequence of nonnegative numbers such that $\sum_j a_j < \infty$, and let $\{\xi_j,\ j \in Z\}$ be a centered sequence of linearly negative quadrant dependent random variables which is uniformly integrable in $L_2$ and satisfying* (1.6). *Let*

$$X_k = \sum_{j=-\infty}^{\infty} a_{k+j}\, \xi_j \quad \textit{and} \quad S_n = \sum_{i=1}^{n} X_i.$$

*Assume*

$$\inf_{n \geq 1} n^{-1} \sigma_n^2 > 0 \tag{1.10}$$

*where* $\sigma_n^2 = \mathrm{Var}\,(S_n)$. *Then*

$$\frac{Sn}{\sigma_n} \xrightarrow{\mathcal{D}} N(0,1) \quad \textit{as} \quad n \to \infty. \tag{1.11}$$

This result extends Theorem 18.6.5 in [**5**] from the i.i.d. case to the linearly negative quadrant dependence case by adding condition (1.6) and improves the central limit theorem of Kim and Baek [**7**] for linear processes generated by LNQD sequences.

**2. Proofs.** We start with the following lemma.

**Lemma 2.1** [**9**]. *Let $\{Z_i,\ 1 \leq i \leq n\}$ be a sequence of linearly negative quadrant dependent random variables with finite second moments. Then*

$$\left| E \exp\left( it \sum_{j=1}^{n} Z_j \right) - \prod_{j=1}^{n} E \exp(itZ_j) \right| \\ \leq Ct^2 \left| \mathrm{Var}\left( \sum_{j=1}^{n} Z_j \right) - \sum_{j=1}^{n} \mathrm{Var}\,(Z_j) \right|$$

*for all $t \in \mathbf{R}$, where $C > 0$ is an arbitrary constant, not depending on $n$.*

*Proof of Theorem* 1.1. Without loss of generality, we assume that $a_{ni} = 0$ for all $i > n$ and $\sup_{n\geq 1} E\,\xi_n^2 = M < \infty$. For every $1 \leq a < b \leq n$ and $1 \leq u \leq b-a$, we have, after some manipulations,

$$\begin{aligned} 0 &\leq \sum_{i=a}^{b-u} a_{ni} \sum_{j=i+u}^{b} a_{nj}\,\mathrm{Cov}\,(\xi_i,\xi_j)^- \\ &\leq \sup_k \Bigg( \sum_{j:|k-j|\geq u} \mathrm{Cov}\,(\xi_k,\xi_j)^- \Bigg) \Bigg( \sum_{i=a}^{b} a_{ni}^2 \Bigg). \end{aligned} \tag{2.1}$$

By the definition of LNQD, we also have, for every $1 \leq a \leq b \leq n$,

$$\mathrm{Var}\left( \sum_{i=a}^{b} a_{ni}\,\xi_i \right) \leq M \sum_{i=a}^{b} a_{ni}^2.$$

We shall construct now a triangular array of random variables $\{Z_{ni},\ 1 \leq i \leq n\}$ for which we shall make use of Lemma 2.1. Fix a small positive $\varepsilon$ and find a positive integer $u = u_\varepsilon$ such that, for every $n \geq u+1$,

$$\begin{aligned} 0 &\leq \left( \sum_{i=1}^{n-u} a_{ni} \sum_{j=i+u}^{n} a_{nj}\,\mathrm{Cov}\,(\xi_i,\xi_j)^- \right) \\ &\leq \varepsilon. \end{aligned} \tag{2.2}$$

This is possible because of (2.1) and (1.6). Denote by $[x]$ the integer part of $x$, and define

$$K = \left[\frac{1}{\varepsilon}\right];$$

$$Y_{nj} = \sum_{i=uj+1}^{u(j+1)} a_{ni}\,\xi_i, \quad j = 0, 1, \dots\,,$$

$$A_j = \left\{ i : 2Kj \leq i < 2Kj+K, \mathrm{Cov}\,(Y_{ni}, Y_{n,i+1})^- \leq \frac{2}{K} \sum_{i=2Kj}^{2Kj+K} \mathrm{Var}\,(Y_{ni}) \right\}.$$

Since $2\,\mathrm{Cov}\,(Y_{ni}, Y_{n,i+1})^- \leq \mathrm{Var}\,(Y_{ni}) + \mathrm{Var}\,(Y_{n,i+1})$, we get that for every $j$ the set $A_j$ is not empty. Now we define the integers $m_1, m_2, \dots, m_n$, recursively. Let $m_0 = 0$ and

$$m_{j+1} = \min\{m : m > m_j,\ m \in A_j\}$$

and define

$$\begin{aligned} Z_{nj} &= \sum_{i=m_j+1}^{m_{j+1}} Y_{ni}, \quad j=0,1,\dots, \\ \mathcal{D}_j &= \{u(m_j+1)+1,\dots,u(m_{j+1}+1)\}. \end{aligned}$$

We observe that

$$Z_{nj} = \sum_{k\in\mathcal{D}_j} a_{nk}\,\xi_k, \quad j=0,1,\dots.$$

By the definition of LNQD the random variables $\{Z_{nj}\}$ are LNQD. From the fact that $m_j \ge 2K(j-1)$ and $m_{j+1} \le K(2j+1)$ every set $\mathcal{D}_j$ contains no more than 3 $Ku$ elements and $m_{j+1}/m_j \to 1$ as $j\to\infty$. Hence, for every fixed positive $\varepsilon$ by (1.2)–(1.5) the array $\{Z_{ni} : i=0,1,\dots,n;\ n\ge 1\}$ satisfies the Lindeberg condition, see Petrov [**11**, Theorem 22, p. 100], that is, $\{Z_{nj}\}$ satisfies

$$\sigma_n^{-1}\sum_{j=1}^{n} EZ_{nj}^2 I(|Z_{nj}|>\varepsilon\sigma_n) \longrightarrow 0 \quad \text{as} \quad n\to\infty \tag{2.3}$$

where $\sigma_n^2 = \operatorname{Var}(\sum_{j=1}^n Z_{nj})$.

We can observe that, by Lemma 2.1 and the construction,

(2.4)

$$\begin{aligned} &\left|E\exp\left(it\sum_{j=1}^n Z_{nj}\right) - \prod_{j=1}^n E\exp(itZ_{nj})\right| \\ &\le Ct^2\left|\left\{\operatorname{Var}\left(\sum_{j=1}^n Z_{nj}\right) - \sum_{j=1}^n \operatorname{Var}(Z_{nj})\right\}\right| \\ &\le Ct^2\left\{2\left(\sum_{i=1}^n \operatorname{Cov}(Z_{ni},Z_{n,i+1})^-\right) + 2\left(\sum_{i=1}^{n-2}\sum_{j=i+2}^{n} \operatorname{Cov}(Z_{ni},Z_{nj})^-\right)\right\} \end{aligned}$$

$$
\begin{aligned}
&\le Ct^2\Bigg[\Bigg\{2\sum_{j=1}^{n}\mathrm{Cov}\,(Y_{n,m_j},Y_{n,m_{j+1}})^- + 2\sum_{i=1}^{n-u}a_{ni}\sum_{j=i+u}^{n}a_{nj}\mathrm{Cov}\,(\xi_i,\xi_j)^-\Bigg\}\\
&\qquad\qquad\qquad\qquad + \Bigg\{2\sum_{i=1}^{n-u}a_{ni}\sum_{j=i+u}^{n}a_{nj}\mathrm{Cov}\,(\xi_i,\xi_j)^-\Bigg\}\Bigg]\\
&= Ct^2\Bigg\{4\sum_{i=1}^{n-u}a_{ni}\sum_{j=i+u}^{n}a_{nj}\mathrm{Cov}\,(\xi_i,\xi_j)^- + 2\sum_{j=1}^{n}\mathrm{Cov}\,(Y_{n,m_j},Y_{n,m_j+1})^-\Bigg\}\\
&\le Ct^2\Bigg\{4\varepsilon + \frac{8}{K}\sum_{i=1}^{n}\mathrm{Var}\,(Y_{ni})\Bigg\}\\
&= Ct^2\Bigg\{4\varepsilon + \frac{8}{K}\sum_{j=1}^{n}\mathrm{Var}\,\Bigg(\sum_{i=uj+1}^{u(j+1)}a_{n_i}\,\xi_i\Bigg)\Bigg\}\\
&\le Ct^2\Bigg\{4\varepsilon + \frac{8M}{K}\sum_{j=1}^{n}\sum_{i=uj+1}^{u(j+1)}a_{n_i}\,\xi_i\Bigg\}\\
&\le C_1t^2\varepsilon\Bigg\{1+\sup_n\sum_{i=1}^{n}a_{ni}^2\Bigg\}\\
&\le C_2t^2\varepsilon \quad \text{for every positive} \quad \varepsilon.
\end{aligned}
$$

Therefore the problem is now reduced to the study of the central limit theorem of a decoupled sequence $\{\widetilde{Z}_{nj}\}$ of independent random variables such that, for each $n$ and $j$, the variable $\widetilde{Z}_{nj}$ is distributed as $Z_{nj}$.

By (2.3) $\{\widetilde{Z}_{nj}\}$ also satisfies the Lindeberg condition, that is, $\{\widetilde{Z}_{nj}\}$ satisfies $\tilde{\sigma}_n^{-1}\sum_{j=1}^n E\widetilde{Z}_{nj}^2 I(|\widetilde{Z}_{nj}| > \varepsilon\tilde{\sigma}_n) \to 0$ as $n \to \infty$ where $\tilde{\sigma}_n^2 = \mathrm{Var}\,(\sum_{j=1}^n \widetilde{Z}_{nj})$, and hence by [**1**, Theorem 7.2]

$$
\tilde{\sigma}_n^{-1}\sum_{j=1}^{n}\widetilde{Z}_{nj} \xrightarrow{\mathcal{D}} N(0,1) \quad \text{as} \quad n\to\infty \tag{2.5}
$$

where $\tilde{\sigma}_n^2 = \mathrm{Var}\,(\sum_{j=1}^n \widetilde{Z}_{nj})$. It follows from (2.3), (2.4) and (2.5) that

$$
\sigma_n^{-1}\sum_{j=1}^{n}Z_{nj} \xrightarrow{\mathcal{D}} N(0,1) \quad \text{as} \quad n\to\infty \tag{2.6}
$$

where $\sigma_n^2 = \mathrm{Var}\,(\sum_{j=1}^n Z_{nj})$, and now the proof is complete by (2.5), (2.6) and [**1**, Theorem 4.2].

*Proof of Corollary* 1.1. Let $A_{ni} = a_{ni}/\sigma_n$. Then we have

$$\max_{1\le i\le n} A_{ni} \longrightarrow 0 \quad \text{as} \quad n\to\infty,$$
$$\sup_n \sum_{i=1}^n A_{ni}^2 < \infty,$$
$$\mathrm{Var}\left(\sum_{i=1}^n A_{ni}\,\xi_i\right) = 1.$$

Hence, by Theorem 1.1 the desired result (1.11) follows.

*Proof of Theorem* 1.2. First note that $\sum_j a_j^2 < \infty$ and, without loss of generality, we can assume $\sup E\,\xi_k^2 = 1$. Let

$$S_n = \sum_{k=1}^n X_k = \sum_{j=-\infty}^{\infty}\left(\sum_{k=1}^n a_{k+j}\right)\xi_j.$$

In order to apply Theorem 1.1, fix $W_n$ such that $\sum_{|j|>W_n} a_j^2 < n^{-3}$, and take $k_n = W_n + n$. Then

$$\frac{S_n}{\sigma_n} = \sum_{|j|\le k_n}\left(\sum_{k=1}^n a_{k+j}\right)\frac{\xi_j}{\sigma_n} + \sum_{|j|>k_n}\left(\sum_{k=1}^n a_{k+j}\right)\frac{\xi_j}{\sigma_n} = T_n + U_n.$$

By the Cauchy-Schwarz inequality and the assumptions we have the following estimate

$$\begin{aligned}
\mathrm{Var}\,(U_n) &\le \sum_{|j|>k_n} \mathrm{Var}\left(\sum_{k=1}^n a_{k+j}\,\frac{\xi_j}{\sigma_n}\right)\\
&\le \sum_{|j|>k_n}\left(\sum_{k=1}^n a_{k+j}/\sigma_n\right)^2 E\,\xi_j^2 \le n\sigma_n^{-2}\sum_{|j|>k_n}\left(\sum_{k=1}^n a_{k+j}^2\right)\\
&\le n^2\sigma_n^{-2}\sum_{|j|>k_n-n} a_j^2 \le n^2\sigma_n^{-2}\sum_{|j|>W_n} a_j^2\\
&\le n^{-1}\sigma_n^{-2} \longrightarrow 0 \quad \text{as} \quad n\to\infty
\end{aligned}$$

which yields

$$U_n \longrightarrow 0 \quad \text{in probability as} \quad n \to \infty. \tag{2.7}$$

By [**1**, Theorem 4.1], it remains to prove that $T_n \xrightarrow{\mathcal{D}} N(0,1)$. Put

$$a_{nk} = \frac{\sum_{j=1}^{n} a_{k+j}}{\sigma_n}. \tag{2.8}$$

From the assumption $\sum_j a_j < \infty$ $(a_j > 0)$, (1.10) and (2.8) we obtain

$$\begin{aligned} \frac{\sup_{-\infty<k<\infty} \sum_{j=1}^{n} a_{k+j}}{\sigma_n} &\longrightarrow 0 \quad \text{as} \quad n \to \infty, \\ \max_{1\le k\le n} a_{n_k} &\longrightarrow 0 \quad \text{as} \quad n \to \infty, \\ \sup_n \sum_{k=1}^{n} a_{nk}^2 &< \infty. \end{aligned}$$

Hence, by Theorem 1.1,

$$T_n \xrightarrow{\mathcal{D}} N(0,1) \tag{2.9}$$

and from (2.7) and (2.9) the desired result (1.10) follows.

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