ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 37, Number 2, 2007

BASIC FUNCTIONAL EQUATIONS OF THE ROGERS-RAMANUJAN FUNCTIONS

SEUNG H. SON

ABSTRACT. The Rogers-Ramanujan functions satisfy some basic functional equations. We prove and use them to produce some identities that Ramanujan recorded.

1. Introduction. The Rogers-Ramanujan functions in the title are defined by

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n},$$
$$H(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n},$$

where

$$(a;q)_n := \prod_{k=0}^{n-1}(1-aq^k), \quad |q|<1. \tag{1.1}$$

These functions satisfy the Rogers-Ramanujan identities [**3**], [**4**, pp. 214–215], [**6**],

$$G(q) = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}, \tag{1.2}$$
$$H(q) = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}, \tag{1.3}$$

where

$$(a;q)_\infty := \lim_{n\to\infty}(a;q)_n. \tag{1.4}$$

Received by the editors on September 21, 2004.

In this article we prove new sets of basic functional equations (3.1)–(3.4) of the functions $G(q)$ and $H(q)$. To show the usefulness of the equations, we construct some identities that Ramanujan recorded and others proved before.

2. Preliminary results. The following identities can be derived from the definitions (1.1) and (1.4).

Lemma 2.1. *For positive integers* $n, N, |q| < 1$ *and* $Q := q^n$,

$$
\begin{aligned}
(a;Q)_\infty &= \prod_{k=0}^{N-1} (aQ^k;Q^N)_\infty,\\
(a;-Q)_\infty &= (a;Q^2)_\infty(-aQ;Q^2)_\infty,\\
(-a;Q)_\infty &= \frac{(a^2;Q^2)_\infty}{(a;Q)_\infty},\\
(-Q;Q)_\infty &= \frac{1}{(Q;Q^2)_\infty}.
\end{aligned}
$$

In his notebook [**1**, pp. 34–38], Ramanujan defines the following theta functions: For $|ab| < 1$, $|q| < 1$, we define

(2.1)

$$
\begin{aligned}
f(a,b) &:= \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2} = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty,\\
\varphi(q) &:= f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q;-q)_\infty}{(q;-q)_\infty},\\
\psi(q) &:= f(q,q^3) = \sum_{n=1}^{\infty} q^{n(n-1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty},\\
f(-q) &:= f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty.
\end{aligned}
$$

The infinite products are generated from the Jacobi triple product identity [**1**, p. 35]. For convenience, we define the function $\chi(q) = (-q;q^2)_\infty$.

By the definition (2.1) one can easily verify the following identities [**1**, p. 45] after rearranging the double sums using $m-n=2j+\delta$ and $m+n=2k-\delta$, $\delta=0,1$.

Lemma 2.2. *If $ab=cd$, then*

$$f(a,b)f(c,d)+f(-a,-b)f(-c,-d)=2f(ac,bd)f(ad,bc),$$
$$f(a,b)f(c,d)-f(-a,-b)f(-c,-d)=2af\left(\frac{b}{c},\frac{abcd}{b/c}\right)f\left(\frac{b}{d},\frac{abcd}{b/d}\right).$$

By using (1.2), (1.3), and Lemma 2.1, we can deduce the following lemma.

Lemma 2.3. *For $|q|<1$,*

$$G(q)H(q)=\frac{(q^5;q^5)_\infty}{(q;q)_\infty}.$$

3. Main functional equations of $G(q)$ and $H(q)$. We find the following functional equations in terms of the theta functions.

Theorem 3.1. *For $|q|<1$,*

$$G(-q)=\frac{f(q^4,q^6)}{f(-q^2)G(q)}=G(q)\,\frac{f(-q,-q^9)}{f(q,q^9)} \tag{3.1}$$

$$H(-q)=\frac{f(q^2,q^8)}{f(-q^2)H(q)}=H(q)\,\frac{f(-q^3,-q^7)}{f(q^3,q^7)}. \tag{3.2}$$

Proof. By (2.1), (1.2), (1.3) and Lemma 2.1, we deduce that

$$\begin{aligned}
&f(-q^2)G(-q)G(q)\\
&\quad=\frac{(q^2;q^2)_\infty}{((-q;-q^5)_\infty(q^4;-q^5)_\infty)((q;q^5)_\infty(q^4;q^5)_\infty)}\\
&\quad=\frac{(q^2;q^{10})_\infty(q^8;q^{10})_\infty(q^{10};q^{10})_\infty}{(-q;q^{10})_\infty(-q^9,q^{10})_\infty((q;q^{10})_\infty(q^6;q^{10})_\infty)((q^4;q^{10})_\infty(q^9;q^{10})_\infty)}
\end{aligned}$$

$$= \frac{((q^2;q^{20})_\infty(q^{12};q^{20})_\infty)((q^8;q^{20})_\infty(q^{18};q^{20})_\infty)(q^{10};q^{10})_\infty}{(q^2;q^{20})_\infty(q^{18};q^{20})_\infty(q^4;q^{10})_\infty(q^6;q^{10})_\infty}$$
$$= \frac{(q^{12};q^{20})_\infty(q^8;q^{20})_\infty}{(q^4;q^{10})_\infty(q^6;q^{10})_\infty}\cdot(q^{10};q^{10})_\infty.$$

Since

$$(q^8;q^{20})_\infty/(q^4;q^{10})_\infty = (-q^4;q^{10})_\infty,$$
$$(q^{12};q^{20})_\infty/(q^6;q^{10})_\infty = (-q^6;q^{10})_\infty,$$

by Lemma 2.1, we conclude that

$$f(-q^2)G(-q)G(q) = (-q^4;q^{10})_\infty(-q^6;q^{10})_\infty(q^{10};q^{10})_\infty = f(q^4,q^6).$$

Similarly we can obtain

$$f(-q^2)H(-q)H(q) = f(q^2,q^8).$$

For the second part of (3.1), we consider the following quotient

$$\frac{G(-q)}{G(q)} = \frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(-q;-q^5)_\infty(q^4;-q^5)_\infty}$$
$$= \frac{(q;q^{10})_\infty(q^6;q^{10})_\infty(q^4;q^{10})_\infty(q^9;q^{10})_\infty}{(-q;q^{10})_\infty(q^6;q^{10})_\infty(q^4;q^{10})_\infty(-q^9;q^{10})_\infty}$$
$$= \frac{(q;q^{10})_\infty(q^9;q^{10})_\infty}{(-q;q^{10})_\infty(-q^9;q^{10})_\infty}\cdot\frac{(q^{10};q^{10})_\infty}{(q^{10};q^{10})_\infty} = \frac{f(-q,-q^9)}{f(q,q^9)}.$$

Similarly, we can prove that

$$\frac{H(-q)}{H(q)} = \frac{f(-q^3,-q^7)}{f(q^3,q^7)}. \qquad \square$$

We also deduce the following functional equations in terms of the theta functions.

Theorem 3.2. *For* $|q|<1$,

$$G(q^4) = \frac{f(q^3,q^7)}{f(-q^2)H(q)} \tag{3.3}$$

$$H(q^4) = \frac{f(q,q^9)}{f(-q^2)G(q)}. \tag{3.4}$$

Proof. By Lemma 2.1, we obtain

$$\begin{aligned}(q^2;q^2)_\infty &= (q^2;q^{10})_\infty(q^4;q^{10})_\infty(q^6;q^{10})_\infty(q^8;q^{10})_\infty(q^{10};q^{10})_\infty,\\ (q^2;-q^5)_\infty &= (q^2;q^{10})_\infty(q^{14};q^{20})_\infty/(q^7;q^{10})_\infty,\\ (-q^3;-q^5)_\infty &= (q^8;q^{10})_\infty(q^6;q^{20})_\infty/(q^3;q^{10})_\infty,\end{aligned}$$

and we find that

$$\begin{aligned}&f(-q^2)H(-q)G(q^4)\\ &\quad= \frac{(q^2;q^2)_\infty}{(q^2;-q^5)_\infty(-q^3;-q^5)_\infty(q^4;q^{20})_\infty(q^{16};q^{20})_\infty}\\ &\quad= \frac{(q^4;q^{10})_\infty(q^6;q^{10})_\infty(q^{10};q_{10})_\infty(q^3;q^{10})_\infty(q^7;q^{10})_\infty}{(q^{14};q^{20})_\infty(q^6;q^{20})_\infty(q^4;q^{20})_\infty(q^{16};q^{20})_\infty}.\end{aligned}$$

After cancelation using

$$\begin{aligned}(q^4;q^{10})_\infty &= (q^4;q^{20})_\infty(q^{14};q^{20})_\infty,\\ (q^6;q^{10})_\infty &= (q^6;q^{20})_\infty(q^{16};q^{20})_\infty,\end{aligned}$$

we conclude that

$$f(-q^2)H(-q)G(q^4) = (q^3;q^{10})_\infty(q^7;q^{10})_\infty(q^{10};q^{10})_\infty = f(-q^3,-q^7).$$

By replacing q by $-q$, we complete the proof of (3.3).

Similarly, we can obtain

$$f(-q^2)G(-q)H(q^4) = f(-q,-q^9),$$

and complete the proof of (3.4) after substituting q for $-q$. $\square$

4. Application of the functional equations. To apply the functional equations (3.1)–(3.4), we select a few identities that are widely known and, using the functional equations, we construct those identities. Ramanujan initially discovered them [**5**] and others proved them. Watson [**8**], Berndt [**2**] and Shen [**7**], and others, contributed in finding some proofs.

Theorem 4.1. *For* $|q| < 1$,

$$G(q)G(q^4) + qH(q)H(q^4) = \chi^2(q) = \frac{\varphi(q)}{f(-q^2)}, \tag{4.1}$$

$$G(q)G(q^4) - qH(q)H(q^4) = \frac{\varphi(q^5)}{f(-q^2)}. \tag{4.2}$$

Proof. By Lemma 2.2,

(4.3)

$$f(q,-q^4)f(-q^2,q^3) + f(-q,q^4)f(q^2,-q^3) = 2f(q^4,q^6)f(-q^3,-q^7),$$

(4.4)

$$f(q,-q^4)f(-q^2,q^3) - f(-q,q^4)f(q^2,-q^3) = 2qf(q^2,q^8)f(-q,-q^9).$$

After replacing q by $-q$, and then subtracting and adding the two equalities (4.3) and (4.4), we obtain

$$f(q^4,q^6)f(q^3,q^7) - qf(q^2,q^8)f(q,q^9) = f(-q,-q^4)f(-q^2,-q^3), \tag{4.5}$$

$$f(q^4,q^6)f(q^3,q7) + qf(q^2,q^8)f(q,q^9) = f(q,q^4)f(q^2,q^3). \tag{4.6}$$

Because of the functional equations (3.1)–(3.4), (4.5) and Lemma 2.3,

$$\begin{aligned}
&G(-q)G(q^4) - qH(-q)H(q^4) \\
&\qquad = \frac{f(q^4,q^6)f(q^3,q^7) - qf(q^2,q^8)f(q,q^9)}{f^2(-q^2)G(q)H(q)} \\
&\qquad = \frac{f(-q,-q^4)f(-q^2,-q^3)}{f^2(-q^2)((q^5;q^5)_\infty/(q;q)_\infty)} \\
&\qquad = \frac{(q;q)_\infty(q^5;q^5)_\infty}{(q^2;q^2)_\infty^2(q^5;q^5)_\infty/(q;q)_\infty} = \frac{(q;q)_\infty^2}{(q^2;q^2)_\infty^2} \\
&\qquad = (q;q^2)_\infty^2 = \chi^2(-q) = (q;q^2)_\infty^2\,\frac{(q^2;q^2)_\infty}{(q^2;q^2)_\infty} \\
&\qquad = \frac{f(-q,-q)}{(q^2;q^2)_\infty} = \frac{\varphi(-q)}{f(-q^2)}.
\end{aligned}$$

After replacing q by $-q$, we complete the proof of (4.1).

Similarly,

$$\begin{aligned}
&G(-q)G(q^4)+qH(-q)H(q^4)\\
&\qquad=\frac{f(q^4,q^6)f(q^3,q^7)+qf(q^2,q^8)f(q,q^9)}{f^2(-q^2)G(q)H(q)}\\
&\qquad=\frac{f(q,q^4)f(q^2,q^3)}{f^2(-q^2)G(q)H(q)}\\
&\qquad=\frac{(q^5;q^5)^2_\infty(-q;q)_\infty/(-q^5;q^5)_\infty}{(q^2;q^2)^2_\infty(q^5;q^5)_\infty/(q;q)_\infty}\\
&\qquad=\frac{(q^5;q^5)_\infty/(-q^5;q^5)_\infty}{(q^2;q^2)_\infty}=\frac{\varphi(-q^5)}{f(-q^2)},
\end{aligned}$$

and by replacing q by $-q$, we finish the proof of (4.2). □

Theorem 4.2. *For* $|q|<1$,

$$G(q)H(-q)+G(-q)H(q)=\frac{2}{\chi^2(-q^2)}=\frac{2\psi(q^2)}{f(-q^2)}, \tag{4.7}$$

$$G(q)H(-q)-G(-q)H(q)=\frac{2q\psi(q^{10})}{f(-q^2)}. \tag{4.8}$$

Proof. From the functional equations (3.1) and (3.2), we deduce that

$$\begin{aligned}
&G(q)H(-q)\pm G(-q)H(q)\\
&\qquad\qquad=G(q)H(q)\left(\frac{f(-q^3,-q^7)}{f(q^3,q^7)}\pm\frac{f(-q,-q^9)}{f(q,q^9)}\right).
\end{aligned} \tag{4.9}$$

By Lemma 2.2,

$$\begin{aligned}
&\frac{G(q)H(q)}{f(q^3,q^7)f(q,q^9)}\left(f(q,q^9)f(-q^3,-q^7)+f(-q,-q^9)f(q^3,q^7)\right)\\
&\qquad=\frac{G(q)H(q)}{f(q^3,q^7)f(q,q^9)}\left(2f(-q^4,-q^{16})f(-q^8,-q^{12})\right).
\end{aligned}$$

By simplification, (1.2)–(2.1) and Lemma 2.1, the product becomes

$$\begin{aligned}
&\frac{2(q^5;q^5)_\infty(q^4;q^{20})_\infty(q^{16};q^{20})_\infty(q^{20};q^{20})_\infty^2(q^8;q^{20})_\infty(q^{12};q^{20})_\infty}{(q;q)_\infty(-q^3;q^{10})_\infty(-q^7;q^{10})_\infty(q^{10};q^{10})_\infty^2(-q;q^{10})_\infty(-q^9;q^{10})_\infty}\\
&\qquad\qquad = \frac{2(q^5;q^{10})_\infty(q^4;q^4)_\infty(q^{20};q^{20})_\infty}{(q;q)_\infty(-q;q^2)_\infty(q^{10};q^{10})_\infty/(-q^5;q^{10})_\infty}\\
&\qquad\qquad = \frac{2(q^5;q^{10})_\infty(-q^5;q^{10})_\infty(q^4;q^4)_\infty(q^{20};q^{20})_\infty}{(q;q^2)_\infty(q^2;q^2)_\infty(-q;q^2)_\infty(q^{10};q^{10})_\infty}\\
&\qquad\qquad = \frac{2(q^{10};q^{20})_\infty(q^4;q^4)_\infty(q^{20};q^{20})_\infty}{(q^2;q^4)_\infty(q^2;q^2)_\infty(q^{10};q^{10})_\infty}\\
&\qquad\qquad = \frac{2(q^4;q^4)_\infty}{(q^2;q^4)_\infty(q^2;q^2)_\infty} = \frac{2\psi(q^2)}{f(-q^2)}\\
&\qquad\qquad = \frac{2}{(q^2;q^4)_\infty^2} = \frac{2}{\chi(-q^2)}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&G(q)H(-q)-G(-q)H(q)\\
&\quad = \frac{G(q)H(q)}{f(q^3,q^7)f(q,q^9)}\left(f(q,q^9)f(-q^3,-q^7)-f(-q,-q^9)f(q^3,q^7)\right)\\
&\quad = \frac{G(q)H(q)}{f(q^3,q^7)f(q,q^9)}\left(2qf(-q^6,-q^{14})f(-q^2,-q^{18})\right)\\
&\quad = \frac{2q(q^5;q^5)_\infty(q^6;q^{20})_\infty(q^{14};q^{20})_\infty(q^{20};q^{20})_\infty^2(q^2;q^{20})_\infty(q^{18};q^{20})_\infty}{(q;q)_\infty(-q^3;q^{10})_\infty(-q^7;q^{10})_\infty(q^{10};q^{10})_\infty^2(-q;q^{10})_\infty(-q^9;q^{10})_\infty}\\
&\quad = \frac{2q(q^5;q^{10})_\infty(q^3;q^{10})_\infty(q^7;q^{10})_\infty(q^{20};q^{20})_\infty^2(q;q^{10})_\infty(q^9;q^{10})_\infty}{(q;q)_\infty(q^{10};q^{10})_\infty}\\
&\quad = \frac{2q(q^{20};q^{20})_\infty^2}{(q^2;q^2)_\infty(q^{10};q^{10})_\infty} = \frac{2q(q^{20};q^{20})_\infty}{(q^2;q^2)_\infty(q^{10};q^{20})_\infty} = \frac{2q\psi(q^{10})}{f(-q^2)}. \qquad \square
\end{aligned}$$

Corollary 4.3. *For* $|q|<1$,

$$\begin{aligned}
\varphi(q)+\varphi(q^5) &= 2f(q,q^9)\,\frac{G(q^4)}{H(q^4)},\\
\varphi(q)-\varphi(q^5) &= 2qf(q^3,q^7)\,\frac{H(q^4)}{G(q^4)}.
\end{aligned}$$

Proof. By adding and subtracting (4.1) and (4.2), we obtain

$$\varphi(q)+\varphi(q^5)=2f(-q^2)G(q)G(q^4),$$
$$\varphi(q)-\varphi(q^5)=2qf(-q^2)H(q)H(q^4).$$

Now we complete the proofs by applying the functional equations (3.4) and (3.3), respectively. □

Corollary 4.4. *For* $|q|<1$,

$$\psi(q^2)+q\psi(q^{10})=f(q^2,q^8)\,\frac{G(q)}{H(q)},$$
$$\psi(q^2)-q\psi(q^{10})=f(q^4,q^6)\,\frac{H(q)}{G(q)}.$$

Proof. By adding and subtracting (4.7) and (4.8), we obtain

$$\psi(q^2)+q\psi(q^{10})=f(-q^2)G(q)H(-q),$$
$$\psi(q^2)-q\psi(q^{10})=f(-q^2)G(-q)H(q).$$

We complete the proofs by applying the functional equations (3.2) and (3.1), respectively. □

Corollary 4.5. *For* $|q|<1$,

$$\varphi^2(q)-\varphi^2(q^5)=4qf(q,q^9)f(q^3,q^7),$$
$$\psi^2(q^2)-q^2\psi^2(q^{10})=f(q^2,q^8)f(q^4,q^6).$$

Proof. By multiplying the two matching identities in the above two corollaries, we can complete the proofs. □

The above identities are of central importance in the theory of modular equations of degree five.

Acknowledgment. The author expresses deep appreciation to Professor Bruce C. Berndt for his invaluable suggestions leading to this result.

REFERENCES

1. B.C. Berndt, *Ramanujan's notebooks, Part* III, Springer-Verlag, New York, 1991.

2. B.C. Berndt, et al., *Ramanujan's forty identities for the Rogers-Ramanujan functions*, Mem. Amer. Math. Soc., to appear.

3. S. Ramanujan, *Proof of certain identities in combinatory analysis*, Proc. Cambridge Philos. Soc. **19** (1919), 214–216.

4. ———, *Collected papers*, Cambridge Univ. Press, Cambridge, 1927; reprinted by Chelsea, New York, 1962; reprinted by the Amer. Math. Soc., Providence, RI, 2000.

5. ———, *The lost notebook and other unpublished papers*, Narosa, New Delhi, 1988.

6. L.J. Rogers, *Second memoir on the expansion of certain infinite products*, Proc. London Math. Soc. **25** (1894), 318–343.

7. L.-C. Shen, *On the additive formulae of the theta functions and a collection of Lambert series pertaining to the modular equations of degree 5*, Trans. Amer. Math. Soc. **345** (1994), 323–345.

8. G.N. Watson, *Proof of certain identities in combinatory analysis*, J. Indian Math. Soc. **20** (1933), 57–69.

Department of Mathematics, University of Colorado, 1420 Austin Bluffs Parkway, Colorado Springs, CO 80933
E-mail address: `sson@uccs.edu`