Rev. Mat. Iberoamericana **19** (2003), 797–812

# Analysis of the free boundary for the $p$-parabolic variational problem $(p \geq 2)$

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**Abstract**

Variational inequalities (free boundaries), governed by the $p$-parabolic equation $(p \geq 2)$, are the objects of investigation in this paper. Using intrinsic scaling we establish the behavior of solutions near the free boundary. A consequence of this is that the time levels of the free boundary are porous (in $N$-dimension) and therefore its Hausdorff dimension is less than $N$. In particular the $N$-Lebesgue measure of the free boundary is zero for each $t$-level.

## 1. Preliminaries

In this paper we consider a variational inequality for the $p$-parabolic operator $(2 \leq p < \infty)$

$$\operatorname{div}(|\nabla u(x)|^{p-2}\nabla u(x)) - \partial_t u,$$

giving rise to a free boundary. Our objective is to analyze the free boundary in the context of regularity theory. To fix the idea, let us start with the formulation of the problem in the weak sense. First some notations:

$\Omega$ is a bounded (smooth) domain in $\mathbb{R}^N$; $\Omega_T = \Omega \times (0,T)$; $V^{1,p}(\Omega_T)$ is the parabolic space [D, page 7],

$$V^{1,p}(\Omega_T) = L^\infty(0,T;L^1(\Omega)) \cap L^p(0,T;W^{1,p}(\Omega)),$$

$f$ and $\theta$ are given bounded functions; the Steklov average $v_h$ of a function is defined by

$$v_h(x,t) = \frac{1}{h}\int_t^{t+h} v(x,\tau)d\tau \qquad \text{for } t \in (0, T-h],$$

and $v_h = 0$ for $t > T - h$.

*2000 Mathematics Subject Classification*: 35K55, 35K85, 35K65, 35R35.
*Keywords*: variational problem, inhomogeneous $p$-parabolic equation, free boundary, porosity.

The variational problem is to find a function

$u \in \mathcal{K}_\theta := \mathcal{K}_\theta(p) = \{w : w \in V^{1,p}(\Omega_T),\ \forall t\ w = \theta \text{ on } \partial_p\Omega_T,\ w \geq 0 \text{ a.e. in } \Omega_T\}$,

such that (for $h > 0$, and $0 < t < t + h < T$)

$$\int_\Omega \partial_t u_h (w-u)dx + \int_\Omega \left(|\nabla u|^{p-2}\nabla u\right)_h \cdot \nabla(w-u)dx + \int_\Omega f_h(w-u)dx \geq 0, \tag{1.1}$$

a.e. in $t \in (0,T)$, and for all $w \in \mathcal{K}_\theta$. In (1.1) we have taken the Steklov average since the time derivative of $u$ may not exist as a function in $V^{1,p}$.

Under certain conditions on $f$, and $\theta$ (see below) we will show that the free boundary of the solutions to the variational problem (1.1) is so-called porous (for each $t$-level cut). We define the porosity below. A consequence of porosity is that the $t$-cuts of the free boundary will have zero Lebesgue measure.

It should also be remarked that the porosity of the free boundary follows as a by-product of our main result, which states that the solution $u$ to (1.1) grows with a certain power away from the free boundary. The proof of the latter, in turn, employs techniques that originate in the author's work with L. Karp [KS]. Cf. also [ASU], [CKS], [KS], [KKPS], [LS] for related results and techniques.

The conditions to be imposed on $f$, and $\theta$ are the following.

**Condition A:** We assume $f$, and $\theta$ are bounded continuous functions on the closure of $\Omega_T$,

$$0 < \lambda_0 \leq f \leq \Lambda_0, \quad \text{in } \Omega_T, \qquad \theta(x,0) = 0,$$

$$f(x,t), \quad \text{monotone non-increasing in } t,$$

and

$$\theta(x,t) \quad \text{monotone non-decreasing in } t.$$

With these conditions one forces the solution $u$ to the variational problem to become monotone non-decreasing in $t$ (see below), i.e., $\partial_t u \geq 0$ in the set $\{u > 0\}$. For $p = 2$ this condition is replaced by $\partial_t u > 0$ due to the strong maximum/comparison principle. The latter is known to fail for the $p$-parabolic equation, see [D, Chapter VI, Lemma 3.1 ].

An example, of the failure of the strong maximum principle, is given by the Fundamental solution obtained by Barenblat [B], which is

$$\mathcal{B}_p = t^{-n/\lambda}\left(c_1 - \frac{p-2}{p}\lambda^{-1/(p-1)}\left(\frac{|x|}{t^{1/\lambda}}\right)^{p/(p-1)}\right)_+^{(p-1)/(p-2)} \qquad (p > 2).$$

Here $\lambda = n(p-2) + p$ and the function is defined when $t > 0$ and $x \in \mathbb{R}^N$. Also $c_1$ is a normalization constant chosen such that $\int \mathcal{B}_p(x,t)dx = 1$, for $t > 0$.

It is noteworthy that above conditions on the functions $f$, and $\theta$ appear naturally in the so-called Stefan problem (see [F], [KiSt]), which has been studied extensively and there is a vast literature on the subject (see the references in [F]). However, the assumptions, in this paper, which lead to $\partial_t u \geq 0$, are only adopted for technical reasons. Indeed it only enters into the proof at one point, where it is mainly used to show that at every $t$ level there is a point where $u$ takes a maximum value comparable with the maximum values of the previous levels of $t$, by the monotonicity property of $u$ ($\partial_t u \geq 0$) we will have that the maximum value at level $t$ is larger than that of level $t-h$ for every $h > 0$.

The difficulty with the $p$-parabolic problem is the inhomogeneity of the operator. In other words the operator is of order $p$ in the spatial directions and of order one in the time direction. This inhomogeneity forces us to use *intrinsic scaling* of the solution, i.e. we consider

$$\frac{u(z + rx, s + r^p S_r^{2-p} t)}{S_r},$$

where $S_r = \sup_{Q_r^-(z,s)} u$, with $Q_r(z,s) = B_r(z) \times (-r^q + s, r^q + s)$ and $Q_r^-(z,s) = B_r(z) \times (-r^q + s, s)$ , with $q = (p-1)/p$. This type of intrinsic scaling already appears in Harnack's inequality for the $p$-parabolic equation (see [D; page 157]). Observe also that by the above definition

$$\Delta_p u_r(x,t) - \partial_t u_r(x,t) = \frac{r^p}{S_r^{p-1}} (\Delta_p u - \partial_t u)(rx, r^p S_r^{2-p} t),$$

and thus the scaling leaves the solution invariant in the sense that if $\Delta_p u - \partial_t u$ is bounded then so is $\Delta_p u_r - \partial_t u_r$, provided $\frac{r^p}{S_r^{p-1}}$ is finite (see Lemma 2.1).

The above intrinsic scaling, in turn, may cause other type of difficulties, one of which is that the scaling of $u$ as above may converge (as $r \to 0$) to a function which vanishes identically, this we don't want to happen since we want to use the limit function in a certain contradictory argument, and we need it to be nonzero.

Another, major obstacle in the analysis is the lack of strong maximum principle, which becomes crucial in the final steps of the proof. We overcome this difficulty by using Hölder's estimate to reduce the final part of the problem to the time independent situation.

Let us gather all properties (needed here) for the solution $u$ to the variational inequality (1.1). For the sake of reference we formulate it as a theorem. Although this theorem can be proven using classical techniques, we couldn't find an exact reference to it for the general case $1 < p < \infty$. For readers' convenience we sketch the proof in the Appendix.

**Classical Theorem.** *Let $1 < p < \infty$. Let also $f$ and $\theta$ satisfy Condition A. Then there exists a unique solution $u$ to the variational problem (1.1) in $\mathcal{K}_\theta$ with*

$$0 \le u \le \|\theta\|_{\infty,\Omega_T} \quad \text{in } \Omega_T, \qquad (1.2)$$
$$\partial_t u \ge 0 \quad \text{in } \{u > 0\}.$$

*Moreover $u$ satisfies*

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) - \partial_t u = g \qquad (1.3)$$

*weakly in $\Omega_T$ with $g \in L^\infty(\Omega_T)$ satisfying*

$$f\chi_{\{u>0\}} \le g \le f\chi_{\overline{\{u>0\}}} \quad \text{a.e. in } \Omega_T, \qquad (1.4)$$

In the sequel we will use the following notation

$$q = \frac{p}{p-1}.$$

We also define the concept of porosity.

**Porosity:** A set $E$ in $\mathbb{R}^N$ is called *porous* with *porosity constant* $\delta$ if there is an $r_0 > 0$ such that for each $x \in E$ and $0 < r < r_0$ there is a point $y$ such that $B_{\delta r}(y) \subset B_r(x) \setminus E$. A porous set has Hausdorff dimension not exceeding $N - C\delta^N$, where $C = C(N) > 0$ is some constant (see e.g. Martio and Vuorinen [MV]). Consequently a porous set has Lebesgue measure zero.

Now we formulate the main theorem in this paper.

**Main Theorem.** *Let $2 \le p < \infty$, and $u$ be a solution to problem (1.1) in $\mathcal{K}_\theta$ with $f$, $\theta$ satisfying Condition A. Then for every compact set $K \subset \Omega_T$ the following hold*

$$c_0 r^q \le \sup_{B_r(x^0)} u(\cdot, t^0) \le C_0 r^q \qquad (x^0, t^0) \in \partial\{u > 0\} \cap K. \qquad (1.5)$$

*Consequently, the intersection $\partial\{u > 0\} \cap K \cap \{t = t^0\}$ is porous (in $\mathbb{R}^N$) with porosity constant*

$$\delta = \delta(\|\theta\|_{\infty,\Omega}, \lambda_0, \Lambda_0, \operatorname{dist}(K, \partial_p\Omega_T), p, N) > 0.$$

*Here $c_0$, depends on $N$, $p$, $\lambda_0$,, and $C_0$ depends on $N$, $p$, $\lambda_0$, $\|\theta\|_{\infty,\Omega_T}$, and $\Lambda_0$.*

We prove this theorem in Section 3. The theorem can be generalized to hold for a larger class of operators of type [D, (1.1)]. There is basically minor (if any at all) modifications in the proofs.

Observe that the porosity of the $t$-sections follow from the upper bound in (1.5) and the lower bound in (1.6) in the above theorem. The estimate from below in (1.5) is not needed, and it is included only for the sake of completeness and future references. It also holds in a more general framework and not only for solutions of problem (1.2), see Lemma 2.1 below.

## 2. A class of functions invariant under intrinsic scaling

The proof of the main theorem is based on the study of the following class of functions. We say that a function $u$ is in $W^{1,p}(Q_1)$, where $Q_1 = B(0,1) \times (-1,1)$ is the unit cylinder in $\mathbb{R}^{N+1}$, belongs to the class $\mathcal{G} = \mathcal{G}(p, N, L, \Lambda_0)$ $(2 \le p < \infty)$ if

$$\|\operatorname{div}(|\nabla u|^{p-2}\nabla u) - \partial_t u\|_{\infty,Q_1} \le \Lambda_0; \tag{2.1}$$

$$0 \le u \le L \qquad \text{a.e. in } Q_1; \tag{2.2}$$

$$u(0,0) = 0; \tag{2.3}$$

$$\partial_t u \ge 0 \qquad \text{a.e. in } Q_1. \tag{2.4}$$

Condition (2.1) is understood in the weak sense, i.e., $\operatorname{div}(|\nabla u|^{p-2}\nabla u) - \partial_t u = h$ weakly for $h \in L^\infty(Q_1)$ with $\|h\|_\infty \le \Lambda_0$. Condition (2.3) makes sense since (2.1) and (2.2) provide that $u \in C_x^{1,\alpha} \cap C_t^{0,\alpha}(Q_{1/2})$ for some $\alpha \in (0,1)$; (see e.g. [D, § IX]).

If the problem is given in $Q_1(z,s)$, then by translation we may consider the problem again in $Q_1(0,0)$, without any change in the norms.

**Lemma 2.1** *Let $u \in W^{1,p}(Q_1)$ be a non-negative continuous function in $Q_1(0,0)$, satisfying*

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) - \partial_t u = f$$

*weakly in $U^+ = \{u > 0\}$ with $f$ satisfying the lower bound in (1.5). Then for every $(z,s) \in \overline{U}^+$ and $r > 0$ with $Q_r(z,s) \subset Q_1$*

$$\sup_{(x,t)\in\partial_p Q_r^-(z,s)} u(x,t) \ge c_0 r^q + u(z,s),$$

*where*

$$c_0 = \min\left(\frac{\lambda_0}{2}, \frac{1}{q}\left(\frac{\lambda_0}{2N}\right)^{1/(p-1)}\right).$$

**Proof.** First suppose that $(z,s) \in U^+$, and for small $\varepsilon > 0$ set

$$w_\varepsilon(x,t) = u(x,t) - u(z,s)(1-\varepsilon),$$

and

$$v(x,t) = \left(\frac{1}{q}\left(\frac{\lambda_0}{2N}\right)^{1/(p-1)}\right)|x-z|^q - \left(\frac{\lambda_0}{2}\right)(t-s).$$

Then $\operatorname{div}(|\nabla v|^{p-2}\nabla v) - \partial_t v = \lambda_0$ and therefore

$$\operatorname{div}(|\nabla w_\varepsilon|^{p-2}\nabla w_\varepsilon) - \partial_t w_\epsilon = \operatorname{div}(|\nabla u|^{p-2}\nabla u) - \partial_t u \ge \operatorname{div}(|\nabla v|^{p-2}\nabla v) - \partial_t v$$

in $U^+ \cap Q_r^-(z,s)$, and $w_\varepsilon \le v$ on $\partial U^+ \cap Q_r^-(z,s)$.

If also $w_\varepsilon \le v$ on $\partial_p Q_r^-(z,s)\cap U^+$, then we may apply the comparison principle to obtain $w_\varepsilon \le v$ in $Q_r^-(z,s)\cap U^+$, which contradicts the fact that $w_\varepsilon(z,s) = \varepsilon u(z,s) > 0 = v(z,s)$. Hence for some point $(y,\tau)\in \partial_p Q_r^-(z,s)$ we must have

$$w_\varepsilon(y,\tau) \ge v(y,\tau) = C_0 r^q.$$

Letting $\varepsilon \to 0$, we obtain the desired result, for all $(z,s)\in U^+$, and by continuity for all $(z,s)\in \overline{U}^+$. The proof is completed. ∎

We bring the reader's attention to the fact that if we use the assumption $\partial_t u \ge 0$, then the lemma can be proved by considering $u$ as a subsolution for the elliptic case, i.e., $\Delta_p u \ge f$.

First we recall the following Hölder's estimate for solutions of our problem or even more general cases (see [D]). Let $G_T = G\times(0,T]$, where $G$ is a bounded domain in $\mathbb{R}^N$. For every pair $(x,t),(y,\tau)\in K$ (a compact in $G_T$) there exist constants $\gamma > 1$ and $\alpha\in(0,1)$ such that

$$|u(x,t)-u(y,\tau)| \le \gamma \|u\|_{\infty,G_T}\left(\frac{|x-y| + \|u\|_{\infty,G_T}^{(p-2)/p}|t-\tau|^{1/p}}{\mathrm{dist}_p(K;\partial_p G_T;p)}\right)^{\alpha} \qquad p>2$$

where

$$\mathrm{dist}_p(K;\partial_p G_T;p) = \inf_{\substack{(x,t)\in K,\\ (y,\tau)\in\partial_p G_T}} \left(|x-y| + \|u\|_{\infty,\partial_p G_T}^{(p-2)/p}|t-\tau|^{1/p}\right).$$

Observe that $\gamma$ does not depend on $\|u\|_{\infty,G_T}$ in our case; see [D, page 41, Theorem 1.1]. The basic reasoning in the proof of the main theorem will be a contradictory argument which uses the intrinsic scaling, compactness argument and Hölder's inequality for functions in our class.

Now we define, the supremum norm of $u$ over the cylinder $Q_r^-(z,s) = B(z,r)\times(s-r^q,s)$ by setting

$$S(r,u,z,s) = \sup_{(x,t)\in Q_r^-(z,s)} u(x,t).$$

When $(z,s)$ is the origin we suppress the point dependent.

Next, for $u\in\mathcal{G}$ we define $\mathbb{M}(u,z,s)$ to be the set of all non-negative integers $j$ such that the following *doubling* condition holds

$$AS(2^{-j-1},u,z,s) \ge S(2^{-j},u,z,s),$$

where

$$A = 2^q \max\left(1, \frac{1}{C_0}\right),$$

with $C_0$ as in Lemma 2.1.

We first show that $0 \in \mathbb{M}(u)$ for all $u \in \mathcal{G}$. This actually follows by the previous lemma in the following way

$$(2.5) \quad S(1,u) \leq 1 = \left(\frac{1}{C_0 2^{-q}}\right) C_0 2^{-q} \leq \left(\frac{1}{C_0 2^{-q}}\right) S(2^{-1},u) = AS(2^{-1},u).$$

With some efforts, our main theorem will follow from the theorem below.

**Theorem 2.2** *There is a positive constant $M_0 = M_0(p,N,L,\Lambda_0)$ such that for every $u \in \mathcal{G}$, there holds*

$$|u(x,t)| \leq M_0(d(x,t))^q \qquad \forall (x,t) \in Q_{1/2},$$

*where*

$$d(x,t) = \sup\{r : \ Q_r(x,t) \subset U^+\} \qquad \text{for } (x,t) \in U^+,$$

*and $d(x,t) = 0$ otherwise.*

First we show a weaker version of Theorem 2.2.

**Lemma 2.3** *There exists a constant $M_1 = M_1(p,N,\Lambda_0)$ such that*

$$S(2^{-j-1},u) \leq M_1 \left(2^{-j}\right)^q,$$

*for all $u \in \mathcal{G}$, and $j \in \mathbb{M}(u)$.*

Observe that the constant $M_1$ here doesn't depend on $L$ (the supremum value of $u$).

**Proof.** We argue by contradiction. Thus we assume that for every $k \in \mathbb{N}$, there are $u_k \in \mathcal{G}$ and $j_k \in \mathbb{M}(u_k)$ such that

$$(2.6) \qquad S(2^{-j_k-1},u_k) \geq k \left(2^{-j_k}\right)^q.$$

Observe that by (2.5) we have $0 \in \mathbb{M}(u) \neq \emptyset$.

Define now

$$(2.7) \qquad \tilde{u}_k(x,t) := \frac{u_k(2^{-j_k}x, 2^{-qj_k}\alpha_k t)}{S(2^{-j_k-1},u_k)},$$

where

$$(2.8) \qquad \alpha_k = \left(\frac{2^{-qj_k}}{S(2^{-j_k-1},u_k)}\right)^{p-2} \leq \left(\frac{1}{k}\right)^{p-2}.$$

The domain of the definition for functions $\tilde{u}_k$ will be the unit cylinder. However, for $p < 2$ this may cause problems. The reason for this is that when $p < 2$, $\alpha_k \to \infty$. This is why the present technique fails for $p < 2$.

From all the above, and the definitions of $\mathbb{M}$ and $\mathcal{G}$ it follows that

$$0 \le \tilde{u}_k \le A \qquad \text{in } Q_1^- \qquad \text{(by doubling)} \tag{2.9}$$

$$\sup_{Q^-_{(1/2)}} \tilde{u}_k = 1 \qquad \text{(by (2.4) and(2.7))} \tag{2.10}$$

$$\tilde{u}_k(0,0) = 0 \qquad \text{(by (2.3))} \tag{2.11}$$

$$\partial_t \tilde{u}_k \ge 0 \qquad \text{in } Q_1^- \qquad \text{(by (2.4))} \tag{2.12}$$

Now by (2.1) and (2.6)

$$\|\operatorname{div}(|\nabla \tilde{u}_k|^{p-2}\nabla \tilde{u}_k) - \partial_t \tilde{u}_k\|_{\infty,Q_1} \le \Lambda_0 k^{1-p}. \tag{2.13}$$

Invoking compactness arguments (see [D, Lemma 14.1 page 75], we infer that a subsequence of $\tilde{u}_k$ converges locally uniformly in $Q_1^-$ to a function $u$. Moreover, the limit function $u \not\equiv 0$, by (2.10), and it satisfies, by (2.9), (2.11)–(2.13)

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) - \partial_t u = 0, \qquad u(0,0) = 0, \qquad u \ge 0, \qquad \partial_t u \ge 0,$$

in $Q_1^-$, i.e., $u$ is a nonzero, nonnegative $p$-parabolic function in $Q_1^-$ that is non-decreasing in time. It also takes a local minimum at the origin. For $p = 2$ this gives a contradiction to the strong minimum principle. For $p > 2$ we need to study the problem further, due to the lack of the strong minimum principle.

Going back to the analysis made above, we see that the supremum of $u$ on the zero time level is larger than or equal to that of the negative time levels, since $\partial_t u \ge 0$. Hence using (2.10) we will have

$$\sup_{B(0,1/2)} u(x,0) = 1.$$

Next we show that $u$ is time independent, i.e.,

$$\partial_t u = 0, \qquad \text{in } Q_1^-. \tag{2.14}$$

Suppose this holds for the moment. Then $u$ is nonzero, nonnegative $p$-harmonic function in the unit ball and it vanishes at the origin.

Indeed, this is a contradiction to the minimum principle for $p$-harmonic functions [HKM], and the proof will be finished in this case as soon as we prove the time independence of $u$, i.e., (2.14). This follows simply by inspection. Indeed, choosing $(x,t),(x,t')\in Q^-_{1/2}$ and using, first the doubling property and then Hölder's inequality with $G_T=Q^-_{2^{-j_k}}$ and $K=Q^-_{2^{-j_k+1}}$, we will have

$$|\tilde u_k(x,t)-\tilde u_k(x,t')|=\frac{|u_k(2^{-j_k}x,2^{-qj_k}\alpha_k t)-u_k(2^{-j_k}x,2^{-qj_k}\alpha_k t')|}{S(2^{-j_k-1},u_k)}\le$$

$$\le A\frac{|u_k(2^{-j_k}x,2^{-qj_k}\alpha_k t)-u_k(2^{-j_k}x,2^{-qj_k}\alpha_k t')|}{S(2^{-j_k},u_k)}\le$$

$$\le A\gamma\frac{\|u_k\|_{Q_{2^{-j_k}}}}{S(2^{-j_k},u_k)}\left(\frac{\|u_k\|^{(p-2)/p}_{Q_{2^{-j_k}}}(2^{-qj_k}\alpha_k)^{1/p}|t-t'|^{1/p}}{\mathrm{dist}_p(K,\partial_p G_T)}\right)^{\alpha},\tag{2.15}$$

where $0<\alpha<1$ is as in Hölder's inequality. Since here $G_T=Q^-_{2^{-j_k}}$ and $K=Q^-_{2^{-(j_k+1)}}$, we have

$$\mathrm{dist}_p(K,\partial_p G_T)\ge 2^{-(j_k+1)q/p}\|u_k\|^{(p-2)/p}_{Q^-_{2^{-j_k}}}.\tag{2.16}$$

It is noteworthy that in (2.16) we have used that $p>2$. Next using (2.16) we reduce the estimate (2.15) to

$$|\tilde u_k(x,t)-\tilde u_k(x,t')|\le A\gamma 2^{q/p}\left(\alpha_k|t-t'|\right)^{\alpha/p}$$

which tends to zero, for $p>2$, and $k\to\infty$. Hence $u$ is $t$-independent, and the proof is completed. ∎

**Proof of Theorem 2.2.** Let us take the first $j$ for which

$$S(2^{-j},u)>2^qM_12^{-qj}.$$

(If there is no such $j$ then we are done.) It follows that

$$S(2^{-j+1},u)\le 2^qM_12^{-q(j-1)}<2^qS(2^{-j},u)\le AS(2^{-j},u),\tag{2.17}$$

i.e. $j-1\in\mathbb{M}$, so that Lemma 2.3 holds for $j-1$. Now we arrive at the following obvious contradiction to (2.17)

$$S(2^{-j},u)\le S(2^{-j+1},u)\le M_12^{-q(j-1)}=2^qM_12^{-qj}.$$

Therefore

$$S(2^{-j},u)\le 2^qM_12^{-qj}\qquad\forall j,$$

which implies

$$\sup_{Q^-_r(0,0)}u\le 2^{2q}M_1r^q\qquad\forall\ r\le 1.$$

To obtain a similar estimate for $u$ over the whole cylinder (and not only over the lower half part) we use a barrier from above. Here is how. Define $w = B_1|x|^q + B_2 t$ where $B_2 = N(qB_1)^{p-1}$ and $B_1 > 0$. Let now $Q_1^+ = B(0,1) \times (0,1)$. Then

$$\Delta_p w - \partial_t w = 0 \leq \Delta_p u - \partial_t u \qquad \text{in } Q_1^+.$$

Since, by choosing $B_1$ large, we will have $w \geq u$ on $\partial_p Q_1^+$, where for the estimate on $\{t = 0\}$ we have used the previous discussion, i.e., $S(r,u) \leq Cr^q$. Hence by the comparison principle we will have $w \geq u$ in $Q_1^+$. Therefore

$$\sup_{Q_r(0,0)} u \leq M_2 r^q.$$

From here the proof of Theorem 2.2 follows. ■

## 3. Proof of the main theorem

Now we complete the proof of the main theorem. Having the estimates from below and above for the function $u$, one can proceed as in [KKPS]. For completeness we carry out the minor changes in the proof of [KKPS].

**Proof of the main theorem.** We assume, without loss of generality, that the compact $K$ in the main theorem is the closed unit cylinder $\overline{Q}_1$, and moreover that $\overline{Q}_2 \subset \Omega_T$.

For $(x,t) \in U^+ \cap \overline{Q}_1$ let $d(x,t)$ be defined as in Theorem 2.2 and take $(x^0, t^0) \in \partial U^+ \cap \overline{Q}_1$ which realizes this distance. Next define

$$\tilde{u}(y) = u(x^0 + y, t^0 + s) \quad \text{for } (y,s) \in Q_1.$$

and apply Classical Theorem in Section 1 to arrive at

$$\|\mathrm{div}(|\nabla\tilde{u}|^{p-2}\nabla\tilde{u}) - \partial_t\tilde{u}\|_\infty \leq \Lambda_0, \qquad 0 \leq \tilde{u} \leq \|\theta\|_{\infty,\Omega}, \qquad \tilde{u}(0,0) = 0.$$

Therefore if $M = \max\{1, \Lambda_0^{1/(p-1)}, \|\theta\|_{\infty,\Omega}\}$, then

$$\frac{\tilde{u}(y, M^{2-p}s)}{M} \in \mathcal{G}.$$

Since $M \geq 1$ and $p \geq 2$ we infer by Theorem 2.2 that

$$u(x,t) = \tilde{u}(x - x^0, t - t^0) \leq MM_0\,(d(x,t))^q \ . \tag{3.1}$$

Next, let $(z,\tau) \in \partial U^+ \cap \overline{Q}_1$. Then for $0 < r < 1$, according to Lemma 2.1 and condition (2.4), there exists $x^1 \in \partial B_r(z)$, such that

$$u(x^1, \tau) \geq C_0 r^q.$$

Now by (3.1)

$$C_0 r^q \le u(x^1,\tau) \le MM_0\,(d(x,\tau))^q\,,$$

which implies that

$$d(x^1,\tau) \ge \delta r, \qquad \delta = \left(\frac{C_0}{MM_0}\right)^{(p-1)/p},$$

or equivalently,

$$B_{\delta r}(x^1) \cap B_r(z) \subset U^+.$$

Note that $\delta \le 1$. Since $x^1 \in \partial B_r(z)$, there is a ball

$$B_{(\delta/2)r}(y) \subset B_{\delta r}(x) \cap B_r(z) \subset B_r(z) \setminus \partial U^+.$$

This shows that $\partial U^+ \cap \{t=\tau\} \cap \overline{B}_1$ is porous with the porosity constant $\delta/2$. The theorem is proved. ■

## 4. General Operators

The technique employed in this paper is quite flexible and can be applied to general situations and to other operators. The mere requirement in the technique is the compactness argument and the strong maximum principle for the limit operator.

The main obstacle of applying the techniques in this paper to the case of $1 < p < 2$ is that this case does not reduce to a time-independent situation as that for $p > 2$.

There are plenty of operators where the above technique should work. we will give some examples of such operators. One needs to interplay between the homogeneity of the operator in its variables and the scaling chosen. Let us give some examples.

**Example 1.** Consider the operator

$$\Delta_p u - \partial_t\left(|u|^{p-2}u\right).$$

Then it is known that this operator has the Harnack inequality (see [T]). In the paper of Peter Lindqvist [L] one can find some interesting discussions concerning other variants of this operator. For this operator one needs to define

$$S_r = \sup_{B_r \times [-r^p,0]} u\ ,$$

then one gets the correct growth which is $r^p$.

**Example 2.** Another operator that is probably easy to handle is

$$\operatorname{div}\left(|u|^{m-1}|\nabla u|^{p-2}\nabla u\right)-\partial_t u,$$

where we need to take $m+p\geq 1$, $p>1$. Here one should consider a scaling of the type

$$\frac{u(rx, r^p S_r^{m+p-1}t)}{S_r},$$

to obtain a growth of the form

$$S_r \leq C r^{p/(p+m-1)}.$$

Recently this technique has been applied by the present author and K. Lee to viscosity solutions for nonlinear operators [LS], where also the regularity of the free boundary has been established.

An interesting question is whether one can relax the conditions $u\geq 0$, $\partial_t u\geq 0$. For $p=2$ this problem has been completely investigated by the present author, L. Caffarelli and A. Petrosyan [CPS]. Similarly the situation where the free boundary hits the fixed boundary is also treated by [ASU].

## 5. Appendix

In this part we will give a sketch of a proof of Classical Theorem presented in the first section.

The existence of a unique solution to the above variational inequality can be shown by using classical techniques such as penalization (see [F], [KiSt] for $p=2$). One introduces a family of functions $\beta_\epsilon(s)$ $(0<\epsilon<1)$ with certain properties such as:

$$\beta_\epsilon \in C^\infty(\mathbb{R}),$$

$$\beta_\epsilon'(s)\geq 0,$$

$$\beta_\epsilon(s)\to -\infty \qquad \text{if } s<0,\ \epsilon\to 0,$$

$$\beta_\epsilon(s)=0 \qquad \text{if } s>\epsilon,$$

$$\beta_\epsilon(s)\leq \Lambda_0, \qquad \text{and} \qquad \beta_\epsilon(0)=-\Lambda_0,$$

where $\Lambda_0$ is the upper bound for $f$. The choice of $\Lambda_0$ is for technical reasons, that will be apparent in the below analysis.

Next we approximate the $p$-Laplacian with a second order smooth operator. We define, namely,

$$\Delta_p^\epsilon u=\operatorname{div}\left((|\nabla u|^2+\epsilon)^{(p-2)/2}\nabla u\right).$$

Then, the penalized problem is to find a solution $u^\epsilon$ to

$$\Delta_p^\epsilon u^\epsilon - \partial_t u^\epsilon - \beta_\epsilon(u^\epsilon) = f_\epsilon, \qquad \text{in } \Omega_T, \tag{5.1}$$

and with the boundary value $u^\epsilon = \theta$ on $\partial_p \Omega_T$. Here $f_\epsilon$ is a smooth enough approximation of $f$. Next, by classical parabolic theory for uniformly elliptic operators, for each $\epsilon$ there is a solution $u^\epsilon$ to (5.1).

As the reader have already noticed, the main result in this paper relies heavily on a basic feature, carried by the solution(s) of the above problem, i.e.,

$$\partial_t u \geq 0. \tag{5.2}$$

To see this let us first take (smooth) functions $f_1, f_2, \theta_1, \theta_2$ such that $f_1 \leq f_2$, $\theta_1 \geq \theta_2$. Let also $u_1$ and $u_2$ denote their corresponding solutions to the penalized problem (5.1). Suppose the relatively open set $U = \{u_2 > u_1\}$ is nonempty. Since on $\partial_p\Omega_T$, $u_1 = \theta_1 \geq \theta_2 = u_2$ there must hold $U \subset \Omega_T$. Next, in $U$, one has

$$\Delta_p^\epsilon u_1 - \partial_t u_1 = \beta_\epsilon(u_1) + f_1 \leq \beta_\epsilon(u_2) + f_2 = \Delta_p^\epsilon u_2 - \partial_t u_2,$$

where the inequality is a consequence of the monotonicity of $\beta$. Now applying the strong comparison principle (for uniformly elliptic operators) we obtain a contradiction. It should be mentioned that the form of the strong comparison principle, applied here, uses the fact that at the maximum point of $u_2 - u_1$ we have $\nabla u_1 = \nabla u_2$, and hence the classical computations work out as usual.

Next, for $t > 0$ and $h > 0$, set

$$\begin{aligned} f_1(x,t) &= f(x,t+h), & \theta_1(x,t) &= \theta(x,t+h), & u_1(x,t) &= u(x,t+h),\\ f_2(x,t) &= f(x,t), & \theta_2(x,t) &= \theta(x,t), & u_2(x,t) &= u(x,t), \end{aligned}$$

and for $t = 0$ set

$$\theta_1(x,0) = u(x,h), \qquad \theta_2(x,0) = \theta(x,0).$$

To apply the above comparison argument we need only to verify that $\theta_1(x,t) \geq \theta_2(x,t)$. However, to enforce this inequality we need to show that $u^\epsilon \geq 0$. But this is a consequence of the strong maximum/minimum principle. Indeed, in the set $\{u^\epsilon < 0\}$ we have $\beta_\epsilon(u^\epsilon) + f \leq 0$ and therefore $u^\epsilon$ becomes supersolution to $\Delta_p^\epsilon$ there. Hence the maximum/minimum principle applies and we must have $\{u^\epsilon < 0\}$ is empty.

For $p = 2$ one can show that $\partial_t u^\epsilon \leq C$, where $C$ is independent of $\epsilon$ (see [F, (9.14)], [KiSt, page 281]). The classical techniques in proving the upper bound for the time derivative for the linear case $p = 2$ is not applicable to the general case.

Observe that the operator $\Delta_p^\epsilon$ satisfies all conditions of $p$-parabolicity (see [D, page 16]). Hence $u^\epsilon \in C^{2,\alpha}(\Omega_T)$ (in both variables, but not uniformly in $\epsilon$). Now by (5.2) and the boundary data we have $u^\epsilon \geq 0$ in $\Omega_T$. Hence by the definition of $\beta_\epsilon$ we must have

$$|\beta_\epsilon(u^\epsilon)| \leq \Lambda_0.$$

From here we conclude that

$$u^\epsilon, \nabla u^\epsilon \in C^\alpha(\Omega_T) \quad u^\epsilon \in V^{1,p}(\Omega_T) \tag{5.3}$$

(uniformly in $\epsilon$); see [D, page 245, 291]. Obviously, from here, we conclude that for a subsequence $\epsilon' \to 0$ we must have a limit function $u$ satisfying the conditions in (5.3).

To verify the variational inequality for the limit function we replace equation (5.1) with its Steklov average and multiply the new equation with $(u^\epsilon - w)$, with $w \geq \delta > 0$, to arrive at

$$\left[-\mathrm{div}\left((|\nabla u^\epsilon|^2 + \epsilon)^{(p-2)/2}\nabla u^\epsilon\right)_h + \partial_t u_h^\epsilon + (\beta_\epsilon(u^\epsilon))_h + (f_\epsilon)_h\right](u^\epsilon - w) = 0 \tag{5.4}$$

Letting $\epsilon < \delta$ we see that $\beta_\epsilon(w) = 0$, and therefore (using the monotonicity of $\beta$) we can disregard from the term $\beta_\epsilon$ in (5.4) by changing the equality sign to an inequality

$$\left[-\mathrm{div}\left((|\nabla u^\epsilon|^2 + \epsilon)^{(p-2)/2}\nabla u^\epsilon\right)_h + \partial_t u_h^\epsilon + (f_\epsilon)_h\right](u^\epsilon - w) \leq 0. \tag{5.5}$$

To arrive at variational inequality (1.1) we set $w = v + \epsilon$ with $v \in \mathcal{K}_\theta$, and integrate (5.5) over $\Omega$ for fixed $t$. Then integrating by parts the first term we will end up at (1.1) for $u^\epsilon$, and with an extra term involving boundary integral

$$\int_{\partial\Omega} \left((|\nabla u^\epsilon|^2 + \epsilon)^{(p-2)/2}\nabla u^\epsilon \cdot \nu\right)_h \epsilon \, d\sigma, \tag{5.6}$$

where $d\sigma$ denotes the boundary element and $\nu$ the unit normal vector to $\partial\Omega$; it is here that we need smoothness for $\partial\Omega$. If the boundary value $\theta$ is smooth enough then (5.6) tends to zero, as $\epsilon$ does so, and we obtain (1.1).

The uniqueness of the solution also follows by employing similar techniques as the one used in proving (5.2). Finally applying similar techniques as that of [KKPS] we will obtain (1.3)–(1.4), we leave the obvious details to the reader.

## References

[ASU] Apushkinskaya, D. E., Shahgholian, H. and Uraltseva, N. N.: Boundary estimates for solutions to the parabolic free boundary problem. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)* **271** (2000). *Kraev. Zadachi Mat. Fiz. i Smezh. Vopr. Teor. Funkts.* **31**, 39–55, 313.

[B] Barenblatt, G. I.: On self-similar motions of a compressible fluid in a porous medium. (Russian)*Akad. Nauk SSSR. Prikl. Mat. Meh.* **16** (1952), 679–698.

[CKS] Caffarelli, L., Karp, L. and Shahgholian, H.: Regularity of a free boundary with application to the Pompeiu problem. *Ann. of Math. (2)* **152** (2000), 269–292.

[CPS] Caffarelli, L., Petrosyan, A. and Shahgholian, H.: Regularity of a free boundary in parabolic potential theory, manuscript.

[D] DiBenedetto, E.: *Degenerate parabolic equations*. Universitext. Springer-Verlag, New York, 1993.

[F] Friedman, A.: *Variational principles and free-boundary problems* (second edition). Robert E. Krieger Publishing Co., Inc., Malabar, Florida, 1988.

[HKM] Heinonen, J., Kilpeläinen, T. and Martio, O.: *Nonlinear potential theory of degenerate elliptic equations*. Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1993.

[KS] Karp, L. and Shahgholian, H.: On the optimal growth of functions with bounded Laplacian. *Electron. J. Differential Equations* 2000, no. 03, 9 pp.

[KKPS] Karp, L., Kilpeläinen, T., Petrosyan, A. and Shahgholian, H.: On the porosity of free boundaries in degenerate variational inequalities. *J. Differential Equations* **164** (2000), no. 1, 110–117.

[KiSt] Kinderlehrer, D. and Stampacchia, G.: *An introduction to variational inequalities and their applications*. Pure and Applied Mathematics **88**. Academic Press, Inc., New York-London, 1980.

[L] Lindqvist, P.: A criterion of Petrowsky's kind for a degenerate quasilinear parabolic equation. *Rev. Mat. Iberoamericana* **11** (1995) no. 3, 569–578.

[LS] Lee, K. and Shahgholian, H.: Regularity of a free boundary for viscosity solutions of nonlinear elliptic equations. *Comm. Pure Appl. Math.* **54** (2001), 43–56.

[MV] Martio, O. and Vuorinen, M.: Whitney cubes, $p$-capacity, and Minkowski content. *Exposition. Math.* **5** (1987), no. 1, 17–40.

[T] Trudinger, N. S.: Pointwise estimates and quasilinear parabolic equations. *Comm. Pure. Appl. Math.* **21** (1968), 205–226.

*Recibido*: 20 de octubre de 2000
*Revisado*: 23 de julio de 2002

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The author was supported by the Swedish Research Council. Thanks go to Peter Lindqvist for invaluable suggestions and hours of discussions on the $p$-parabolic equation, and to the referee for his/her view points which led to a major revision of the original version of the paper.